University of Trieste

Department of Mathematics and Geosciences
Bachelor degree in Mathematics

SKEIN RELATIONS AND POLYNOMIAL INVARIANTS OF KNOTS AND LINKS

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Introduction

La teoria dei nodi è una branca della topologia dedicata allo studio di nodi e link, ovvero immersioni rispettivamente di una o più 1-sfere in $\mathbb{R}^3$. Due link sono equivalenti se possono essere trasformati l'uno nell'altro attraverso un'isotopia ambiente. Il problema fondamentale, quindi, è quello di distinguere due link diversi e a ciò è dovuta l'introduzione di invarianti, ovvero oggetti algebrici che non cambiano per link equivalenti.

L'obiettivo di questa tesi è lo studio di tre particolari invarianti: il polinomio di Alexander, il polinomio di Jones e il polinomio HOMFLY. L'introduzione di invarianti polinomiali, a partire dal 1923 con quello di Alexander e poi nel 1984 con quello di Jones, ha segnato un grande passo avanti nella classificazione di nodi e link: tali polinomi, attraverso le formule di skein, hanno la proprietà di essere "concettualmente facili" da calcolare e risulta ancora più semplice confrontarli. Quest'ultima osservazione è uno dei motivi principali che ha portato alla sviluppo del polinomio di Alexander, il quale deriva dal gruppo di un nodo che risulta più difficile da comparare.

L'elaborato è quindi strutturato nel modo seguente. Il Capitolo 1 presenta i concetti fondamentali della teoria, introducendo in particolare la nozione di diagramma, le mosse di Reidemeister e alcuni invarianti che saranno utilizzati nei capitoli successivi. Gran parte della tesi è dedicata al Capitolo 2, in cui si è sviluppato il polinomio di Alexander per nodi attraverso un approccio geometrico. La definizione si è poi estesa a link orientati, arrivando così ad una relazione cosiddetta di skein che caratterizza il polinomio. Nel Capitolo 3 si introduce la parentesi di Kauffman con un approccio combinatorio, la quale viene poi modificata per ottenere il polinomio di Jones. Questo viene in seguito caratterizzato da una nuova formula di skein. Infine il Capitolo 4 è dedicato al polinomio HOMFLY, di cui i polinomi precedenti si rivelano essere un caso particolare.

Knot theory is a branch of topology dedicated to the study of knots and links, which are embeddings of one or more 1-sphere in $\mathbb{R}^3$ respectively. Two links are equivalent if they can be related by an ambient isotopy. The fundamental problem of the theory is that of distinguish different links and from this fact follows the introduction of invariants, which are algebraic objects that remain unchanged under ambient isotopies of the link.

The aim of this thesis is the study of three particular invariants: the Alexander polynomial, the Jones polynomial and the HOMFLY polynomial. The introduction of polynomial invariants, starting with the Alexander polynomial in 1923 and then in 1984 with the Jones one, represented a significant breakthrough in the classification of knots and links: such polynomials have the great advantage of being "conceptually simply" to compute and they are much more easy to compare. This observation is one of the main reasons that led to the development of the Alexander polynomial, which comes from the knot group that is difficult to compare.

The paper is organised as follows. Chapter 1 presents the fundamental concepts of the theory, introducing the notion of link diagram, the Reidemeister moves and some invariants that will be useful in the following chapters. The main part of the thesis is dedicated to Chapter 2, in which the Alexander polynomial for knots is developed with a geometric approach. Then we extend the definition to oriented links, obtaining a so-called skein relation which characterizes the polynomial. In Chapter 3 we introduce the Kauffman bracket with a combinatorial approach, which will be then modified in order to obtain the Jones polynomial. This one will be then characterized by a new skein relation. In conclusion, Chapter 4 is dedicated to the HOMFLY polynomial, of which the previous polynomials are particular cases.

Chapter 1

Fundamental concepts of knot theory

In this chapter we will give a definition of knot and link, following the line of [Burde and Zieschang, 2003]. Intuitively, we can think of a knot as a closed curve lying in a 3-dimensional space. If we consider more curves, we obtain a link. It is customary to define links as living in $\mathbb{S}^3$, for it is compact. Moreover, since $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, it is equivalent to considering links in $\mathbb{R}^3$.

We will then elaborate on some of the links' properties. An explanation of what it means for two links to be equivalent and how we are able to draw diagrams of them will be given. We will also introduce the Reidemeister moves, which show us the fundamental three ways a link diagram can be changed without altering the link itself. Finally, we will show some examples of link invariants taken from [Rolfsen, 1976].

1.1 Knots and links

Definition 1.1.1. A *knot* is the image of an embedding $h\colon \mathbb{S}^1 \to \mathbb{S}^3$. More generally, a *link* of m components is the image of embeddings of a disjoint union of m circles into $\mathbb{S}^3$, $h\colon \bigsqcup_1^m \mathbb{S}^1 \to \mathbb{S}^3$.

Therefore, a knot is a link of one component. We write K and L to denote a knot and a link respectively. Sometimes we will refer to a link as the embedding rather than its image.
Figure 1.1 shows four diagrams of knots and links that will be used in examples throughout this thesis. From left to right, they are called 3_1, $\bar{3}_1$ (the *mirror image* of 3_1, *i.e.* its reflection through a plane), the Hopf link and 5_1. We call 3_1 the left trefoil, $\bar{3}_1$ the right trefoil and 5_1 the cinquefoil. We will later point out how we can draw these diagrams and how the naming is done.

Let us now define what it means for two links to be equivalent. Before doing so, we recall the following

Definition 1.1.2. Let us consider X, Y topological spaces and $h, h'\colon X \to Y$ embeddings. A continuous map $H\colon Y \times [0,1] \to Y$ is defined to be an *ambient isotopy* taking h to h' if

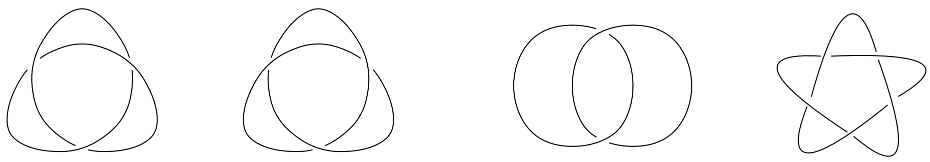


Figure 1.1: From left to right: 3_1, $\bar{3}_1$, the Hopf link and 5_1.

- $H(\cdot,0)$ is the identity map,
- for every fixed t, $H(\cdot,t)$ is a homeomorphism from Y to itself,
- $H(h(\cdot),1) = h'(\cdot)$.

Definition 1.1.3. Two links L and L' are equivalent if there exists an ambient isotopy between them. We call such equivalence class *link type*. In the case of knots, we will talk about knot type.

Figure 1.2 depicts a simple example of equivalent knots. The knot type in the picture is called *unknot* and it is denoted with 0_1.

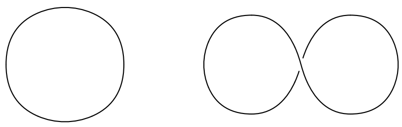


Figure 1.2: The unknot.

A particular class of links is that of *polygonal links*, for which every component is made up of a finite number of consecutive segments in $\mathbb{R}^3$ called edges, with extremes called vertices. A link which is equivalent to a polygonal one is called *tame*, otherwise it is called *wild*. A famous example of wild knot is the Fox knot in Figure 1.3.

Since wild links come up with pathological behaviours, this will be the only reference to this kind of links. From now on, we will omit to specify the adjective qualifying the link, always implying that a tame one is considered. Finally, we also mention the following

Theorem 1.1.4. *A link parametrized by a function of class C^1 is tame.*

A proof can be found in [Crowell and Fox, 1977].

1.2 Reidemeister moves

In depicting knots and links, we consider projections of them as subsets of $\mathbb{R}^3$ onto a plane. Definitions are found in [Rolfsen, 1976] and [Burde and Zieschang, 2003].

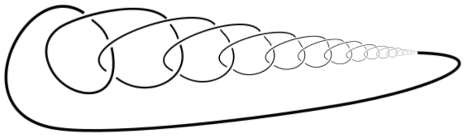


Figure 1.3: The Fox knot.

Definition 1.2.1. Let L be a polygonal link in $\mathbb{R}^3$. Let P be a plane and $\pi\colon \mathbb{R}^3 \to P$ the orthogonal projection. We say that P is regular for L provided that every $\pi^{-1}(x)$, $x \in P$, intersects L in 0, 1 or 2 points and, if 2, neither of them is a vertex.

Proposition 1.2.2. *Given a link L and a plane P, it is possible to make P regular for the link by arbitrarily small perturbation of either P or L.*

Hence, with regular projections we only need to worry about two points projecting to the same point. At such double points, called *crossings*, our convention is to draw the segment containing points closer to the plane as a broken line and the segment containing the points further away from the plane as a solid line (see Figure 1.4).

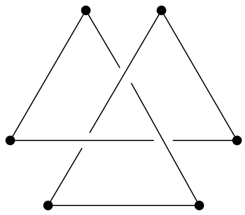


Figure 1.4: A regular projection of the left trefoil.

In a regular projection, we call *arc* the projection of an arc in the link. After a regular projection, we can depict the link with smooth rather than polygonal arcs with an *isotopy of the link projection*, that is an isotopy of the arcs. In this fashion, we capture all information about the link in the planar projection with the above convention about double points, obtaining a *link diagram*.

Manipulating link diagrams can be a daunting task. The work of Reidemeister and independently Alexander and Briggs in the 1920s streamlined this task significantly. The idea was to formalize the procedure of showing that two diagrams represent the same link and this was achieved by isolating the three moves shown in Figure 1.5, called *Reidemeister moves*.

Those moves represent ambient isotopies that take place in a specified region of the link diagram: inside the region, the diagram looks as in the left-hand side (right-hand side respectively) of one of the picture in Figure 1.5. After the move, the region looks as in the right-hand side (or left-hand side) of the same picture, while outside it everything remains the same.

Definition 1.2.3. We call two link diagrams D and D' equivalent if they can be transformed into the other by a sequence of Reidemeister moves and planar isotopies.

What make the Reidemeister moves so effective is the following

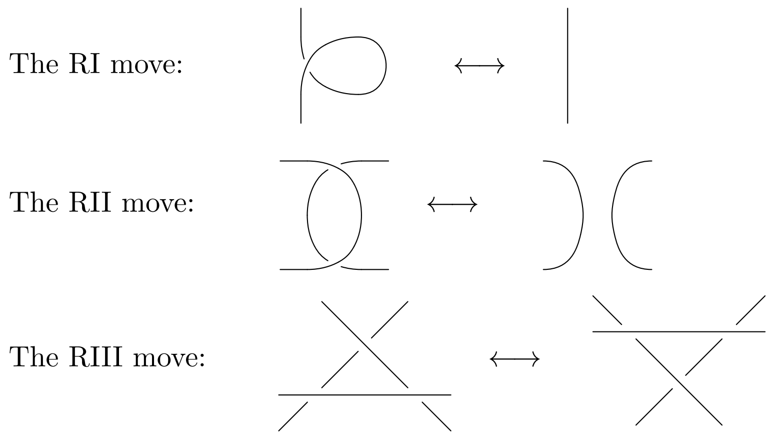


Figure 1.5: Reidemeister moves.

Theorem 1.2.4. *Two links L and L' are equivalent if and only if all their diagrams are equivalent.*

A proof can be found in [Burde and Zieschang, 2003]. We can express the theorem as

$$\frac{\{\text{links}\}}{\text{amb. isotopies of } \mathbb{R}^3} \longleftrightarrow \frac{\{\text{link diagrams}\}}{\text{RI, RII, RIII, planar isotopies of the diagram}},$$

and it allows us to "define" the notion of link type (and consequently, the notion of link invariant) as the equivalence class of link diagrams, modulo Reidemeister moves and planar isotopies of the diagram. For the purpose of studying link type, the above relation is fundamental because, while the left-hand side is topological, we can deal with the right-hand side in a combinatorial way.

1.3 Knot and link invariants

In order to prove that two links are equivalent, we can show a step-by-step process of deformation between the two links. Conversely, it is not a simple problem to show that two links are not equivalent. It can, however, be proven by means of invariants. For a set S, we call the map

$$\mathscr{I} : \{\text{links}\} \longrightarrow S$$

an *invariant* of links if it satisfies $\mathscr{I}(L) = \mathscr{I}(L')$ for any two equivalent links L and L'. If $\mathscr{I}$ is only defined for knots, it is called a knot invariant. We will see some knot and link invariants that will allow us to distinguish between the various links in Figure 1.1.

One of the most famous invariants is the crossing number.

Definition 1.3.1. Given a link L, we define the *crossing number* $c(L)$ of the link as the minimum number of crossings in any regular projection of the link.

Crossing number is a link invariant. Restricting our attention to knots, it is easy to see that the only zero-crossing knot is the unknot, while one and two-crossing knots do not exist. The only three-crossing knots are the trefoil and its mirror image and so on. Knot tables employ the crossing number as an ordering scheme: hence the Rolfsen notation c_i, where c refers to the crossing number and i distinguishes among same crossing number knots. For links of m components, we will write c_i^m. The number of prime knots with a given crossing number is of interest. A knot is prime if it cannot be written as the knot sum of two non-trivial knots. The precise definition can be found in [Rolfsen, 1976]. The sequence, which ignores mirror images, starts with the number of knots of crossing number 3 and continues on. The first terms are

$$1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, 46972, \ldots$$

It is noteworthy that nowadays no one has found a closed form for such sequence. Figure 1.6 contains pictures of all knots up to seven crossings.

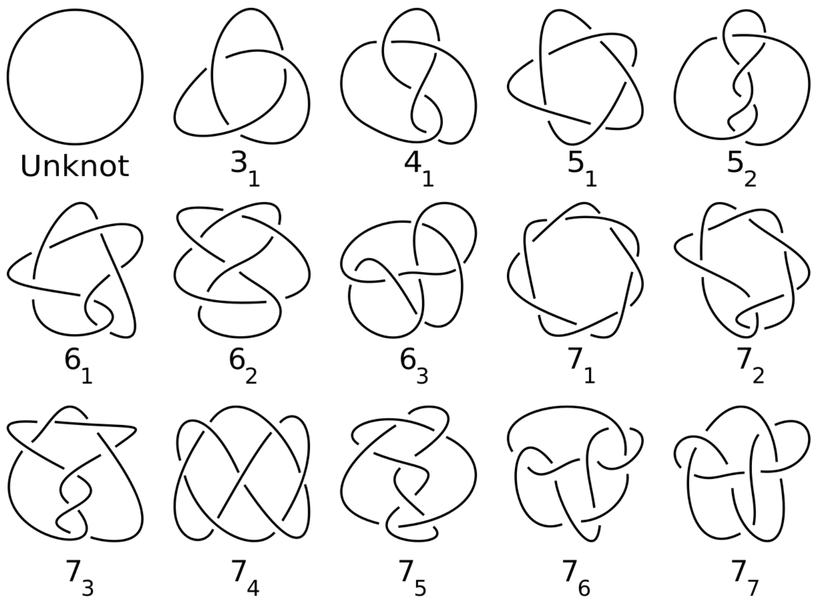


Figure 1.6: Knots up to seven crossings.

Another useful knot invariant is the group of a link.

Definition 1.3.2. The group of link L is defined as the fundamental group of its complement: $\pi_1(\mathbb{S}^3 \setminus L)$.

Since the link complement is connected, different choices of basepoints yield to isomorphic groups. For this reason is common use to omit the explicit reference to the basepoint. An application of van Kampen's theorem leads to the isomorphism $\pi_1(\mathbb{S}^3 \setminus L) \cong \pi_1(\mathbb{R}^3 \setminus L)$. In addition, since complements of equivalent links are homeomorphic, their groups are isomorphic. Thus, the group of a link is an invariant.

Contrary to the fundamental group, the integral homology of the link complement $X = \mathbb{S}^3 \setminus L$ is quite useless, since it depends only on the number of components. In particular, thanks to Hurewicz theorem,

$$\frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \cong H_1(X) \cong \mathbb{Z}^m,$$

where $[\pi_1(X), \pi_1(X)]$ is the commutator subgroup of $\pi_1(X)$. This observation will be useful in Chapter 2.

Since the homology of X does not give any information about L and in general is difficult to deal with the group of a link, we will construct in Chapter 2 some invariants from $\pi_1(X)$ which are easier to compare, including the Alexander polynomial.

We want to define now a link invariant which measure how linked up two components are. This will allow us to distinguish between the Hopf link and the unlink of two components.

For this purpose, it is useful to introduce the notion of *oriented link*. This is simply defined as the image of an embedding of the disjoint union of oriented circles in $\mathbb{S}^3$. The notion of equivalence has to be adjusted: two oriented links are equivalent if there exist an ambient isotopy between them which respects the orientation. A link diagram inherits the orientation of the link.

The Reidemeister moves for oriented diagrams are nothing but the original ones, with all possible configurations of orientation. A possible set of generating moves is shown in Figure 1.7. Theorem 1.2.4 implies the same bijection between oriented link types and oriented diagrams modulo $\overrightarrow{\mathrm{RI}}$,$\overrightarrow{\mathrm{RII}}$,$\overrightarrow{\mathrm{RIII}}$ moves and planar isotopies of the diagram.

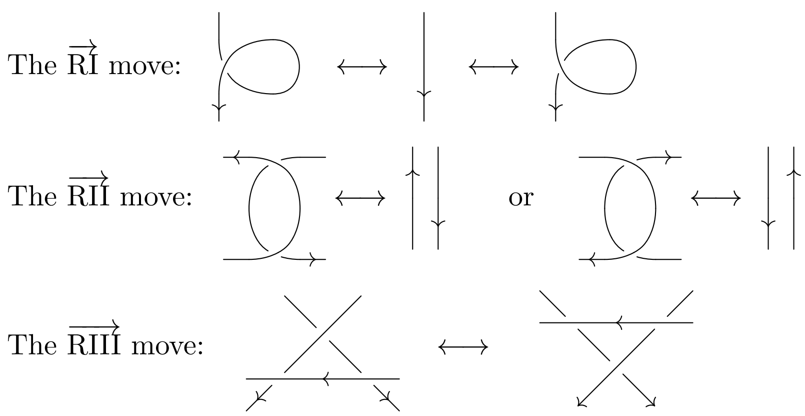
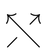


Figure 1.7: Reidemeister moves for oriented diagrams.

As an application, we introduce the linking number as follows. We call the crossings  and  of an oriented diagram a *positive* and *negative* crossing respectively. For a crossing p, we give the sign $\epsilon(p) = \pm 1$ if it is a positive or a negative one respectively.

Definition 1.3.3. Let us consider a diagram of a two-component oriented link. Call the oriented components J and K. The *linking number* is defined as

$$\mathrm{lk}(J,K) = \frac{1}{2} \sum_{p \in J \sqcap K} \epsilon(p),$$

where $J \sqcap K$ is the set of crossings of J with K.

Note that we do not count crossings of the same component. Furthermore,

$$\mathrm{lk}(J,K) = \mathrm{lk}(K,J),$$
$$\mathrm{lk}(-J,K) = -\,\mathrm{lk}(J,K),$$

where $-J$ is J with reverse orientation.

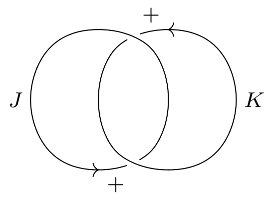
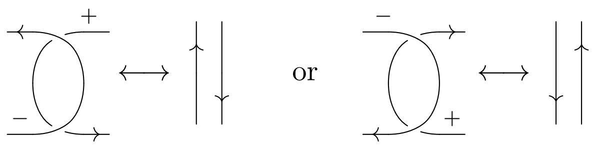


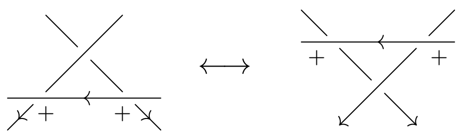
Figure 1.8: Linking number of an oriented Hopf link: $\mathrm{lk}(J,K) = 1$.

Proposition 1.3.4. *The linking number is well-defined (*i.e.*, it does not depends on the diagram) and is a two-component link invariant.*

Proof. We verify that lk is invariant under the $\overrightarrow{\mathrm{RI}}$, $\overrightarrow{\mathrm{RII}}$ and $\overrightarrow{\mathrm{RIII}}$ moves. For the first move, it does not affect both components, so it leaves the linking number unchanged. For the second move, if it regards different components, we find the same contribution to the sum, hence we have no change in the linking number.



Finally, considering all possible combination of the two components and once + and − are assigned to each crossing, it is clear that $\overrightarrow{\mathrm{RIII}}$ does not change the linking number. An example is given in the picture below.



This concludes the proof.

With this link invariant, we can see that the unlink of two components is different from the Hopf link, since they have linking number 0 and 1 respectively (see Figure 1.8).

We may extend the definition of linking number to 1-cycles.

Definition 1.3.5. Let x, y be disjoint oriented closed curves in $\mathbb{S}^3$. Then $[x], [y]$ be 1-cycles in and $\mathbb{S}^3$. Choose a 2-chain $c \in C_2(\mathbb{S}^3)$ so that $x = \partial(c)$. Then $[c \cap y]$ is a 0-cycle, well-defined up to homology. Since $H_0(\mathbb{S}^3) \cong \mathbb{Z}$, $[c \cap y]$ corresponds to an integer, denoted by $\mathrm{lk}([x], [y])$, which we call linking number of $[x]$ and $[y]$.

Proposition 1.3.6. *The linking number has the following properties.*

- *If J, K are oriented knots, than $\mathrm{lk}([J], [K]) = \mathrm{lk}(J, K)$.*
- *Let $[x_0], [y_0], [x_1], [y_1]$ be 1-cycles. If there are homotopies $x_0 \simeq_H x_1$, $y_0 \simeq_G y_1$ such that*
$$\mathrm{Im}(H(\cdot, t)) \cap \mathrm{Im}(G(\cdot, t)) = \varnothing$$
for all t, then
$$\mathrm{lk}([x_0], [y_0]) = \mathrm{lk}([x_1], [y_1]).$$
- *The function lk defined above disjoint cycles is a symmetric bilinear form.*

A proof can be found in [Rolfsen, 1976].

Chapter 2

The Alexander polynomial

A significant part of this thesis is devoted to the Alexander polynomial, first presented in [Alexander, 1928]. There are different ways to compute this invariant, some involving algebraic techniques and others with a more geometric or combinatorial approach.
We will first present a geometric way to acquire the Alexander polynomial specifically for knots, following [Rolfsen, 1976]. Then, we will extend the definition to oriented links as in [Cromwell, 2004]. Finally, we will give a more combinatorial characterization of the polynomial (*i.e.* the Conway polynomial), which will bring us closer to the Jones one presented in Chapter 3.

Let us consider the complement of a knot: $X = \mathbb{S}^3 \setminus K$. Than X is path-connected, locally path-connected and semilocally simply connected. From covering theory, if we take the commutator subgroup $C = [\pi_1(X), \pi_1(X)]$ of the knot group, then there exist a unique covering map $p\colon X_\infty \to X$, up to equivalence of coverings, such that

$$C = p_*(\pi_1(X_\infty)).$$

Since C is a normal subgroup of the knot group, p is regular. Hence we have

$$\mathrm{Aut}(X_\infty, p) \cong \frac{\pi_1(X)}{p_*(\pi_1(X_\infty))} = \frac{\pi_1(X)}{C}.$$

Thanks to Hurewicz theorem, $\pi_1(X)/C \cong H_1(X)$. By an observation in Chapter 1, the first homology group of the knot complement is the infinite cyclic group. Therefore

$$\mathrm{Aut}(X_\infty, p) \cong \mathbb{Z}.$$

This fact clarifies the notation. Since the group of covering automorphisms is the infinite cyclic one, we call X_∞ the *infinite cyclic cover* of the knot complement.

The theory of coverings gives us the covering space with $\mathbb{Z}$ as a group of covering automorphisms. The following paragraph is dedicated to an explicit geometric construction of this space.

2.1 The infinite cyclic cover of a knot complement

Let us recall the following

Definition 2.1.1. Let X be a topological space. A subset $A \subset X$ is said to be bicollared if exists a continuous map $b\colon A \times [-1,1] \to X$ such that $b(a,0) = a$, $\forall a \in A$.

From now on, we will call surface a connected 2-manifold (see Appendix A).

Definition 2.1.2. Let L be a link. A *Seifert surface* for L is a bicollared and compact surface whose boundary is L. If L has an orientation, the bicollar must agree with the orientation of the link.

The following theorem provides an algorithm for the construction of a Seifert surface of a given link.

Theorem 2.1.3 (Seifert algorithm)**.** *Any link has a Seifert surface.*

Proof. If L is not oriented, choose an orientation for every component of L. Let D be an oriented link diagram of L. At each crossing in the diagram, we make the following local change: delete the crossing and reconnect the loose ends in the only way compatible with the orientation. A modified diagram D_* is obtained, which is a set of non-intersecting oriented circles called *Seifert circuits*.
So D_* is the boundary of a set of oriented discs in the plane (they inherit the orientation of the oriented circles). Although discs may be nested, we can make them disjoint by pushing their interior slightly off the plane, starting with the outermost ones and working inward.
We continue by connecting the discs with half-twisted strips at the places where the crossings used to be. The strips are twisted in such a way that they correspond to the type of crossing there was before. The resulting surface is bicollared and compact, but it may occur that it is not connected. In that case, we can join components with tubes, so that the bicollar agrees with the orientation of L. This concludes the proof.

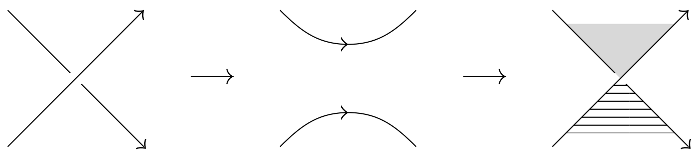


Figure 2.1: Adding an half-twisted strip to in the neighbourhood of a crossing.

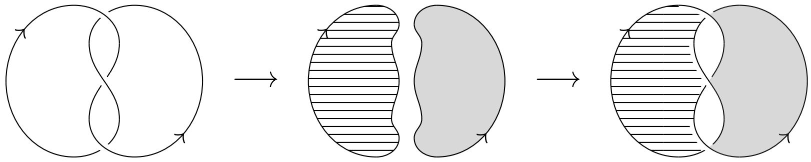
It is important to notice that there could exist different Seifert surfaces for the same link. Furthermore, Seifert algorithm can give different surfaces, depending on one's choice of components' orientation.

The previous theorem assures that the following definition is well-posed.

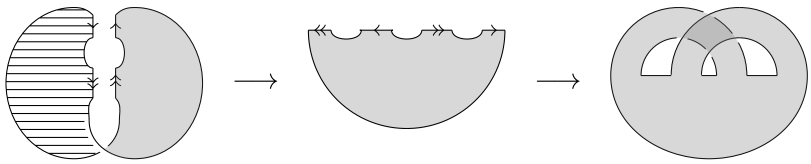
Definition 2.1.4. The *genus* of a link L, denoted with $g(L)$, is the minimum genus of all Seifert surfaces for L.

The genus is a link invariant. It must be pointed out that minimal surface may not be unique and that Seifert algorithm does not give necessarily a minimal surface for the link. The genus has many properties and important consequences in knot theory. See [Rolfsen, 1976] for further readings.

As an example, we compute a (minimal) Seifert surface of the left trefoil. We consider the following non-standard projection.



The resulting surface is a punctured torus, as it can be seen by the following manipulation.



The surface is minimal because it cannot be of genus 0 (otherwise the trefoil would be the unknot and we will prove in Chapter 3 that this is not the case). Hence, $g(3_1) = 1$.

Let us now consider the complement of a knot $X = \mathbb{S}^3 \setminus K$. Let M be a Seifert surface for K and let $b\colon \mathring{M} \times [-1,1] \to \mathbb{S}^3$ be an open bicollar of $\mathring{M}$. We define

$$\begin{aligned} N &= b(\mathring{M} \times (-1,1)) \qquad & Y &= \mathbb{S}^3 \setminus (K \cup N) \\ \mathring{M}^+ &= b(\mathring{M} \times \{1\}) \qquad & \mathring{M}^- &= b(\mathring{M} \times \{-1\}). \end{aligned} \tag{2.1}$$

The space Y, which can be regard as X cut open along M. It has a boundary with two components, each homeomorphic to $\mathring{M}$:

$$\partial Y = \mathring{M}^- \cup \mathring{M}^+.$$

In order to construct X_∞, let us form countably many copies of $(Y, \mathring{M}^-, \mathring{M}^+)$, denoted $(Y_i, \mathring{M}_i^-, \mathring{M}_i^+)$ and $i \in \mathbb{Z}$. From the disjoint union

$$\widetilde{X} = \bigsqcup_{i \in \mathbb{Z}} Y_i,$$

we define the space $X_\infty = \widetilde{X}/\sim$, where $\sim$ is the equivalence relation which identify $\mathring{M}_i^-$ with $\mathring{M}_{i+1}^+$.

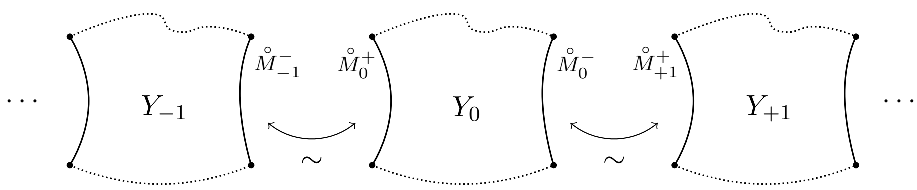


Figure 2.2: Schematic construction of X_∞.

It is clear that we have a covering map $p\colon X_\infty \to X$. Furthermore, we have a covering automorphism $\tau\colon X_\infty \to X_\infty$ which takes the internal points of Y_i to the corresponding ones in Y_{i+1} and $\mathring{M}_{i-1}^- \sim \mathring{M}_i^+$ to $\mathring{M}_i^- \sim \mathring{M}_{i+1}^+$.
$\mathrm{Aut}(X_\infty, p)$ acts transitively on X_∞: if $e_n, e_m \in p^{-1}(x)$, with $e_n \in Y_n$ and $e_m \in Y_m$, then $\tau^{m-n}(e_n) = e_m$. With the same argument, τ generates $\mathrm{Aut}(X_\infty, p)$: if we consider another covering automorphism h which takes $e_n \in Y_n$ to $e_m \in Y_m$ (with $e_n, e_m \in p^{-1}(x)$), then $h = \tau^{m-n}$. In addition, τ has infinite order by construction. This proves that p is regular and $\mathrm{Aut}(X_\infty, p) \cong \mathbb{Z}$.

2.2 Defining the Alexander polynomial

Let us consider the first homology group of the infinite cyclic cover: $H_1(X_\infty)$. It has a natural structure of abelian group, so $\mathbb{Z}$ acts as a ring on it. In addition, $\mathrm{Aut}(X_\infty, p)$ acts as a group. As a consequence, the group ring $\Lambda = \mathbb{Z}[\mathrm{Aut}(X_\infty, p)]$ acts as a ring on $H_1(X_\infty)$ in the following natural way.
Let us fix a generator τ of $\mathrm{Aut}(X_\infty)$ (only two choices are possible). For every $f \in \Lambda$,

$$f = \sum_{i=r}^{s} a_i \, \tau_*^i \qquad r, s \in \mathbb{Z} \text{ with } r \leq s,$$

we define the product of f with an element $\alpha \in H_1(X_\infty)$ by the formula

$$f\alpha = \sum_{i=r}^{s} a_i \, \tau_*^i(\alpha).$$

With the previous definition, $H_1(X_\infty)$ is a (left unitary) Λ-module. Λ can be seen as the ring of Laurent polynomial with integer coefficients, $\mathbb{Z}[t, t^{-1}]$, where the variable $t = \tau_*$ corresponds to a "translation" along X_∞. Therefore we will write

$$f\alpha = \sum_{i=r}^{s} a_i \, t^i \alpha,$$

for $f = \sum_{i=r}^{s} a_i \, t^i \in \Lambda$.

Definition 2.2.1. Let X_∞ the infinite cycling cover of a knot K. We call the Λ-module $H_1(X_\infty)$ the *Alexander invariant* of K.

Definition 2.2.2. Any presentation matrix for the Alexander invariant $H_1(X_\infty)$ of a knot K is called an *Alexander matrix* for K. The order ideal in Λ is called the *Alexander ideal* and any generator Δ_K is called the *Alexander polynomial* (see Appendix C for definitions).

We shall now prove that the previous definition is well-posed, *i.e.* the Alexander invariant is finitely presentable as a Λ-module and that there exists a square Alexander matrix. Note that Δ_K is defined up to multiplication by units of Λ, which are the monomials $\pm t^i$.

2.2.1 Seifert matrix

This paragraph is dedicated to the construction of a particular Alexander matrix, which will give us a proof of the well-posedness of Definition 2.2.2 and a method for the computation of Δ_K.

Let K be an oriented knot, M a Seifert surface in $\mathbb{S}^3$ for K. Let $b\colon \mathring{M} \times [-1,1] \to \mathbb{S}^3$ be an bicollar for $\mathring{M}$, so that it agrees with the orientation of K. Let us take $[x] \in H_1(\mathring{M})$, with $x\colon [0,1] \to \mathring{M}$ a representative loop. In addition, we define

$$x^+ = b(x,1) \qquad x^- = b(x,-1),$$

which can be seen as the image of x under the shift maps

$$s^\pm \colon \mathring{M} \longrightarrow \mathring{M} \times \{\pm 1\}.$$

We note that for every $[x], [y] \in H_1(\mathring{M})$, x and y^+ are disjoint closed oriented curves in $\mathbb{S}^3$ (with orientation induced my the standard orientation of $[0,1]$). Hence, the following definition is well-posed.

Definition 2.2.3. We call the function $\Theta\colon H_1(\mathring{M}) \times H_1(\mathring{M}) \to \mathbb{Z}$, defined by

$$\Theta([x],[y]) = \operatorname{lk}(x, y^+),$$

a *Seifert form* for K. Thanks to Corollary A.0.7, we can choose a basis $e_1, \dots, e_{2g}$ for the $\mathbb{Z}$-module $H_1(\mathring{M})$ (where g is the genus of M) and define the *Seifert matrix* V to be the $2g \times 2g$ integral matrix with entries

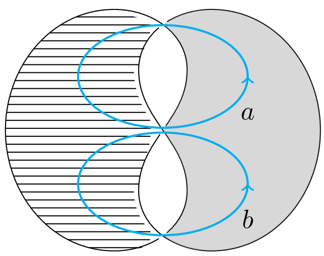
$$V_{i,j} = \Theta(e_i, e_j).$$

The Seifert matrix depends upon the choices of the surface M and the basis e_i (the requirement on K to be oriented avoids the further ambiguity about the choice of the bicollar). In addition, thanks to Proposition 1.3.6, we have that

$$\Theta([x],[y]) = \operatorname{lk}(x^-, y^+) = \operatorname{lk}(x^-, y).$$

This observation will be useful in the next discussion.

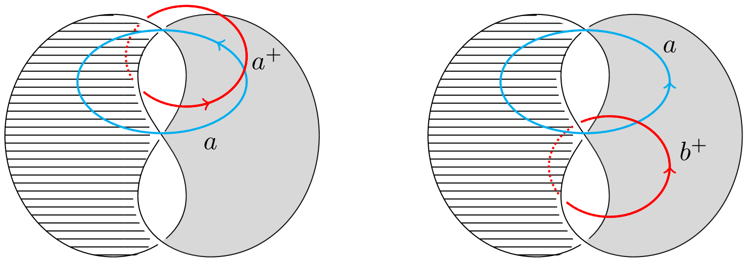
Let us compute a Seifert matrix for the left trefoil. We choose a basis for the Seifert surface found in the previous section as in the picture below.



The knot is oriented so that the filled area is the positive side of the bicollar, while the striped one is the negative side. Then the related Seifert matrix is

$$V = \begin{pmatrix} \operatorname{lk}(a,a^+) & \operatorname{lk}(a,b^+) \\ \operatorname{lk}(b,a^+) & \operatorname{lk}(b,b^+) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

as it can be seen, for example, for $V_{1,1}$ and $V_{1,2}$ from the picture below.



Although the Seifert matrix is not a knot invariant, the connection with the Alexander matrix is given by the following

Theorem 2.2.4. *If V is a Seifert matrix for an oriented knot K, then $V^T - tV$ is an Alexander matrix for K.*

Here T denotes the transpose matrix and tV is the matrix V with every entry multiplied by t. It is clear that the matrix $V^T - tV$ is a $2g \times 2g$ matrix over Λ. This proves that the Alexander invariant is finitely presentable as Λ-module and that there exists a square Alexander matrix.
An immediate consequence of the theorem is that

$$\Delta_K(t) = \det(V^T - tV).$$

Furthermore, since

$$\Theta([x],[y]) = \mathrm{lk}(x, y^+) = \mathrm{lk}(y, x^-),$$

V^T is a Seifert matrix for $-K$, which denotes K with the opposite direction. Hence,

$$\Delta_K = \Delta_{-K}.$$

This fact agrees with the definition of the Alexander polynomial, since the infinite cyclic cover of the knot complement does not require orientation on the knot.

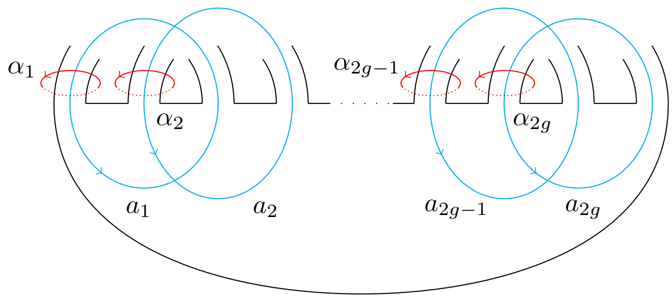
Let us now prove the theorem. Before that, some preparatory lemmata are needed.

Lemma 2.2.5. *Let M be a Seifert surface for a knot K. Then there exists a basis $[a_1], \dots, [a_{2g}]$ for $H_1(\mathring{M})$ as $\mathbb{Z}$-module and a basis $[\alpha_1], \dots, [\alpha_{2g}]$ of $H_1(\mathbb{S}^3 \setminus M)$ as $\mathbb{Z}$-module such that*

$$\mathrm{lk}(a_i, \alpha_j) = \delta_{i,j}.$$

We call $\{[\alpha_j]\}$ the dual basis for $\{[a_j]\}$.

Part of the proof. Since $\mathring{M}$ is homeomorphic to a disk with $2g$ handles, we can consider the basis $[a_j]$ represented by loops which passes throughout the disk and one handle. Then we can choose the α_j in $\mathbb{S}^3 \setminus M$ as in the picture below. It can be shown that the α_j form a basis for $H_1(\mathbb{S}^3 \setminus M)$.



Let us consider the space Y defined by Equation 2.1. Since Y is homotopy equivalent to $\mathbb{S}^3 \setminus M$, we can consider the $\{[\alpha_j]\}$ as generators of $H_1(Y)$. On the other hand, Y can be seen as a subspace of X_∞, identifying Y with Y_0 through the immersion $i_Y : Y \hookrightarrow X_\infty$. Then we have the homology classes $\{[\widetilde{\alpha}_j]\}$, where $\widetilde{\alpha}_j = i_Y(\alpha_j)$.

Lemma 2.2.6. *The set $\{t^i[\widetilde{\alpha}_1], \dots, t^i[\widetilde{\alpha}_{2g}]\}_{i\in\mathbb{Z}}$ generates $H_1(X_\infty)$ as a $\mathbb{Z}$-module. Therefore, $H_1(X_\infty)$ is generated by $[\widetilde{\alpha}_1], \dots, [\widetilde{\alpha}_{2g}]$ as Λ-module.*

Proof. Let us define the space U as Y_0 with a small neighbourhood of M_0^- in Y_{-1} and a small neighbourhood of M_1^+ in Y_{+1}, which is an open subset of X_∞. Then $\mathcal{U} = \{\tau^i(U)\}_{i\in\mathbb{Z}}$ is an open cover of X_∞.
If we consider a loop $\gamma : [0,1] \to X_\infty$, then for compactness of the support of the curve $\gamma^* = \gamma([0,1])$ there exist integers r, s such that

$$\gamma^* \subset \bigcup_{i=r}^{s} \tau^i(U).$$

Let us call $k = r - s$ the range of γ. We prove by induction over k that $[\gamma] \in H_1(X_\infty)$ is a linear combination of the $t^i[\widetilde{\alpha}_j]$.
If $k = 0$, then $\gamma^* \subset \tau^{\bar{\imath}}(U)$ for some integer $\bar{\imath}$. Hence, $\tau^{-\bar{\imath}}(\gamma^*) \subset U$ and

$$[\tau^{-\bar{\imath}}(\gamma)] = \sum_{j=1}^{2g} c_j\, [\widetilde{\alpha}_j].$$

If we apply $t^{\bar{\imath}}$, we obtain

$$[\gamma] = \sum_{j=1}^{2g} c_j\, t^{\bar{\imath}}[\widetilde{\alpha}_j].$$

Suppose now that the proposition is true for loops with range lower than k. We can suppose that $\gamma^* \cap \tau^s(U)$, $\gamma^* \cap \tau^r(U) \neq \varnothing$. Let us define the loops $\gamma_0, \gamma_-, \gamma_+$ as in Figure 2.3.

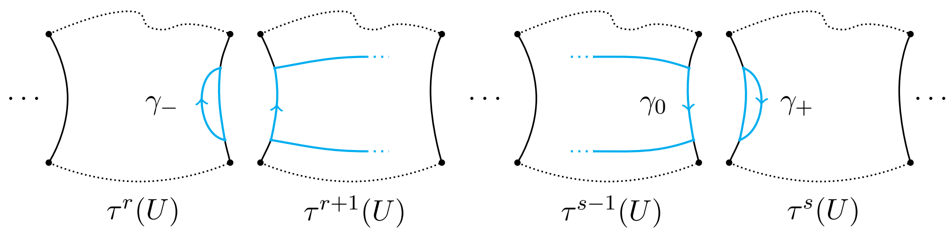


Figure 2.3: Division of γ.

Hence, we have

$$[\gamma] = [\gamma_-] + [\gamma_0] + [\gamma_+],$$

where $\gamma_-, \gamma_0, \gamma_+$ are loops with range lower than k. By induction hypothesis, we obtain that $[\gamma]$ is a $\mathbb{Z}$-linear combination of the $t^i[\widetilde{\alpha}_j]$.
The last part follows from the fact that t^i is a unit in Λ. This concludes the proof.

Let us now consider a loop a in $\mathring{M}$, with M a Seifert surface for an oriented knot. Then we have the curves $a^{\pm}$ in $\mathbb{S}^3 \setminus M$ and, as above, the loops $\widetilde{a}^{\pm}$ in X_∞. They can be seen as the images of a under the maps

$$\begin{aligned} f^{\pm}\colon \mathring{M} \xrightarrow{s^{\pm}} \mathring{M} \times \{\pm 1\} \longrightarrow \mathring{M}_0^{\pm} \subset X_\infty \\ a \longmapsto a^{\pm} \longmapsto \widetilde{a}^{\pm}. \end{aligned}$$

Lemma 2.2.7. *The Alexander invariant, as a Λ-module generated by the $[\widetilde{\alpha}_j]$, has*

$$t[\widetilde{a}_j^+] = [\widetilde{a}_j^-] \qquad \textit{with } j = 1, \dots, 2g$$

as defining relations. Hence,

$$H_1(X_\infty) \cong \Big([\widetilde{\alpha}_j] \,\Big|\, t[\widetilde{a}_j^+] - [\widetilde{a}_j^-] \Big) \qquad \textit{with } j = 1, \dots, 2g.$$

Proof. Let us recall the open cover $\mathcal{U} = \{\tau^i(U)\}_{i \in \mathbb{Z}}$ of the infinite cyclic cover of K defined in Lemma 2.2.6. It is convenient to consider a curve $\Gamma \in X_\infty$ which connects the pieces $\tau^i(U)$, such that $p(\Gamma)$ is a loop in the knot complement X linking K once. Defining the spaces

$$V = \Big(\bigcup_{i \text{ even}} \tau^i(U) \Big) \cup \Gamma \qquad W = \Big(\bigcup_{i \text{ odd}} \tau^i(U) \Big) \cup \Gamma,$$

we have that

$$X_\infty = \mathring{V} \cup \mathring{W} = \Big(\bigcup_{i \text{ even}} \tau^i(U) \Big) \cup \Big(\bigcup_{i \text{ odd}} \tau^i(U) \Big).$$

Thus, (V, W) is an excision couple for X_∞. In addition, the intersection $V \cap W$ is the union of $\mathring{M}_i^-$ with a small nighbourhood for each i and the curve Γ, which is path-connected. The Mayer-Vietoris sequence in reduced homology is

$$\cdots \longrightarrow H_1(V \cap W) \xrightarrow{\varphi} H_1(V) \oplus H_1(W) \xrightarrow{\psi} H_1(X_\infty) \longrightarrow 0,$$

where the maps

$$\varphi = (i_V)_* \oplus (i_W)_* \qquad \psi = (i_W)_* - (i_V)_*$$

are induced by the immersions $i_A\colon A \hookrightarrow X_\infty$, $i_B\colon B \hookrightarrow X_\infty$. The maps are also a Λ-homomorphisms, since the immersions commute with τ.
From the fact that ψ is surjective, it follows that

$$H_1(X_\infty) \cong \frac{H_1(V) \oplus H_1(W)}{\ker(\psi)} = \frac{H_1(V) \oplus H_1(W)}{\operatorname{Im}(\varphi)} = \operatorname{coker}(\varphi).$$

This will allow us to find a Λ-module presentation for the Alexander invariant. Now $H_1(V \cap W)$ is an infinitely generated abelian group,

$$H_1(V \cap W) \cong \mathbb{Z}\,\langle\, t^i[\widetilde{a}_j^-] \,\rangle \qquad \text{with } i \in \mathbb{Z},\ j = 1, \dots, 2g,$$

since $V \cap W$ consists of one copy of $\mathring{M}_0^-$ for every power of t (the curve Γ does not affect it). On the other hand,

$$H_1(V) \cong \mathbb{Z}\,\langle\, t^i[\widetilde{a}_j^-], t^i[\widetilde{a}_j^+], t^i[\widetilde{\alpha}_j] \,\rangle \qquad \text{with } i \text{ even},\ j = 1, \dots, 2g,$$

since V consists of one copy of U for every even power of t. For the same reason,

$$H_1(W) \cong \mathbb{Z}\,\langle\, t^i[\widetilde{a}_j^-], t^i[\widetilde{a}_j^+], t^i[\widetilde{\alpha}_j]\,\rangle \qquad \text{with } i \text{ odd, } j = 1,\ldots,2g.$$

Let us consider a generator $t^i[\widetilde{a}_j^-]$ of $H_1(V \cap W)$. We want to calculate $\varphi(t^i[\widetilde{a}_j^-])$. Since φ commutes with t, it suffices to evaluate $\varphi([\widetilde{a}_j^-])$. We note that

$$(i_V)_*([\widetilde{a}_j^-]) = [\widetilde{a}_j^-],$$

since $\widetilde{a}_j^- \in \mathring{M}_0^- \subset V$, while

$$(i_W)_*([\widetilde{a}_j^-]) = t[\widetilde{a}_j^+],$$

since $\widetilde{a}_j^- = \tau(\widetilde{a}_j^+) \in \mathring{M}_1^+ \subset W$. Hence,

$$\varphi(t^i[\widetilde{a}_j^-]) = t^i\,[\widetilde{a}_j^-] \oplus t^{i+1}\,[\widetilde{a}_j^+]$$

and, thanks to the map ψ, it follows that a $\mathbb{Z}$-module presentation of $H_1(X_\infty)$ is

$$H_1(X_\infty) \cong \frac{\mathbb{Z}\,\langle\, t^i[\widetilde{a}_j^-], t^i[\widetilde{a}_j^+], t^i[\widetilde{\alpha}_j]\,\rangle}{\mathbb{Z}\,\langle t^{i+1}\,[\widetilde{a}_j^+] - t^i\,[\widetilde{a}_j^-]\rangle} \qquad \text{with } i \in \mathbb{Z},\ j = 1,\ldots,2g.$$

Thanks to Lemma 2.2.6, we can eliminate $[\widetilde{a}_j^-]$ and $[\widetilde{a}_j^+]$ from the generators:

$$H_1(X_\infty) \cong \frac{\mathbb{Z}\,\langle\, t^i[\widetilde{\alpha}_j]\,\rangle}{\mathbb{Z}\,\langle t^{i+1}\,[\widetilde{a}_j^+] - t^i\,[\widetilde{a}_j^-]\rangle} \qquad \text{with } i \in \mathbb{Z},\ j = 1,\ldots,2g.$$

Finally, as a Λ-module,

$$H_1(X_\infty) \cong \frac{\Lambda\,\langle\,[\widetilde{\alpha}_j]\,\rangle}{\Lambda\,\langle t\,[\widetilde{a}_j^+] - [\widetilde{a}_j^-]\rangle} = \Big([\widetilde{\alpha}_j]\,\Big|\,t[\widetilde{a}_j^+] - [\widetilde{a}_j^-]\Big) \qquad \text{with } j = 1,\ldots,2g.$$

since t is a unit.

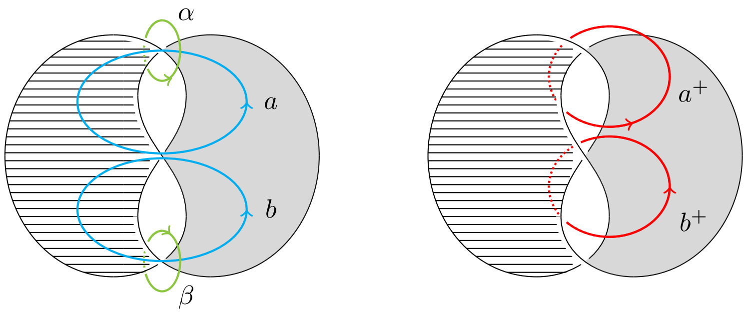
Corollary 2.2.8. *The Alexander polynomial is well-defined invariant of oriented knots.*

Proof. Since the Alexander invariant is finitely presentable and has a square presentation matrix, it is well defined. In addition, since the order ideal is an invariant of $H_1(X_\infty)$, the Alexander polynomial is a knot invariant.

As an example, let us compute a presentation matrix for the Alexander invariant of the left trefoil. If we choose the basis of the previous example, we obtain

$$H_1(X_\infty) \cong \Big([\widetilde{\alpha}], [\widetilde{\beta}]\,\Big|\,t[\widetilde{a}^+] - [\widetilde{a}^-], t[\widetilde{b}^+] - [\widetilde{b}^-]\Big),$$

where α and β are dual to a and b.



As it can be seen from the diagrams,

$$[\widetilde{a}^+] = [\widetilde{\alpha}] \qquad [\widetilde{b}^+] = [\widetilde{\beta}] - [\widetilde{\alpha}],$$

and in the same way

$$[\widetilde{a}^-] = [\widetilde{\alpha}] - [\widetilde{\beta}] \qquad [\widetilde{b}^-] = [\widetilde{\beta}].$$

Thus,

$$H_1(X_\infty) \cong \Big([\widetilde{\alpha}], [\widetilde{\beta}] \,\Big|\, (t-1)[\widetilde{\alpha}] + [\widetilde{\beta}], (t-1)[\widetilde{\beta}] - t[\widetilde{\alpha}] \Big).$$

We can calculate now a presentation matrix for $H_1(X_\infty)$:

$$P = \begin{pmatrix} t-1 & 1 \\ -t & t-1 \end{pmatrix}.$$

Therefore the Alexander polynomial of the left trefoil is

$$\Delta_{3_1}(t) = \det(P) = t^2 - t + 1.$$

In the previous presentation of $H_1(X_\infty)$, we could also use the first relator to eliminate $[\widetilde{\beta}] = (1-t)[\widetilde{\alpha}]$. Finally we obtain

$$\begin{aligned} H_1(X_\infty) &\cong \Big([\widetilde{\alpha}] \,\Big|\, (t-1)(1-t)[\widetilde{\alpha}] - t[\widetilde{\alpha}] \Big) \\ &= \Big([\widetilde{\alpha}] \,\Big|\, (t^2 - t + 1)[\widetilde{\alpha}] \Big) \\ &\cong \frac{\Lambda}{(t^2 - t + 1)}. \end{aligned}$$

Since the relator is

$$\rho = (t^2 - t + 1)[\widetilde{\alpha}],$$

we have the presentation matrix $P' = (t^2 - t + 1)$ and the Alexander polynomial of the left trefoil

$$\Delta_{3_1}(t) = t^2 - t + 1,$$

in accordance with what found before.

We are now ready to prove the statement which connects the Seifert matrix to the Alexander invariant.

Proof of Theorem 2.2.4. Let M be a bicollared Seifert surface for an oriented knot K and let V be the Seifert matrix associated to some basis $[a_1], \dots, [a_{2g}]$ of $H_1(\mathring{M})$. Then, thanks to Lemma 2.2.5, we can choose a dual basis $[\alpha_1], \dots, [\alpha_{2g}]$ of $H_1(\mathbb{S}^3 \setminus M)$. The same duality remains true for the corresponding 1-cicles in $H_1(X_\infty)$. Thus, for every $[\gamma] = \sum c_j[\widetilde{\alpha}_j]$ in $H_1(\mathbb{S}^3 \setminus M)$ we find that

$$\begin{aligned}
\operatorname{lk}(\gamma, \widetilde{a}_i) = \operatorname{lk}([\gamma], [\widetilde{a}_i]) &= \operatorname{lk}\Big(\sum_{j=1}^{2g} c_j\,[\widetilde{\alpha}_j], [\widetilde{a}_i]\Big) \\
&= \sum_{j=1}^{2g} c_j \operatorname{lk}([\widetilde{\alpha}_j], [\widetilde{a}_i]) \\
&= \sum_{j=1}^{2g} c_j\, \delta_{i,j} = c_i.
\end{aligned}$$

On the other hand, thanks to Lemma 2.2.6, as a Λ-module $H_1(X_\infty)$ is generated by $[\widetilde{\alpha}_1], \dots, [\widetilde{\alpha}_{2g}]$ with relators

$$[\widetilde{a}_i^-] - t\,[\widetilde{a}_i^+] \qquad i = 1, \dots, 2g.$$

Writing them out in terms of the generators, we obtain

$$[\widetilde{a}_i^-] - t\,[\widetilde{a}_i^+] = \sum_{j=1}^{2g} \operatorname{lk}(\widetilde{a}_i^-, \widetilde{a}_j)\,[\widetilde{\alpha}_j] - t \sum_{j=1}^{2g} \operatorname{lk}(\widetilde{a}_i^+, \widetilde{a}_j)\,[\widetilde{\alpha}_j] \qquad i = 1, \dots, 2g.$$

Since $\operatorname{lk}(\widetilde{a}_i^-, \widetilde{a}_j) = \operatorname{lk}(\widetilde{a}_j, \widetilde{a}_i^+) = V_{i,j}$, the equations become

$$[\widetilde{a}_i^-] - t\,[\widetilde{a}_i^+] = \sum_{j=1}^{2g} \Big(V_{i,j}\,[\widetilde{\alpha}_j] - t\,V_{j,i}\,[\widetilde{\alpha}_j]\Big) \qquad i = 1, \dots, 2g.$$

Thus, a presentation matrix for $H_1(X_\infty)$ is precisely $V^T - tV$ and the theorem is proven.

We can check the formula by computing the Alexander polynomial via the Seifert matrix of the left trefoil found before. We have

$$\begin{aligned}
\Delta_{3_1}(t) &= \det(V^T - tV) \\
&= \begin{vmatrix} 1-t & +t \\ -1+t & 1-t \end{vmatrix} \\
&= t^2 - t + 1,
\end{aligned}$$

which is the same result as before.

2.2.2 More on the Alexander polynomial

Even though the Alexander polynomial is only defined up to multiplication by $\pm t^n$, it is a powerful invariant. In this section we will use the notation $f(t) \doteq g(t)$ to denote $f(t) = \pm t^n g(t)$. Here are some properties that follow naturally from the previous theorem.

Proposition 2.2.9. *The Alexander polynomial of a knot K satisfies*

$$\Delta_K(1) \doteq 1.$$

Proof. Let V be an $n \times n$ Seifert matrix for K. We note that, with the usual basis $[a_j]$, $V_{i,j} = 0$ for $i \neq j$ or $i \neq j \pm 1$. Hence $\Delta_K(1) = \det(V^T - V)$ has non-zero entries only on the $i = j \pm 1$ positions. These correspond to

$$(V^T - V)_{j\pm 1,j} = \pm\Big(\mathrm{lk}(a_{j+1}, a_j^+) - \mathrm{lk}(a_{j+1}, a_j^+)\Big),$$

which are $\pm$ the algebraic number of intersections of a_j and a_{j+1} in M. Since this value is 1, we obtain that $V^T - V$ consists of g blocks of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

down the diagonal and zero elsewhere. Thus, $\Delta_K(1) \doteq \det(V^T - V) = 1$.

In the previous section we have defined the Alexander polynomial for knots, since the group of a link of $m > 1$ components has not the infinite cyclic group as first homology group. Despite of that, Theorem 2.2.4 suggests a way to define the Alexander polynomial for oriented links. We note that a more symmetric presentation matrix for the Alexander invariant of a knot is $\det(t^{1/2}V - t^{-1/2}V^T)$, since V is a $2g \times 2g$ matrix and

$$\begin{aligned} \det(t^{1/2}V - t^{-1/2}V^T) &= \big(-t^{-1/2}\big)^{2g} \det(V^T - tV) \\ &= (-t)^{-g} \det(V^T - tV) \doteq \Delta_K(t). \end{aligned}$$

Let now L be an oriented link, M a Seifert surface for L. We can choose a bicollar for M so that it agrees with the orientation of the link. We define the Seifert form $\Theta\colon H_1(\mathring{M}) \times H_1(\mathring{M}) \to \mathbb{Z}$ as before by the formula

$$\Theta([x], [y]) = \mathrm{lk}(x, y^+).$$

Since $H_1(\mathring{M})$ is a finitely generated abelian group, we can find an integral square matrix V associated to Θ, called Seifert matrix.

Definition 2.2.10. Let V be a Seifert matrix for an oriented link L. We define the *Conway-normalized Alexander polynomial* of L as

$$\Delta_L(t) = \det(t^{1/2}V - t^{-1/2}V^T).$$

Note that with this definition $\Delta_L \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ and that there is no ambiguity of multiplication by units. In addition, thanks to the remark above, the definition reduces to the Alexander polynomial in the case of knots.
However, from the above definition it is not clear whether the Conway-normalized Alexander polynomial is well-defined (*i.e.* it does not depend on any choice of the Seifert surface and of the basis of its first homology group) and that it is an invariant for oriented links. Nevertheless, this is the case.

The demonstration proceeds understanding how to relate different Seifert surfaces of a link, which reveals to be a finite sequence of ambient isotopies and the addition or removal of tubes, where each addition of a tube is required to preserve the orientability of the surface. Then, it establishes a relation between the corresponding Seifert matrices, which comes out to produce the same polynomial.

Definition 2.2.11. Let A be an integral square matrix. An *elementary enlargement* of A is a matrix B of the form

$$B = \begin{pmatrix} & & & \xi_1 & 0 \\ & A & & \vdots & \vdots \\ & & & \xi_n & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & 0 & 0 \\ \eta_1 & \cdots & \eta_n & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

for some column ξ or row η. The matrix A is called an *elementary reduction* of B. Two integral square matrices are said to be *S-equivalent* if they are related by a sequence of elementary enlargements, reductions and matrix congruences.

Proposition 2.2.12. *Let V and U be Seifert matrix related to equivalent links. Then V and U are S-equivalent.*

Here the elementary enlargements and reductions of the Seifert matrix reflects the addition and removal of tubes in the surface, while the matrix congruence is simply a change of basis in $H_1(\mathring{M})$. A complete proof of the statement can be found in [Cromwell, 2004].

Corollary 2.2.13. *The Conway-normalized Alexander polynomial is a well-defined invariant of oriented links.*

Proof. It suffices to show that $\det(t^{1/2}V - t^{-1/2}V^T)$ is invariant under S-equivalent matrices.
If V and U are congruent matrices, *i.e.* there exists an invertible matrix P such that $P^TVP = U$, then, setting $x = t^{1/2}$,

$$\begin{aligned} \det(x\,U - x^{-1}\,U^T) &= \det\big(x\,P^TVP - x^{-1}\,(P^TVP)^T\big) \\ &= \det\big(P^T(x\,V - x^{-1}\,V^T)P\big) \\ &= \det(P^T)\,\det(x\,V - x^{-1}\,V^T)\,\det(P) \\ &= \det(x\,V - x^{-1}\,V^T). \end{aligned}$$

In the last step we used the fact that if P is an invertible integer matrix, then $\det(P) = 1$.
Let now U be an elementary enlargement of V of the form

$$U = \begin{pmatrix} & & & \xi_1 & 0 \\ & V & & \vdots & \vdots \\ & & & \xi_n & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(x\,U - x^{-1}\,U^T) = \begin{vmatrix} & & & x\,\xi_1 & 0 \\ & x\,V - x^{-1}\,V^T & & \vdots & \vdots \\ & & & x\,\xi_n & 0 \\ -x^{-1}\,\xi_1 & \cdots & -x^{-1}\,\xi_n & 0 & x \\ 0 & \cdots & 0 & -x^{-1} & 0 \end{vmatrix}$$

and, expanding the determinant firstly along the last column and secondly along the last row, we obtain (supposing V of even dimension)

$$\begin{aligned} \det(x\,U - x^{-1}\,U^T) &= -x \begin{vmatrix} & & & x\,\xi_1 \\ & x\,V - x^{-1}\,V^T & & \vdots \\ & & & x\,\xi_n \\ 0 & \cdots & 0 & -x^{-1} \end{vmatrix} \\ &= -x\,(-x^{-1})\,\det(x\,V - x^{-1}\,V^T) \\ &= \det(x\,V - x^{-1}\,V^T). \end{aligned}$$

The calculations are analogue V is of odd dimension and if V is an elementary enlargement of the second type.

An immediate consequence of the above definition is a lower bound in the genus of an oriented link. Let us define the *breadth* of a Laurent polynomial to be the difference between the highest and lowest exponents with non zero coefficients. We note that, setting $x = t^{1/2}$, then $\Delta_L(x) \in \mathbb{Z}[x, x^{-1}]$.

Proposition 2.2.14. *For every oriented link L of m components and genus $g = g(K)$, setting $x = t^{1/2}$,*

$$\frac{1}{2}\,\text{breadth}(\Delta_L(x)) \leq 2g + m - 1.$$

Proof. Let M be a minimal Seifert surface for L. Then $H_1(\mathring{M})$ has a basis of $d = 2g + m - 1$ elements. Therefore, the associated Seifert matrix V is $d \times d$ and so is $xV - x^{-1}V^T$. Since $\Delta_L(x) = \det(xV - x^{-1}V^T)$, breadth$(\Delta_L(x)) \leq 2d$.

In the next propositions we will sometimes set $x = t^{1/2}$ in order to simplify the notation.

Proposition 2.2.15. *If L is an oriented link of m components, then*

$$\begin{aligned} &\Delta_L = \Delta_{-L} \\ &\Delta_L = (-1)^{m-1}\Delta_{\bar{L}}, \end{aligned}$$

where $-L$ and $\bar{L}$ are the opposite and the mirror image of L.

Proof. If V is a Seifert matrix associated to a Seifert surface M for L, then V^T is a Seifert matrix for $-L$, since the only change is in the sign of the bicollar of M. Then,

$$\begin{aligned}\Delta_{-L}(t) &= \det(x\,V^T - x^{-1}\,V)\\ &= \det\big((x\,V - x^{-1}\,V^T)^T\big)\\ &= \det(x\,V - x^{-1}\,V^T) = \Delta_L(t).\end{aligned}$$

For the second equation, we note that, with the same notation, $-V$ is a Seifert matrix for $\bar{L}$, since in the calculation of the linking number we have changed the positive and negative crossings. Then, knowing that V is a $(2g+m-1)\times(2g+m-1)$ matrix,

$$\begin{aligned}\Delta_{\bar{L}}(t) &= \det(-x\,V + x^{-1}\,V)\\ &= \det\big(-(x\,V - x^{-1}\,V^T)\big)\\ &= (-1)^{2g+m-1}\det(x\,V - x^{-1}\,V^T) = (-1)^{m-1}\Delta_L(t).\end{aligned}$$

In particular, the Alexander polynomial cannot distinguish between a knot and its mirror image.

Proposition 2.2.16. *For every oriented link L of m components,*

$$\begin{aligned}\Delta_L(-t) &= (-1)^{m-1}\Delta_L(t)\\ \Delta_L(t^{-1}) &= (-1)^{m-1}\Delta_L(t).\end{aligned}$$

Proof. The relations follow from calculations analogue to those of the previous proposition.

For knots, ignoring the Conway normalization, we can write the second equation as

$$\Delta_K(t) \doteq \Delta_K(t^{-1}).$$

The relation can be seen as a consequence of the symmetry in the choice of τ_* rather than $-\tau_*$ as generator of the infinite cyclic cover.

Corollary 2.2.17. *The Alexander polynomial of a knot K satisfies*

$$\Delta_K(t) \doteq c_0 + c_1(t^{-1}+t) + c_2(t^{-2}+t^2) + \cdots,$$

with c_0 odd.

Proof. By Proposition 2.2.16,

$$\Delta_K(t) \doteq b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n,$$

where $b_{n-i} = \pm b_i$ with the same choice of sign for all i.
If n is odd, then $\Delta_K(1) = b_0 + b_1 + b_2 + \cdots + b_n$ would be even, which contradicts Proposition 2.2.9. Thus n is even. Now if $b_{n-i} = -b_i$ for all i, then $b_{n/2} = -b_{n/2}$ and it implies that $b_{n/2} = 0$ and so $\Delta_K(1) = 0$, again a contradiction. Hence $b_{n-i} = b_i$ and $b_{n/2}$ is odd, otherwise $\Delta_K(1)$ would be even. Therefore, within multiplication by units of Λ, $\Delta_K(t)$ is of the required form.

2.3 The Conway polynomial

The Seifert surface description and the original definition of the Alexander polynomial offer interesting insight, but are generally difficult routes to compute Δ_L. In [Conway, 1967] was firstly presented a *skein relation*, which is a more useful description for computation.

The main result is the following

Theorem 2.3.1. *There exists a unique polynomial invariant*

$$\nabla \colon \{\text{oriented links}\} \to \mathbb{Z}[z]$$

satisfying:

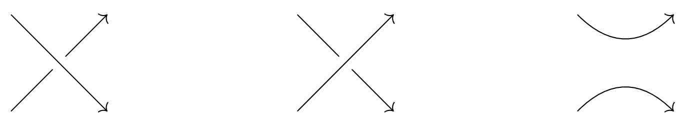
(a) the normalization

$$\nabla_{\bigcirc} = 1;$$

(b) the skein relation

$$\nabla_{L_+} - \nabla_{L_-} = z\,\nabla_{L_0},$$

where L_+, L_-, L_0 are three oriented links differing only locally according to the diagrams below.



L_+ L_- L_0

The invariant ∇_L is called the Conway polynomial *and it is related to the Conway-normalized Alexander polynomial by*

$$\Delta_L(t) = \nabla_L(t^{-1/2} - t^{1/2}).$$

A proof of a more general result, which includes the existence and uniqueness part of the theorem, is postponed to Chapter 4. Assuming it to be true, we shall prove that $\Delta_L(t) = \nabla_L(t^{-1/2} - t^{1/2})$. Thus, it suffices to prove that the Conway-normalized Alexander polynomial satisfies the axioms.

Definition 2.3.2. A link L is called *splittable* if it is separated by a 2-sphere embedded in $\mathbb{S}^3$. We will write $L = L_1 \sqcup L_2$.

Lemma 2.3.3. *If L is a splittable oriented link, then $\Delta_L = 0$.*

Proof. Let $L = L_1 \sqcup L_2$, M_i a Seifert surface for L_i and V_i the corresponding Seifert matrix. Then a Seifert surface M for L can be obtained connecting M_1 and M_2 with a tube. A basis for $H_1(\mathring{M})$ can be obtained taking the union of the basis of $H_1(\mathring{M}_1)$ and $H_1(\mathring{M}_2)$, together with a meridian m of the tube. Since

$\Theta([a_j],[m]) = \mathrm{lk}(a_j, m^+) = 0$ for every element in the union of the basis of $H_1(\mathring{M}_1)$ and $H_1(\mathring{M}_2)$, we obtain the Seifert matrix

$$V = \begin{pmatrix} V_1 & & \\ & V_2 & \\ & & 0 \end{pmatrix}.$$

It follows that

$$\Delta_L(t) = \det(t^{-1/2}V^T - t^{1/2}V) = 0.$$

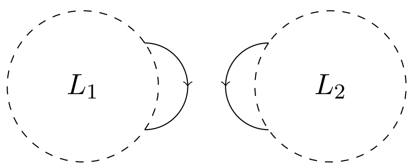
Proposition 2.3.4. *The Conway-normalized Alexander polynomial satisfies the skein relation*

$$\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\,\Delta_{L_0}.$$

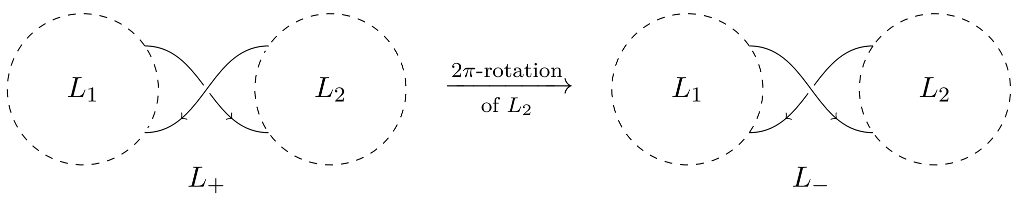
Proof. Let D_+, D_-, D_0 be the diagrams related to the skein relation. We have three cases.

Suppose that D_+ is a disconnected diagram. Then D_- and D_0 must also be disconnected, which implies that L_+, L_-, L_0 are all splittable links. Hence, thanks to Lemma 2.3.3, the skein relation holds. The same argument applies if D_- is disconnected.

Suppose now that D_0 is a disconnected diagram, while D_+, D_- are connected. The situation for L_0 is the one depicted below, were the dashed circles contain the remaining part of the diagram.



Then L_0 is a spittable link, while L_+ and L_- are of the same type:



Then the skein relation holds.

Finally, suppose that D_+, D_-, D_0 are all connected and let M_+, M_-, M_0 be the respective Seifert surfaces constructed via the Seifert algorithm. Let $[a_1], \dots, [a_n]$ be a basis for $H_1(\mathring{M}_0)$. Each loop a_i is also a loop in $M_\pm$. We can complete to a basis for $H_1(\mathring{M}_+)$ with a loop b passing once through the twisted band added in D_+ and back to the rest of the surface. The same argument applies to L_-, with a loop c passing once through the added band analogously. If V_0 is the Seifert matrix for

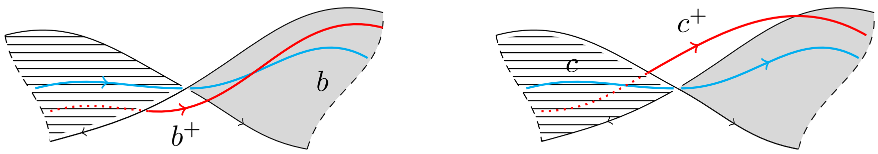


Figure 2.4: Sign of the crossings for M_+ and M_-.

L_0 relative to the loops a_i, the Seifert matrices for L_+ and L_- relative to the bases $a_1, \dots, a_n, b$ and $a_1, \dots, a_n, c$ are

$$V_+ = \begin{pmatrix} & & & \nu_1 \\ & V_0 & & \vdots \\ & & & \nu_n \\ \lambda_1 & \cdots & \lambda_n & \mu \end{pmatrix} \qquad V_- = \begin{pmatrix} & & & \nu_1 \\ & V_0 & & \vdots \\ & & & \nu_n \\ \lambda_1 & \cdots & \lambda_n & \mu+1 \end{pmatrix},$$

where we have set

$$\begin{aligned} \lambda_i &= \operatorname{lk}(b, a_i^+) = \operatorname{lk}(c, a_i^+) \\ \nu_i &= \operatorname{lk}(a_i, b^+) = \operatorname{lk}(a_i, c^+) \\ \mu &= \operatorname{lk}(b, b^+) = \operatorname{lk}(c, c^+) - 1. \end{aligned}$$

Let us prove that the above relations hold.
By construction, b and c are the same loop, except that b passes through a negatively oriented half-twisted band and c passes through a positively oriented half-twisted one. So, the considered linking numbers can only differ at the local crossing in the added bend. This proves that

$$\operatorname{lk}(b, a_i^+) = \operatorname{lk}(c, a_i^+) \qquad \operatorname{lk}(a_i, b^+) = \operatorname{lk}(a_i, c^+).$$

Consider now the crossing induced in the bend. The loop b crosses over b^+, so it contributes with a $-1/2$ to $\operatorname{lk}(b, b^+)$. However, c crosses under c^+, adding $1/2$ to $\operatorname{lk}(c, c^+)$. This proves that

$$\operatorname{lk}(c, c^+) - \operatorname{lk}(b, b^+) = 1.$$

Setting $x = t^{1/2}$ and expanding the determinants $\det(x\, V_+^T - x^{-1}\, V_+)$ and $\det(x\, V_-^T - x^{-1}\, V_-)$ about the last column and subtracting, we note that almost everything cancels and we obtain the thesis:

$$\begin{aligned} \Delta_{L_+} - \Delta_{L_-} &= \mu(x - x^{-1}) \det(x\, V_0^T - x^{-1}\, V_0) \\ &\qquad - (\mu+1)(x - x^{-1}) \det(x\, V_0^T - x^{-1}\, V_0) \\ &= -(x - x^{-1}) \det(x\, V_0^T - x^{-1}\, V_0) \\ &= (x^{-1} - x)\Delta_{L_0}. \end{aligned}$$

Now, the Alexander polynomial is an invariant for oriented links and (as it can be immediately verified)

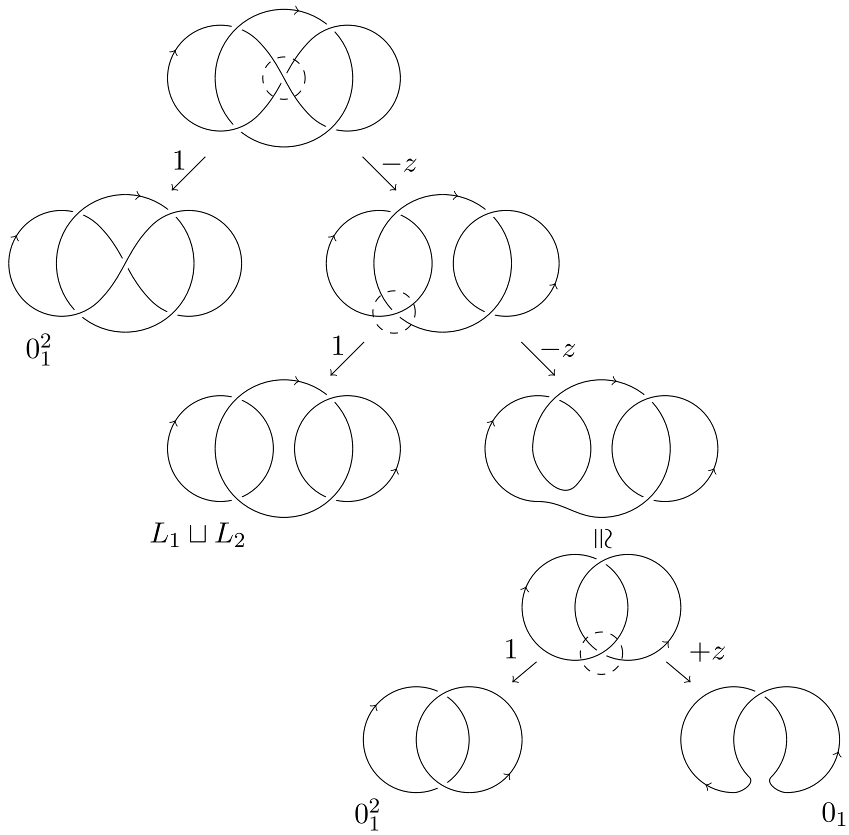
$$\Delta_{\bigcirc} = 1.$$

Furthermore, Theorem 2.3.1 assures that, applying the skein relation, we obtain an expression of the form

$$\Delta_L(t) = \sum_{i \in I} a_i (t^{1/2} - t^{-1/2})^i \qquad I \subset \mathbb{N} \text{ finite}, \ a_i \in \mathbb{Z}.$$

Thus, the Alexander polynomial can be write as an integer polynomial in the variable $t^{-1/2} - t^{1/2}$ and, with the substitution $z = t^{-1/2} - t^{1/2}$, we obtain that $\nabla_L(z) = \Delta(z)$.

Let us apply the skein relation to compute the Conway polynomial of an oriented *Whitehead link*, denoted with 5_1^2.



where 0_1^m is the unlink of $m > 1$ components. Hence,

$$\begin{aligned} \nabla_{5_1^2} &= \nabla_{0_1^2} - z\,\nabla_{L_1 \sqcup L_2} + (-z)^2\,\nabla_{0_1^2} + z(-z)^2\,\nabla_{0_1} \\ &= z^3. \end{aligned}$$

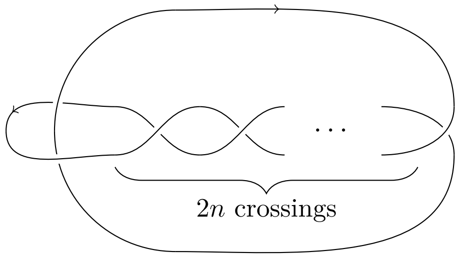
In the middle, we have also computed the Conway polynomial of an oriented Hopf link 2_1^2:

$$\nabla_{2_1^2} = z.$$

An interesting application of the Conway polynomial is the following

Proposition 2.3.5. *There are infinitely many knot types.*

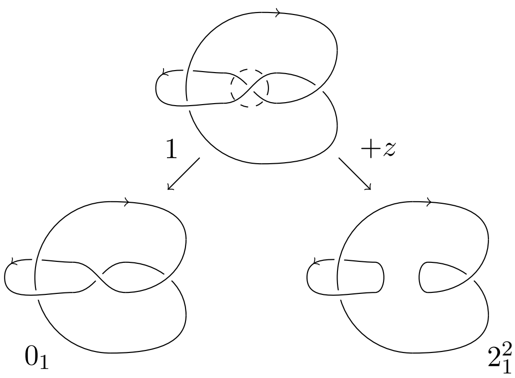
Proof. Let us show by induction that, denoting with $K^{(n)}$ the oriented knot below,



then

$$\nabla_{K^{(n)}}(z) = 1 + nz^2.$$

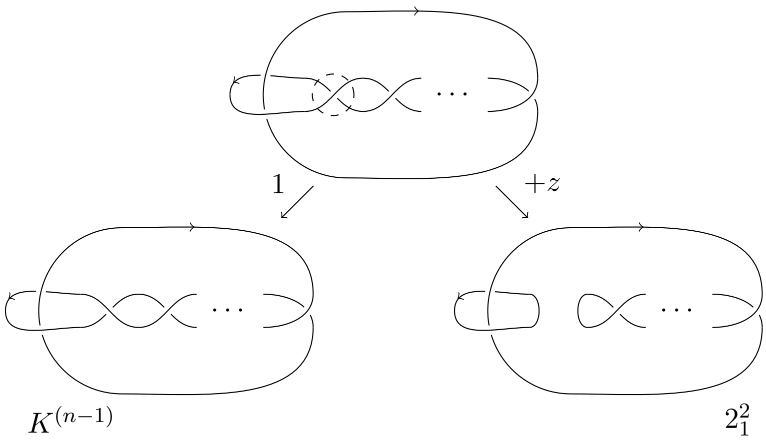
For $n = 1$, we have the *figure-eight knot*



where 2_1^2 is the same oriented Hopf link encountered in the previous example. Hence, since $\nabla_{2_1^2} = z^2$,

$$\nabla_{K^{(1)}} = \nabla_{0_1} + z\,\nabla_{2_1^2} = 1 + z^2.$$

Suppose now that $\nabla_{K^{(n-1)}} = 1 + (n-1)z^2$.



Therefore

$$\begin{aligned}\nabla_{K^{(n)}} &= \nabla_{K^{(n-1)}} + z\,\nabla_{2_1^2}\\ &= 1 + (n-1)z^2 + z^2\\ &= 1 + nz^2.\end{aligned}$$

Since all polynomials are different, we have that the $K^{(n)}$ are not equivalent.

Chapter 3

The Jones polynomial

In this section we will present the Jones polynomial. Starting with the Kauffman bracket as in [Kauffman, 1987], then we will modify it in order to achieve an invariant for oriented links, namely the Jones polynomial.

3.1 The Kauffman bracket

Definition 3.1.1. Let D be a link diagram. We define the *Kauffman bracket* $\langle D \rangle$ to be the element of the ring $\mathbb{Z}[A, B, d]$ by means of the following axioms.

(i) $\langle \bigcirc \rangle = 1$.

(ii) $\langle \bigcirc \sqcup D \rangle = d \, \langle D \rangle$, with $D \neq \varnothing$.

(iii) It satisfies the skein relation

$$\langle \times \rangle = A \, \langle \asymp \rangle + B \, \langle \,)(\, \rangle,$$

which is a relation among the Kauffman bracket of diagrams differing only locally as shown.

Note that axiom (iii) is defined looking at the crossing in such a way that the overpass goes from top-left to bottom-right. There are two such possibilities and it is possible to switch from one to another by rotating the crossing of an angle π. Nevertheless this ambiguity does not matter, since the diagrams on the right-hand-side of the skein relation are also the same when rotated by an angle of π.

Rules (ii) and (iii) then imply that the value of $\langle \cdot \rangle$ on a disjoint collection of m circles is d raised to $m - 1$:

$$\langle \underbrace{\bigcirc \sqcup \cdots \sqcup \bigcirc}_{m \text{ times}} \rangle = d^{m-1}. \tag{3.1}$$

It is not immediately clear that the bracket is uniquely determined. This problem is resolved in the following paragraphs, which give us an explicit formula for the bracket, independent of any choice.

3.1.1 A state model

Let us label the crossings in an n-crossing diagram D arbitrarily by 1 up to n.

Definition 3.1.2. A *state* is defined as a function $s\colon \{1,2,\ldots,n\} \to \{A,B\}$. We construct the diagram s from D by smoothing each crossing of D in the way indicated by Figure 3.1, depending on whether the crossing has been given label A or B by the function s. So D has 2^n different states.

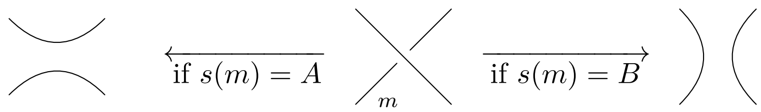


Figure 3.1: Smoothing the m-th crossing of D.

We define $\langle D|s\rangle$ for a diagram D and one of its state s by the formula

$$\langle D|s\rangle = A^i B^j,$$

where $i = \#s^{-1}(A)$ and $j = \#s^{-1}(B)$.

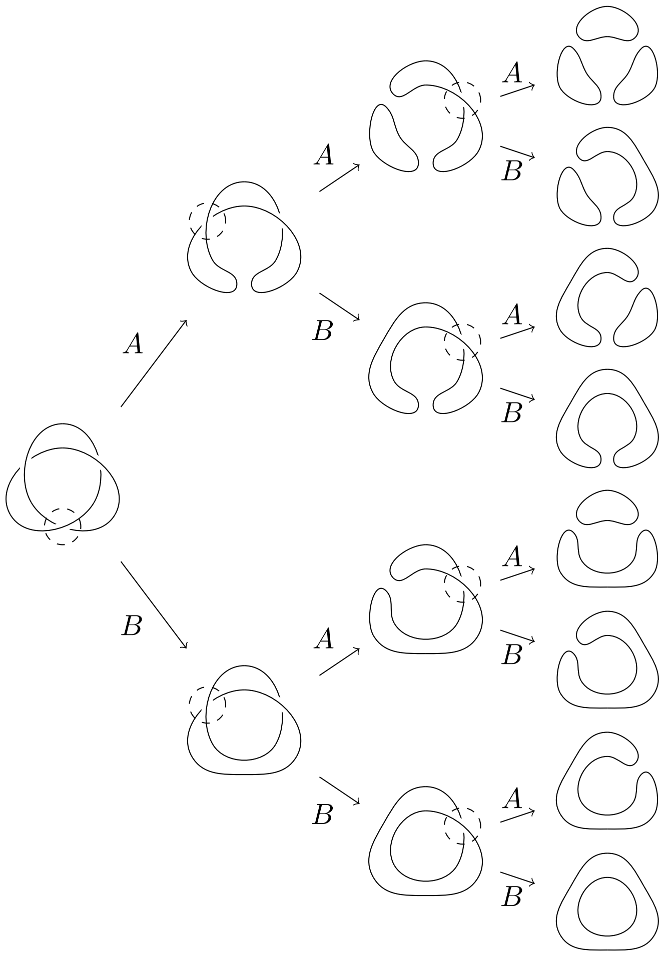
Proposition 3.1.3. *The Kauffman bracket $\langle D\rangle$ is well-defined by the axioms in Definition 3.1.1. It is given by the* state sum formula

$$\langle D\rangle = \sum_s \langle D|s\rangle\, d^{|s|-1}, \tag{3.2}$$

where $|s|$ denotes the number of circles in the splitting of s and the sum is over all possible states of the diagram.

Proof. Applying the skein relation at the crossing labelled as 1, we reduce $\langle D\rangle$ to a linear combination of the brackets of two other diagrams, each with crossings numbered from 2 to n. Applying again the skein relation to each of these diagrams at the crossing numbered with 2, we obtain a linear combination of four diagrams, each with crossings numbered from 3 to n. Repeating, we boil down to a linear combination of 2^n brackets of crossingless diagrams, indexed by states s, and each with a factor $\langle D|s\rangle$. Finally an $|s|$-component crossingless diagrams has bracket $d^{|s|-1}$ for Equation 3.1.
All we have done is fix some ordering for applying the skein relations and introduce notation so as to write an explicit formula for the result; having done that, we see that the formula does not involve the chosen ordering of crossings. It is also clear that if we were to remove circles at earlier stages, we would get the same result. So we do indeed have a well-defined Kauffman bracket, not depending on the order of application of skein relations.

As an example, we compute the Kauffman bracket of the standard diagram of the left trefoil.



Therefore,

$\langle$  $\rangle = A^3d^2 + 3A^2Bd + 3AB^2 + B^3d.$

The picture above is an example of *skein tree*, which allows the computation of the braket.

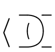
3.1.2 Reidemeister moves' impact

The Kauffman bracket is not a link invariant because it changes under Reidemeister moves. In this section we will see how it behaves under all three moves and consequently how to adjust A, B, d to obtain an invariant.

Type moves II and III

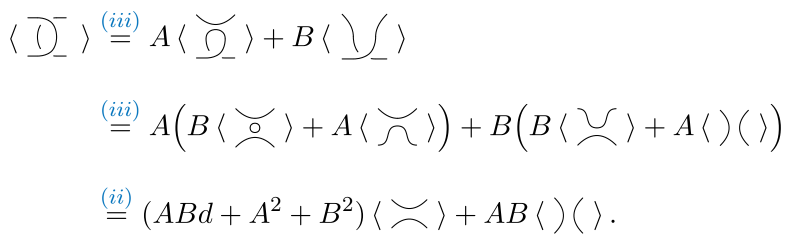
Lemma 3.1.4. *The following formula holds:*

$\langle$  $\rangle = (ABd + A^2 + B^2)\,\langle$  $\rangle + AB\,\langle\,)(\,\rangle\,.$

Hence we have RII invariance for $\langle\cdot\rangle$, i.e. $\langle$  $\rangle = \langle\,)(\,\rangle$ *for all diagrams, if*

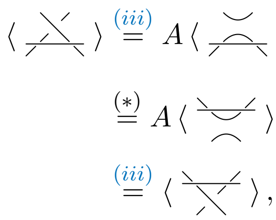
$$B = A^{-1} \quad \textit{and} \quad d = -(A^2 + A^{-2}). \tag{3.3}$$

Proof. We can see that



Lemma 3.1.5. *RII invariance for $\langle \cdot \rangle$ implies RIII invariance.*

Proof.

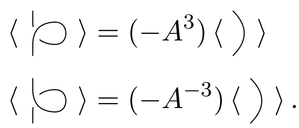


where in $(*)$ we have used RII invariance.

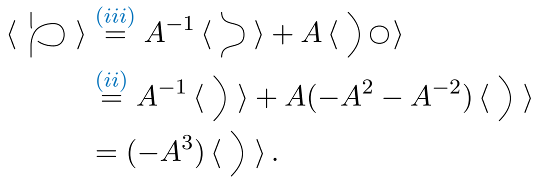
Thus we see that by choosing $B = A^{-1}$ and $d = -(A^2 + A^{-2})$, $\langle D \rangle$ becomes a Laurent polynomial in A and it is an invariant under moves of type II and III. It is not invariant under the type I moves, but it behaves as follows.

Type I move

Proposition 3.1.6. *With the choices indicated in Equation 3.3, then*



Proof. Let us prove the first equation:



The second one can be proven with analogous calculations.

From now on, we assume that B and d are chosen as indicated in Equation 3.3.

3.2 Defining the Jones polynomial

3.2.1 The Jones polynomial via Kauffman bracket

In order to turn the Kauffman bracket into a polynomial invariant, we have to modify it and its behaviour under type I move. Let us define another diagram function, the writhe.

Definition 3.2.1. Given a diagram D of an oriented link L, we define the *writhe* $w(D)$ as the sum of the signs of the crossings.

The writhe of a knot diagram does not depend on the orientation, for if we change the orientation of the knot, each crossing preserves its sign: the arrows point in the opposite direction, but since they do it on both lines, the writhe stays the same. This is not a general fact for links with more than one component, where there are several orientations that can be changed.

To obtain an invariant of ambient isotopy for oriented knots and links, we define a Laurent polynomial f_L by the formula

$$f_L = (-A)^{-3w(D)} \langle D \rangle , \tag{3.4}$$

where D denotes a diagram of L. The bracket is defined on oriented diagrams by ignoring the orientation.

Theorem 3.2.2. *The polynomial $f_L \in \mathbb{Z}[A, A^{-1}]$ defined above is an invariant for oriented links.*

Proof. Recall that $\langle D \rangle$ does not change under $\overrightarrow{\text{RII}}$ and $\overrightarrow{\text{RIII}}$ by Lemmata 3.1.4 and 3.1.5. If D is changed by a type II move, then the writhe $w(D)$ remains unchanged, because the two crossings in move II cancel each other out: one of them is positive and the other is negative depending on the orientation chosen. If D is then changed by a Reidemeister move III, the writhe again stays the same, since the number of crossings is preserved and there are the same types of crossings before and after the move. So f_L is an invariant under Reidemeister moves II and III.

Let us now have a look at what happens under type I move. When D is changed, then the writhe is changed by a $+1$, if we add a positive kink, or a -1, if we get rid of a positive kink, because the crossing being created or deleted is a positive one. Let us call D' the diagram when an extra positive kink has been added. Then $w(D') = w(D) + 1$ and by Proposition 3.1.6 $\langle D' \rangle = (-A^3) \langle D \rangle$, so we have that

$$\begin{aligned}(-A)^{-3w(D')} \langle D' \rangle &= (-A)^{-3(w(D)+1)}(-A^3) \langle D \rangle \\ &= (-A)^{-3w(D)} \langle D \rangle .\end{aligned}$$

The calculations are similar if we get rid of a positive kink or if we deal with a negative kink. This shows that 3.4 is also an invariant under Reidemeister move I.

We can now define the Jones polynomial as follows.

Definition 3.2.3. The *Jones polynomial* V_L of an oriented link L is the Laurent polynomial in $t^{1/2}$, with integer coefficients, defined by

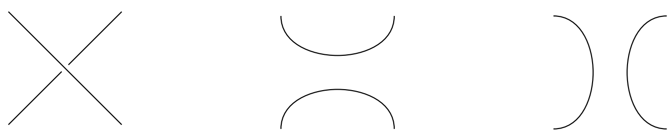
$$V_L(t) = f_L(t^{-1/4}). \tag{3.5}$$

By Theorem 3.2.2, the Jones polynomial is an invariant over oriented links. It can happen, though, that two different links have the same Jones polynomial, so we do not have an if-and-only-if statement. Furthermore, since the writhe of a knot diagram does not depend on the chosen orientation, the Jones polynomial of an oriented knot does not either.

At first glance, it seems that V_L should belong to $\mathbb{Z}[t^{1/4}, t^{-1/4}]$, since the substitution $A^{-2} = t^{1/2}$ implies $A = t^{-1/4}$, but we can show by induction on the number n of crossings in a diagram that f_L does indeed belong $\mathbb{Z}[A^2, A^{-2}]$.
As base case, consider a diagram with zero crossings. Then the writhe is 0 and Equation 3.1 shows that the statement is true for $n = 0$.
Now, fix a diagram D with n and suppose the statement true for all diagrams with less crossings. Choose a crossing, say m, that looks like in Figure 3.2a. Let D', D'' be the diagrams that are the same as D except for the crossing m, where D' looks like in Figure 3.2b and D' like in Figure 3.2c. Then D' and D'' have $n-1$ crossings and $w(D') = w(D'') = w(D) \pm 1$, depending on the sign of the crossing. Let us suppose that m is a positive crossing. Than $w(D') = w(D'') = w(D) - 1$ an, by property (iii) of Kauffman bracket

$$\begin{aligned} f_L &= (-A)^{-3w(D)} \langle D \rangle \\ &= (-A)^{-3w(D)} (A \langle D' \rangle + A^{-1} \langle D'' \rangle) \\ &= (-A)^{-3(w(D')+1)} A \langle D' \rangle + (-A)^{-3(w(D'')+1)} A^{-1} \langle D'' \rangle \\ &= -(-A)^{-2} (-A)^{-3w(D')} \langle D' \rangle - (-A)^{-4} (-A)^{-3w(D'')} \langle D'' \rangle . \end{aligned}$$

By induction, $(-A)^{-3w(D')} \langle D' \rangle$ and $(-A)^{-3w(D'')} \langle D'' \rangle \in \mathbb{Z}[A^2, A^{-2}]$, then the statement follows.



(a) **(b)** **(c)**

Figure 3.2

3.2.2 A skein relation

We will show now a characterisation of the Jones polynomial via a skein relation. This definition was actually presented before the one introduced in the previous paragraph and can be found in [Jones, 1985].

Proposition 3.2.4. *There exist a unique invariant of oriented links*

$$V \colon \{\textit{oriented links}\} \to \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

satisfying:

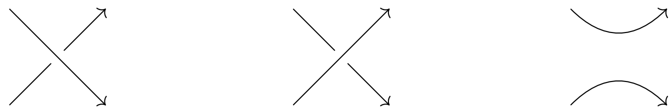
(a) the normalization

$$V_{\bigcirc} = 1;$$

(b) the skein relation:

$$t^{-1} V_{L_+} - t V_{L_-} = (t^{1/2} - t^{-1/2}) V_{L_0}, \tag{3.6}$$

whenever L_+, L_- *and* L_0 *are three oriented links differing only locally according to the diagrams below.*



L_+ L_- L_0

Figure 3.3: Skein diagrams.

In addition, V is the Jones polynomial.

Part of the proof. As for the Conway polynomial, we delay to Chapter 4 the proof of the fact that the axioms uniquely determines a polynomial in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$. Assuming that, we prove now that the Jones polynomial satisfies the normalization and the skein relation. By the definition of the Kauffman bracket, we have that

$$\langle \bigcirc \rangle = 1$$

and $w(\bigcirc) = 0$. Thus,

$$V_{\bigcirc} = 1.$$

For the skein relation, we have again from the Kauffman axioms that

$$\langle \; \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle,$$

$$\langle \; \rangle = A^{-1} \langle \asymp \rangle + A \langle)(\rangle.$$

Hence

$$A \langle \; \rangle - A^{-1} \langle \; \rangle = (A^2 - A^{-2}) \langle \asymp \rangle.$$

Since $w(L_+) = w(L_0) + 1$, it follows that

$$\begin{aligned} f_{L_+} &= (-A)^{-3w(L_+)} \langle \; \rangle \\ &= (-A)^{-3(w(L_0)+1)} \langle \; \rangle \\ &= (-A)^{-3}(-A)^{-3w(L_0)} \langle \; \rangle, \end{aligned}$$

or equivalently

$$A^4 f_{L_+} = -A(-A^{-3w(L_0)}) \langle \; \rangle.$$

Similarly it is possible to prove that

$$-A^{-4} f_{L_-} = A^{-1}(-A^{-3w(L_0)}) \langle$$  $$\rangle .$$

Adding the two equations, we get

$$A^4 f_{L_+} - A^{-4} f_{L_-} = -(-A^{-3w(L_0)})(A \langle$$  $$\rangle - A^{-1} \langle$$  $$\rangle)$$

$$= (A^{-2} - A^2)(-A^{-3w(L_0)}) \langle$$  $$\rangle$$

$$= (A^{-2} - A^2) f_{L_0}.$$

The substitution $A^{-2} = t^{1/2}$ concludes the proof.

3.3 Some properties of the Jones polynomial

A first important property of the Jones polynomial is that it can recognises a knot from its mirror image.

Proposition 3.3.1. *Let $\bar{K}$ denote the mirror image of K. Then*

$$V_{\bar{K}}(t) = V_K(t^{-1}).$$

Proof. Let us prove that

$$\langle \bar{K} \rangle (A) = \langle K \rangle (A^{-1}).$$

We note that switching all crossings results in the replacement of every appearance of A by its inverse A^{-1} in the expansion 3.2 of the bracket. This proves the formula. Since the writhe of a mirror image is the negative of the writhe of the original, the thesis follows as well.

We can compute now the Jones polynomial of the left trefoil. We have seen that, with choices 3.3,

$$\langle D \rangle = \langle$$  $$\rangle = -A^3 - A^{-5} + A^7.$$

Since we are dealing with the left trefoil, whose crossings are all negative ones, we have that $w(D) = -3$ and then

$$\begin{aligned} V_{3_1} &= (-A^{-3w(D)}) \langle D \rangle \Big|_{A^{-2}=t^{1/2}} \\ &= (-A)^{-3(-3)}(-A^3 - A^{-5} + A^7) \Big|_{A^{-2}=t^{1/2}} \\ &= -(A^2)^8 + (A^2)^6 + (A^2)^2 \Big|_{A^{-2}=t^{1/2}} \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

Thanks to Proposition 3.3.1, we have immediately that $V_{\bar{3}_1} = -t^4 + t^3 + t$. Since $V_{3_1} \neq V_{\bar{3}_1}$, we can conclude that the left trefoil is different from the right one. Furthermore, noting that $V_{0_1} = 1$, we have that the trefoil is different from the unknot.

Proposition 3.3.2. *If 0_1^m is the unlink of m components, then for every oriented link L*

$$V_{L\sqcup 0_1^m}(t) = (-t^{1/2} - t^{-1/2})^{m-1}\, V_L(t).$$

Proof. Firstly, let us prove that

$$V_{0_1^m}(t) = (-t^{1/2} - t^{-1/2})^{m-1}.$$

We know that the Kauffman bracket of the standard diagram of the unlink of m components is

$$\langle \bigcirc \sqcup \cdots \sqcup \bigcirc \rangle = (-A^2 - A^{-2})^{m-1}.$$

With the same diagram (and any possible orientation of the unlink)

$$w(\bigcirc \sqcup \cdots \sqcup \bigcirc) = 0,$$

since there are no crossings. Hence, with $A^2 = t^{1/2}$, we have the formula.
Let us now prove that

$$V_{L\sqcup\bigcirc} = -(t^{1/2} + t^{-1/2})\, V_L(t).$$

Let us fix an oriented diagram D of L. By axiom (ii),

$$\langle D \sqcup \bigcirc \rangle = -(A^2 - A^{-2})\, \langle D \rangle\, .$$

On the other hand, $D \sqcup \bigcirc$ has the same type of crossings of D. Then

$$w(D \sqcup \bigcirc) = w(D)$$

and, with $A^2 = t^{1/2}$ and the formula above, we have the thesis.

Lemma 3.3.3 (Unknotting algorithm)**.** *For every diagram of a link L of m components, we can reverse a finite number of crossing in order to obtain a diagram of the unlink of m components.*

Proof. Let us assign an orientation to the link, order its components and choose a basepoint in the diagram (different from a crossing) for every component of L. Starting at the basepoint of the first component and proceeding following the link orientation, we can change overpasses to underpasses, so that every crossing is first encountered as an underpass. Continue throughout the basepoints of the second and all subsequent components in the same way, changing crossings so that every crossing is first encountered as an underpass. This finite process geometrically separates and unknots the components, creating an unlink of m components.
We can also see that the number of changes is less or equal to the number of crossing in the diagram.

Proposition 3.3.4. *For every oriented link L of m components,*

$$V_L(1) = (-2)^{m-1}.$$

Proof. For $t = 1$, the skein relation becomes

$$V_{L_+}(1) = V_{L_-}(1).$$

It follows that changing overcrossings to undercrossings or vice versa has no effect on the value of $V_L(1)$. Now, we can apply the skein relation to a finite number of crossing in order to obtain a diagram of the unlink as in the previous lemma. Indeed, by Proposition 3.3.2

$$V_L(1) = V_{0_1^m}(1) = (-t^{1/2} - t^{-1/2})^{m-1}\Big|_{t=1} = (-2)^{m-1}.$$

Chapter 4

The HOMFLY polynomial

In the previous chapters we have expressed the Conway and the Jones polynomials by means of a skein formula. It will now be shown that those are two particular cases of a more general polynomial invariant in two variables, the so-called HOMFLY polynomial (or sometimes HOMFLY-PT polynomial, from the initials of its co-discoverers [Freyd et al., 1985] and [Przytycki and Traczyk, 1987]).

Using the approach of [Lickorish and Millett, 1987], the main result of this chapter is the following

Theorem 4.0.1. *There exists a unique polynomial invariant*

$$\mathcal{P} \colon \{\textit{oriented links}\} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}],$$

called the HOMFLY polynomial*, satisfying:*

(a) the normalization

$$\mathcal{P}_{\bigcirc} = 1;$$

(b) the skein relation

$$l\,\mathcal{P}_{L_+} + l^{-1}\,\mathcal{P}_{L_-} + m\,\mathcal{P}_{L_0} = 0.$$

It can be immediately seen that we can recover the Alexander and the Jones polynomial as

$$\begin{aligned} \Delta_L(t) &= \mathcal{P}_L\Big(i, i\,(t^{1/2} - t^{-1/2})\Big) \\ V_L(t) &= \mathcal{P}_L\Big(i\,t^{-1}, -i\,(t^{1/2} - t^{-1/2})\Big), \end{aligned}$$

where i is the formal square root of -1.

In order to prove the theorem, we will proceed as follows. Firstly, we will show that $\mathcal{P}_L$ can be calculated for every oriented link L, after some choices for the computation. Secondly, we will prove that the final result is independent of such choices. In the following, we will employ some nomenclature.

- A link is *ordered* if an order is given to its components. Any diagram of L inherits such an order.

- A link diagram is *based* if a basepoint (different from a crossing) is specified for every component.

- The set of oriented ordered based link diagrams with at most n crossings is denoted by $\mathscr{D}_n$. We set $\mathscr{D} = \bigcup_n \mathscr{D}_n$.

- An element $D \in \mathscr{D}$ is said to be *ascending* if, when traversing the components of D in their given order and from their basepoints in the direction specified by the orientation, every crossing is first encountered as an undercrossing. Every ascending diagram represents an unlink.

With this definitions, Lemma 3.3.3 states that every $D \in \mathscr{D}_n$ can be modified, changing overpasses to underpasses, so that the resulting diagram $\alpha(D)$ is an ascending one. In addition $\alpha(D)$, which is called the *standard ascending* of D, has the same number of components and the same number of crossings of the starting diagram, so that $\alpha(D) \in \mathscr{D}_n$.
In order to prove the theorem, and in particular to apply the skein relation, we will need to specify that the polynomial is evaluated on a particular oriented diagram. To emphasize this fact, we will write $\mathcal{P}_D$, where D is an oriented link diagram. Then the skein relation will be written as

$$l\,\mathcal{P}_{D_+} + l^{-1}\,\mathcal{P}_{D_-} + m\,\mathcal{P}_{D_0} = 0.$$

The result above gets rid of this ambiguity, since it establishes that equivalent oriented links possess equal HOMFLY polynomials. Thus, the computation can be done on every diagram of such a link.

The proof of Theorem 4.0.1 presented here will be given by induction on the number n of crossings of a diagram. For $n = 0$ there is nothing to prove. We will assume now the following

Inductive hypothesis $\mathcal{H}(n-1)$. For each $D \in \mathscr{D}_{n-1}$ there is an associated $\mathcal{P}_D \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ such that

1. is independent of the choices of ordering of the components and of basepoints;

2. is invariant under oriented Reidemeister (Figure 4.1) moves which do not increase the number of crossings beyond $n-1$;

3. satisfies the skein relation relating diagrams in $\mathscr{D}_{n-1}$.

In addition, we require that

4. if D is any ascending diagram of c components, then, setting $\mu = -(l+l^{-1})m^{-1}$,

$$\mathcal{P}_D = \mu^{c-1}.$$

From now on, we assume the truth of $\mathcal{H}(n-1)$ and we will define a polynomial $\mathcal{P}_D$ with $D \in \mathscr{D}_n$ which satisfies $\mathcal{H}(n)$.

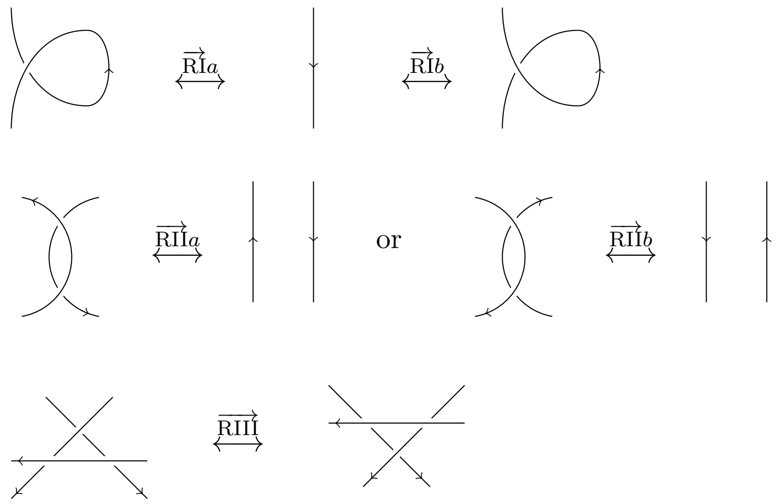


Figure 4.1: Oriented Reidemeister moves.

Definition 4.0.2. Let $D \in \mathscr{D}_n \setminus \mathscr{D}_{n-1}$ with c components. If D is an ascending diagram, we set

$$\mathcal{P}_D = \mu^{c-1}.$$

Otherwise, we can employ the unknotting algorithm together with the skein relation written in the forms

$$\begin{aligned}\mathcal{P}_{D_+} &= -l^{-1}m\,\mathcal{P}_{D_0} - l^{-2}\,\mathcal{P}_{D_-}\\ \mathcal{P}_{D_-} &= -lm\,\mathcal{P}_{D_0} - l^{2}\,\mathcal{P}_{D_+},\end{aligned}$$

applying $\mathcal{H}(n-1)$ to each D_0 (with arbitrary choices of orders of components and of basepoints, since $D_0 \in \mathscr{D}_{n-1}$) and the definition μ^{c-1} for the resulting standard ascending $\alpha(D)$. In such a way, we obtain an element of $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$.

We shall prove now that, with the above definition, $\mathcal{P}_D$ satisfies $\mathcal{H}(n)$. Let us denote the ordered components of the link as $K_1, \dots, K_c$ and label with natural numbers $1, \dots, n$ the crossings, considering the order given by the unknotting algorithm.

Lemma 4.0.3. *If the crossings of D that differ from those of $\alpha(D)$ are changed in any sequence to achieve the standard ascending, then the corresponding calculation using the skein relation yields $\mathcal{P}_D$.*

Proof. Without loss of generality, we can alter the sequence of unknotting moves by interchanging the first two crossing switches, which the algorithm requires at, say, the crossing labelled i and then at the crossing labelled j. Let $\sigma_k D$ and $\eta_k D$ be the same as D, except that the kth crossing is switched in $\sigma_k D$ (from overpass to underpass) and nullified in $\eta_k D$, with $k = i, j$. This is nothing but the notation D_+, D_-, D_0 with the additional information about what crossing of D we are looking at and ignoring the sign of the crossing.

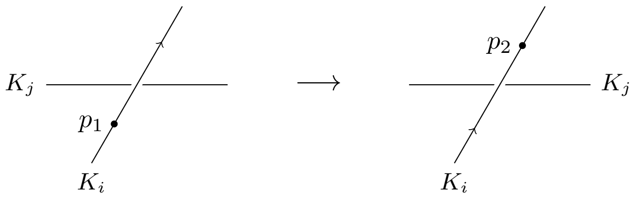


Figure 4.2: Crossing r in the diagrams of D_1 and D_2, for K_i crossing over K_j.

Since $\eta_k D \in \mathscr{D}_{n-1}$, the choices have no effects on $\mathcal{P}_{\eta_k D}$. Let us denote with ϵ_k the sign of the kth crossing. Applying the skein relation to the standard sequence of changes $D \to \sigma_i D \to \sigma_j \sigma_i D$, we obtain

$$\begin{aligned}\mathcal{P}_D &= -l^{\epsilon_i} m\, \mathcal{P}_{\eta_i D} - l^{2\epsilon_i}\, \mathcal{P}_{\sigma_i D}\\ &= -l^{\epsilon_i} m\, \mathcal{P}_{\eta_i D} - l^{2\epsilon_i} \left(-l^{\epsilon_j} m\, \mathcal{P}_{\eta_j \sigma_i D} - l^{2\epsilon_j}\, \mathcal{P}_{\sigma_j \sigma_i D}\right)\\ &= -l^{\epsilon_i} m\, \mathcal{P}_{\eta_i D} + l^{2\epsilon_i + \epsilon_j} m\, \mathcal{P}_{\eta_j \sigma_i D} + l^{2(\epsilon_i + \epsilon_j)}\, \mathcal{P}_{\sigma_j \sigma_i D}.\end{aligned}$$

On the other hand, with the reverse sequence $D \to \sigma_j D \to \sigma_i \sigma_j D$, we find the expression

$$-l^{\epsilon_j} m\, \mathcal{P}_{\eta_j D} + l^{2\epsilon_j + \epsilon_i} m\, \mathcal{P}_{\eta_i \sigma_j D} + l^{2(\epsilon_i + \epsilon_j)}\, \mathcal{P}_{\sigma_i \sigma_j D}.$$

Now, note the commutativity of the successive changes $\sigma_j \sigma_i = \sigma_i \sigma_j$, thus the last terms of the two expressions are equal. In addition, with the skein relation of $\mathcal{H}(n-1)$ for $\eta_i D$ and $\eta_j D$,

$$\begin{aligned}\eta_i D &= -l^{\epsilon_j} m\, \mathcal{P}_{\eta_j \eta_i D} - l^{2\epsilon_j}\, \mathcal{P}_{\sigma_j \eta_i D}\\ \eta_j D &= -l^{\epsilon_i} m\, \mathcal{P}_{\eta_i \eta_j D} - l^{2\epsilon_i}\, \mathcal{P}_{\sigma_i \eta_j D}.\end{aligned}$$

Since $\sigma_j \eta_i = \eta_i \sigma_j$, $\sigma_i \eta_j = \eta_j \sigma_i$ and $\eta_i \eta_j = \eta_j \eta_i$, substituting these expressions above we find that the two ways to compute $\mathcal{P}_D$ are equivalent.
By induction on the value of the difference between the number of crossings of D and that of $\alpha(D)$, we have the thesis.

Referring to the link projection associated to a link diagram, we call *segment* any component of the complement of the double points.

Proposition 4.0.4. *$\mathcal{P}$ is independent of the choice of basepoints.*

Proof. It suffices to show that moving a basepoint from a segment to the following one (along the orientation of the component) does not affect the polynomial.
Suppose that the basepoint lying on a component K_i is moved from p_1 to p_2, passing a crossing of K_i with K_j. Let D_1 and D_2 be the two elements of $\mathscr{D}_n$ (up to planar isotopies of the diagram) that have basepoints on K_i at p_1 and p_2 respectively and are otherwise the same. We can distinguish two cases.

- Let $i \neq j$. In this case $\alpha(D_1) = \alpha(D_2)$ and hence $\mathcal{P}_{D_1} = \mathcal{P}_{D_2}$ as, by the previous lemma, the choice of the sequence of crossing changes is irrelevant.

- Let $i = j$. In this case $\alpha(D_1)$ and $\alpha(D_2)$ differ only at the considered crossing, say r, since in D_2 it is not the first encountered crossing. Let us suppose that K_i crosses over K_j (see Figure 4.2). For the previous lemma, $\mathcal{P}_{D_1}$ can be computed changing firstly all the other relevant crossings. The resulting diagram is $\sigma_r\alpha(D_1)$ and we obtain $\mathcal{P}_{D_1} = f(\mathcal{P}_{\sigma_r\alpha(D_1)})$, where f is some function coming out from recursion. On the other hand, $\mathcal{P}_{D_2} = f(\mathcal{P}_{\alpha(D_2)})$, since we have applied the skein relation repeatedly in the same order to the same crossings.

  Now $\mathcal{P}_{\alpha(D_1)} = \mu^{c-1}$ and $\mathcal{P}_{\eta_r\alpha(D_1)} = \mu^c$, since $\eta_r\alpha(D_1) \in \mathscr{D}_{n-1}$ and it is an ascending diagram (every crossing is first encountered as an underpass). By the definition of $\mathcal{P}$ in $\mathscr{D}_n$,

$$\begin{aligned}
\mathcal{P}_{\sigma_r\alpha(D_1)} &= -l^{-2\epsilon_r}\,\mathcal{P}_{\alpha(D_1)} - l^{-\epsilon_r} m\,\mathcal{P}_{\eta_r\alpha(D_1)} \\
&= -c\,\mu^{c-1} - l^{-\epsilon_r} m\,\mu^c \\
&= \mu^{c-1}\left(-l^{-2\epsilon_r} + l^{-\epsilon_r} m(l+l^{-1})m^{-1}\right) \\
&= -\mu^{c-1}\left(-l^{-2\epsilon_r}\,\underbrace{l^{-\epsilon_r+1} + l^{-\epsilon_r-1}}_{=1+l^{-2\epsilon_r}}\right) \\
&= \mu^{c-1}.
\end{aligned}$$

  Since $\mathcal{P}_{\sigma_r\alpha(D_1)} = \mathcal{P}_{\alpha(D_2)} = \mu^{c-1}$, substituting them in f we obtain $\mathcal{P}_{D_1} = \mathcal{P}_{D_2}$. Finally, if K_i crosses under K_j, we can repeat the argument considering $\alpha(D_1)$ and $\sigma_r\alpha(D_2)$.

Proposition 4.0.5. *$\mathcal{P}$ satisfies the skein formula relating diagrams in $\mathscr{D}_n$.*

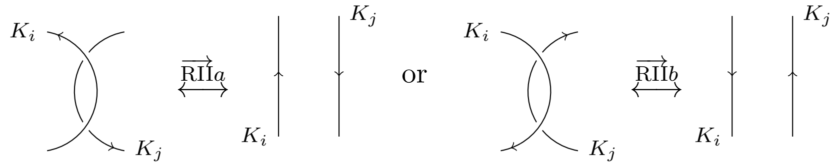
Proof. The skein formula relates diagrams in $\mathscr{D}_n$ if and only if D_+ (or equivalently D_-) are in $\mathscr{D}_n$. Hence, D_+, D_- are in $\mathscr{D}_n$. Then the formula

$$l\,\mathcal{P}_{D_+} + l^{-1}\,\mathcal{P}_{D_-} + m\,\mathcal{P}_{D_0} = 0$$

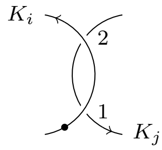
can always be seen as the first step in the calculation of $\mathcal{P}_{D_+}$ or $\mathcal{P}_{D_-}$ (for a certain component, not necessarily the first one), after that we have moved the basepoint in an appropriate way.

Proposition 4.0.6. *$\mathcal{P}$ is invariant under oriented Reidemeister moves which do not increase the number of crossings beyond n;*

Proof. *$\overrightarrow{RI}$ moves.* Note that for both $\overrightarrow{\mathrm{RI}}a$ and $\overrightarrow{\mathrm{RI}}b$, we can choose an appropriate basepoint in such a way that the crossings in the left diagram are first encountered as an underpass. Thus, the definition of $\mathcal{P}$ gives the same result.
$\overrightarrow{RII}$ moves. In this case we must take into account that arcs in the diagram can be part of two different components, say K_i and K_j, $i \leq j$. Excluding the simplest cases in which an appropriate choice of basepoint for K_i assures that every crossing is first encountered as an underpass, we are left with the case below, with $i < j$.



Consider now the $\overrightarrow{\mathrm{RII}}a$ move. Let us denote the left diagram with D and the right one with D'. We can choose a basepoint on K_i and label the crossings as below.



Then the crossings have signs $\epsilon_1 = -1$ and $\epsilon_2 = -1$. Computing $\mathcal{P}$ we obtain

$$\begin{aligned} \mathcal{P}_D &= -l^2\big(-l^{-2}\,\mathcal{P}_{\sigma_2\sigma_1 D} - l^{-1}m\,\mathcal{P}_{\eta_2\sigma_1 D}\big) - lm\,\mathcal{P}_{\eta_1 D} \\ &= \mathcal{P}_{\sigma_2\sigma_1 D} + lm\,\mathcal{P}_{\eta_2\sigma_1 D} - lm\,\mathcal{P}_{\eta_1 D} \\ &= \mathcal{P}_{\sigma_2\sigma_1 D} + lm\big(\mathcal{P}_{\eta_2\sigma_1 D} - \mathcal{P}_{\eta_1 D}\big). \end{aligned}$$

Now, we can see that in $\sigma_2\sigma_1 D$ we can choose a basepoint for K_i so that every crossing is first encountered as an underpass. Thus, $\mathcal{P}_{\sigma_2\sigma_1 D} = \mathcal{P}_{D'}$. On the other hand, applying $\overrightarrow{\mathrm{RI}}$ invariance, we obtain that $\mathcal{P}_{\eta_2\sigma_1 D} = \mathcal{P}_{\eta_1 D}$ (see Figure 4.3). Hence, $\mathcal{P}_D = \mathcal{P}_{D'}$. The same argument applies to $\overrightarrow{\mathrm{RII}}b$.

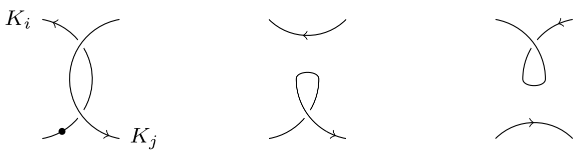
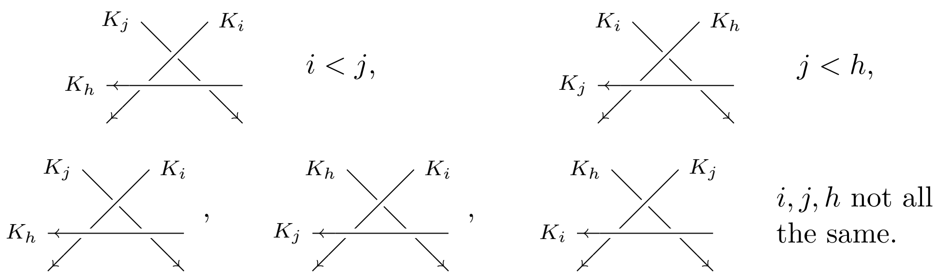


Figure 4.3: From left to right: $\sigma_2\sigma_1 D$ with the appropriate basepoint, $\eta_2\sigma_1 D$ and $\eta_1 D$.

$\overrightarrow{RIII}$ move. Here we have to deal with at most three components of the link, say K_i, K_j and K_h with $i \leq j \leq h$. Excluding again the cases in which an appropriate choice of basepoints for K_i and possibly K_j assures that every crossing is first encountered as an underpass, we are left with the six cases below (we have represented only the left diagram in the $\overrightarrow{\mathrm{RIII}}$ move).



Consider the first case. Let us call D the diagram on left of the $\overrightarrow{\mathrm{RIII}}$ move and D' the transformed diagram on the right. We can choose the basepoints on K_i as below.

Let us consider the central crossing, which must be changed for the unknotting algorithm. Since it is a positive crossing in both cases, we have

$$\mathcal{P}_D = -l^{-1}m\,\mathcal{P}_{\eta D} - l^{-2}\,\mathcal{P}_{\sigma D}$$
$$\mathcal{P}_{D'} = -l^{-1}m\,\mathcal{P}_{\eta D'} - l^{-2}\,\mathcal{P}_{\sigma D'}.$$

Now ηD is the same as $\eta D'$, up to planar isotopies of the diagram, so $\mathcal{P}_{\eta D} = \mathcal{P}_{\eta D'}$. On the other hand, in σD we can choose a basepoint in K_i such that all the crossings in the considered portion of the diagram are first encountered as underpasses. Hence, we can calculate $\mathcal{P}$ of σD and $\mathcal{P}$ of its transformed under $\overrightarrow{\text{RIII}}$ in the same way. But such a transformed diagram is precisely $\sigma D'$. Thus, $\mathcal{P}_{\sigma D} = \mathcal{P}_{\sigma D'}$ and finally $\mathcal{P}_D = \mathcal{P}_{D'}$.
The same argument applies to all other cases.

The next results will discuss *non-standard ascending* diagrams, that is, elements of $D \in \mathscr{D}_n$ which are standard ascending of other diagrams with the same orientation of D but different ordering of the components (and possibly different basepoints).

Lemma 4.0.7. *Suppose that D is a non-standard ascending element of $\mathscr{D}_n$. Let e be a closed 2-cell in the projection plane such that*

- *$e \cap D$ is the union of an arc a in ∂e and a finite number of arcs (to be called* transversals*), properly embedded in e,*
- *no basepoint is in e,*
- *each transversal crosses a in one point,*
- *no pair of transversals crosses in more than one point.*

Let $b = \overline{\partial e \setminus a}$ and let $\hat{D}$ be the result of substituting b for a in D, with b crossing over or under each transversal with the same choice as a. Then $\mathcal{P}_{\hat{D}} = \mathcal{P}_D$.

Proof. Let us prove the lemma by induction on the number ν of transversals. For the case $\nu = 0$, we have no transversals. This is the case of planar isotopies of the projection, which do not change the polynomial since they do not affect the crossings. Let us suppose that the proposition is true for $(\nu - 1)$ transversals. Let N and S denote the endpoints of a.
Among the transversals which meet both a and b, define the distance from N as the number of endpoints. Then t separates e into two cells. Take the one which contains N, minus a sufficiently small neighbourhood of a (see Figure 4.4).
Such a cell satisfies the induction hypothesis, with t playing the role of a. Thus, we can move t toward N without changing the diagram. Now, thanks to Proposition 4.0.6, we can change the projection with planar isotopies and Reidemeister moves of the third type along the remaining neighbourhood of a (starting at $t \cap a$), so as to leave the polynomial unchanged and eliminate the intersection of t and e, as in

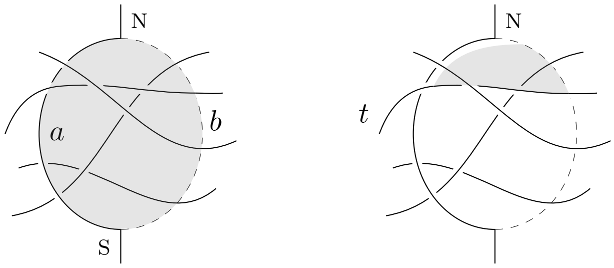


Figure 4.4

Figure 4.5. The fact that D is ascending ensures that at each such a move the three arcs concerned with the move are indeed as required for $\overrightarrow{\mathrm{RIII}}$. The resulting diagram is still ascending and has $\nu - 1$ transversals. Thus, by induction, we may replace a with b without changing the polynomial.

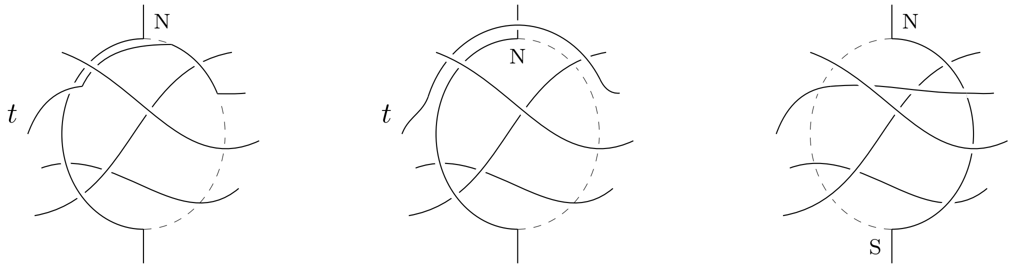


Figure 4.5: Moving t outside the cell (left and centre) and $\hat{D}$ (right).

Finally, we can employ the reverse of the above process, which leaves the polynomial unchanged and restores t to its original position, obtaining $\hat{D}$.

Lemma 4.0.8. *Suppose that D is a non-standard ascending element of $\mathscr{D}_n$. Let e be a closed 2-cell in the projection plane such that*

- *$e \cap D$ is the union of an arc a in ∂e and a finite number of arcs, the* transversals, *properly embedded in e,*
- *no basepoint is in e,*
- *one transversal, say t, crosses a in two points,*
- *if $b = \overline{\partial e \setminus a}$, each other transversal crosses a, b and t at one point.*

Let $\hat{D}$ be the result of substituting b for a in D, with b crossing over or under each transversal with the same choice as a. Then $\mathcal{P}_{\hat{D}} = \mathcal{P}_D$.

Proof. We can choose e' a sub-cell of e as in Figure 4.6 so that e' contains only one point of $t \cap a$. We then apply Lemma 4.0.7 in e' to move $a \cap e'$ and finally Reidemeister moves invariance of $\mathcal{P}$ to move the remaining part of a.

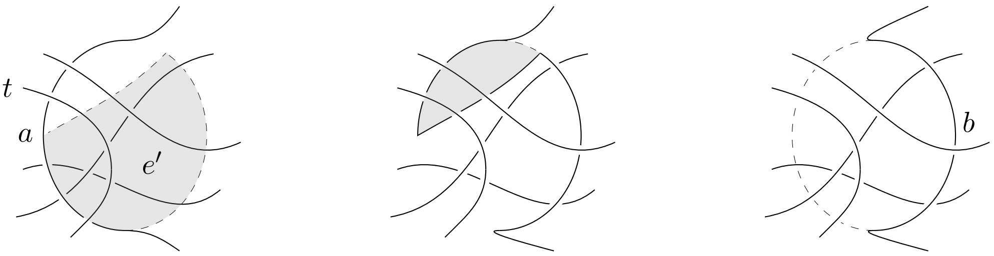


Figure 4.6

Lemma 4.0.9. *If $D = \bigsqcup_{i=1}^{r} D_i \in \mathscr{D}_n$ is disconnected as projection, then*

$$\mathcal{P}_D = \mu^{r-1} \prod_{i=1}^{r} \mathcal{P}_{D_i}.$$

Proof. If $r = 1$ there is nothing to prove. Now, if the statement is true for $r-1$, then we can think of D as

$$\left(\bigsqcup_{i=1}^{r-1} D_i \right) \sqcup D_r$$

and the thesis follows by induction.

A *diagram loop* is a simple closed curve in a link projection. Thus, it is the projection of some sub-arc of the link when the loop starts and ends at a double point of the projection or the projection of an entire component which has no self-crossing.

Proposition 4.0.10. *For every ascending diagram $D \in \mathscr{D}_n$ with c components,*

$$\mathcal{P}_D = \mu^{c-1}.$$

Proof. We can suppose that D is connected as a projection, since if $D = \bigsqcup_1^r D_i$, then $\mathcal{P}_D = \mu^{r-1} \prod_1^r \mathcal{P}_{D_i}$ (with the order on D_i induced by that of D) and repeat the argument on $\mathcal{P}_{D_i}$.
Choose an innermost diagram loop of the projection of D. For the hypothesis about connectedness of D, it is the projection of some sub-arc of the link which starts and ends at a double point of the projection. Then there are two cases.

- If this loop contains no crossing of the projection (other than the double point where the loop starts and stops) it can be removed by a RI move without changing the polynomial, obtaining an ascending diagram of c components $\widetilde{D} \in \mathscr{D}_{n-1}$. Then, by induction, $\mathcal{P}_D = \mathcal{P}_{\widetilde{D}} = \mu^{c-1}$.

- Otherwise, there are transversals across the loop and, if necessary, we can move the basepoints so that no one lies inside the loop (nor on it, unless the loop is the projection of a whole component). Thus, within the loop there is an innermost occurrence of arcs a and t and a cell e as in Lemma 4.0.8. Hence, a pair of crossings can be removed, changing D to another ascending projection $\hat{D}$ with the same polynomial and only $(n-2)$ crossings. As before the inductive hypothesis implies that $\mathcal{P}_D = \mu^{c-1}$.

Proposition 4.0.11. *$\mathcal{P}$ is independent of the choice of order of the components.*

Proof. Let D' be the same diagram, but with some other ordering of its components. Then we can give to the components of $\alpha(D')$ the original order, obtaining a non-standard ascending diagram E. Note that E depends upon both D and D', but the crucial fact is that it has the same components' ordering of D.
Now, we can compute $\mathcal{P}_D$ with a sequence of skein relations and applications of $\mathcal{H}_{n-1}$, starting from $\mathcal{P}_{\alpha(D)} = \mu^{c-1}$ and changing the crossings in any sequence (Proposition 4.0.5) from $\alpha(D)$ to D. In particular, the crossings can be changed following the sequence $\alpha(D) \to E \to D$. We can formalise the relations that will occur through the diagram

$$\mathcal{P}_{\alpha(D)} \xmapsto{f} \mathcal{P}_E \xmapsto{g} \mathcal{P}_D.$$

We know that by definition $\mathcal{P}_{\alpha(D)} = \mu^{c-1}$ and, thanks to the previous proposition, $\mathcal{P}_E = \mu^{c-1}$. Thus, f is the identity and

$$\mathcal{P}_D = g(\mu^{c-1}).$$

On the other hand, we can start from $\alpha(D')$ and change the crossings in order to obtain D'. The sequence of changes $\alpha(D') \to D'$ is nothing but that of $E \to D$. Thus, $\mathcal{P}_{D'} = g(\mathcal{P}_{\alpha(D')}) = g(\mu^{c-1})$ (with the same function g as before) and finally

$$\mathcal{P}_D = \mathcal{P}_{D'}.$$

In conclusion, we can prove the main result of the chapter.

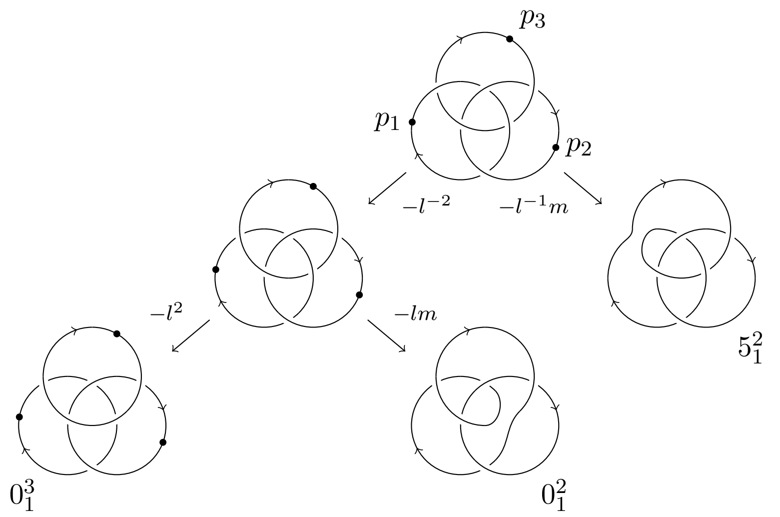
Proof of Theorem 4.0.1. As a consequence of the above propositions, we have proved $\mathcal{H}(n)$. Thus, by induction

$$\mathcal{P} \colon \mathscr{D} \to \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$$

is well-defined and is an invariant of the equivalence class of the oriented links, since every oriented link has a projection in some $\mathscr{D}_n \subset \mathscr{D}$ (after any choice of ordering of the components and basepoints), any two projections of equivalent links are in some $\mathscr{D}_n$, and are equivalent by a finite sequence of Reidemeister moves that does not increase the number of crossings beyond a certain number n_0 of crossings. In addition, thanks to Propositions 4.0.10 and 4.0.5, $\mathcal{P}$ satisfies normalisation condition and the skein relation.

Suppose now that there is another such a function $\mathcal{Q}$, which is different from $\mathcal{P}$. Then there exists an oriented link L with minimal crossing number n (≥ 3) such that $\mathcal{P}_L \neq \mathcal{Q}_L$. Applying the skein relation, we can express both $\mathcal{P}_L$ and $\mathcal{Q}_L$ as the same $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$-linear combination of polynomials of diagrams in $\mathscr{D}_{n-1}$ and an unlink. But $\mathcal{P} = \mathcal{Q}$ on $\mathscr{D}_{n-1}$ (since L is minimal in the crossing number) and, as it can be easily seen from the axioms, both $\mathcal{P}$ and $\mathcal{Q}$ agree on unlinks. Thus, we have reached a contradiction.
This concludes the proof.

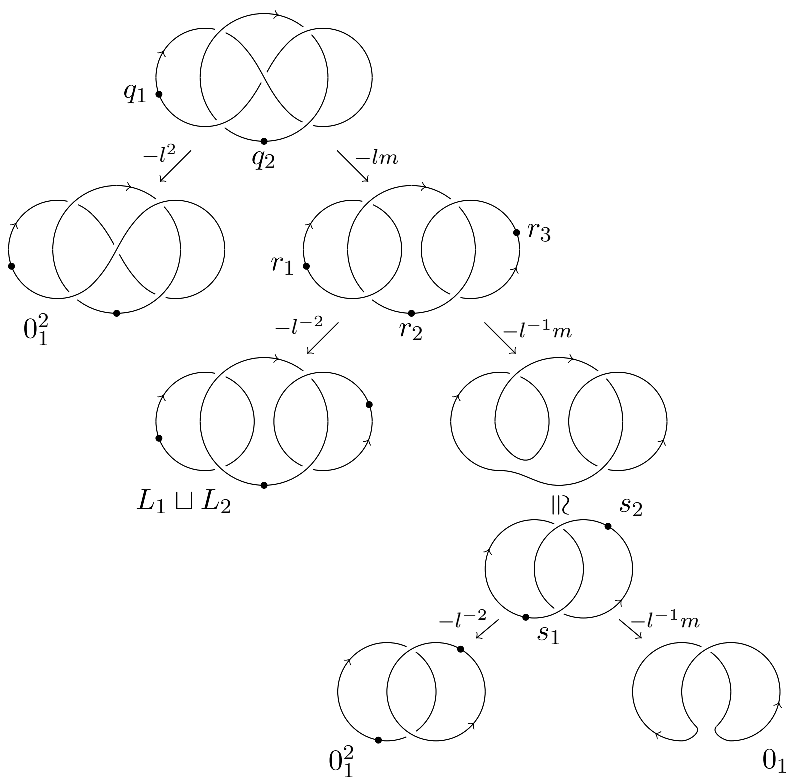
Let us apply the algorithm to compute the HOMFLY polynomial of an oriented *Borromean link*, denoted with 6^3_1. We will write $p_1, p_2, \dots$ for the ordered basepoints.



Here 5^2_1 an oriented *Whitehead link*. Then we have

$$\begin{aligned}\mathcal{P}_{6^3_1} &= \mathcal{P}_{0^3_1} + l^{-1}m\,\mathcal{P}_{0^2_1} - l^{-1}m\,\mathcal{P}_{5^2_1}\\ &= \mu^2 + l^{-1}m\mu - l^{-1}m\,\mathcal{P}_{5^2_1}.\end{aligned}$$

With analogue calculations, one can find the HOMFLY polynomial of 5^2_1.



Here, $L_1 \sqcup L_2$ is the union of an unknot and an oriented Hopf link 2_1^2. In the above calculation, we have also found that

$$\begin{aligned}\mathcal{P}_{2_1^2} &= -l^{-2}\,\mathcal{P}_{0_1^2} - l^{-1}m\,\mathcal{P}_{0_1}\\ &= -l^{-2}\mu - l^{-1}m.\end{aligned}$$

Thus, thanks to Lemma 4.0.9,

$$\mathcal{P}_{L_1\sqcup L_2} = \mu\mathcal{P}_{0_1}\mathcal{P}_{2_1^2} = -l^{-2}\mu^2 - l^{-1}m\mu.$$

and then

$$\begin{aligned}\mathcal{P}_{5_1^2} &= -l^2\,\mathcal{P}_{0_1^2} + l^{-1}m\,\mathcal{P}_{L_1\sqcup L_2} - l^{-2}m^2\,\mathcal{P}_{0_1^2} - l^{-1}m^3\,\mathcal{P}_{0_1}\\ &= -l^2\mu - l^{-2}m^2\mu - l^{-3}m\mu^2 - l^{-2}m^2\mu - l^{-1}m^3\\ &= -l^2\mu - 2l^{-2}m^2\mu - l^{-3}m\mu^2 - l^{-1}m^3.\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{P}_{6_1^3} &= \mu^2 + l^{-1}m\mu - l^{-1}m\,(-l^2\mu - 2l^{-2}m^2\mu - l^{-3}m\mu^2 - l^{-1}m^3)\\ &= (1 + l^{-4}m^2)\mu^2 + (l^{-1}m + lm + 2l^{-3}m^3)\mu + l^{-2}m^4.\end{aligned}$$

Chapter 5

Conclusions

The key question in knot theory is to tell whether two given knots or links are equivalent or not.

The Alexander polynomial turned out to be a very useful invariant. It has a long history and there are several ways to approach it. The viewpoint of homology theory was taken in this thesis. This resulted not only in a quite complex definition for the polynomial, but also in nice properties. One of the most important consequences is that the breadth of the Alexander polynomial for a knot gives a lower bound for the genus of the knot. Further, the Alexander polynomial can distinguish prime knots up to crossing number eight, excluding mirror images. Furthermore, the geometrical meaning of the variable t is clear: it is a translation automorphism of the infinite cycling cover of the knot complement.

The Alexander polynomial also has a downside. Unlike the Jones polynomial, it cannot distinguish between a knot and its mirror image. There turn out to be infinitely many non-trivial knots with Alexander polynomial equal to one (see [Rolfsen, 1976]). Furthermore the Alexander polynomial is only defined up to multiplication with a unit $\pm t^i$. This gives rise to a Conway-normalized version. The latter has the advantage that a skein relation can be deduced for it. Although conceptually easy, the skein relation is computationally disadvantageous, since there exist algorithms involving an algebraic approach which run in polynomial time.

The Jones polynomial turned out to be even more powerful than the Alexander one in distinguishing knots. It distinguishes knots from their mirror images and there exists a skein relation for this polynomial. A negative result for computational complexity is that he calculation of the Jones polynomial is expected to be an exponential process (see [Welsh, 1993] for a more precise statement). In addition, prime knots up to nine crossing number have distinct Jones polynomials.

However, a full understanding of the polynomial is still missing. Until now, there is no geometrical construction of the polynomial via the fundamental group, the homology groups and covering spaces as for the Alexander one. Nevertheless, a 3-dimensional understanding of the Jones polynomial was discovered in [Witten, 1989] in relation to a $2+1$ dimensional topological quantum field theory. Another unsolved question is whether the Jones polynomial detects unknottedness (see [Andersen et al., 2002]), while there exist infinite families of links with more than one component with trivial Jones polynomial.

Open questions for the HOMFLY polynomial are quite the same as those for the Jones polynomial. In particular, we still miss a pure 3-dimensional definition of the invariant and it is unknown whether there exist non-trivial knots or links with HOMFLY equal to one.

In conclusion, it can be said that knot theory is a very promising field. It has its roots in mathematics, but it has also many important applications in theoretical physics, chemistry and biology. To mention just few examples, it has applications in topological quantum field theory, statistical mechanics and DNA recombination. The interplay between these different fields through knot theory is fascinating.

There are still many unsolved questions, which remain a great challenge for future generations of mathematicians.

Appendix A

Classification of compact surfaces

In this section, we will give a summary description of compact surfaces' classification. See [Munkres, 2000] and [Massey, 1991] as a reference.

Definition A.0.1. A 2-manifold is an Hausdorff, first-countable space X such that each point $x \in X$ has a neighbourhood homeomorphic with an open set of $\mathbb{R}^2$. We call *surface* a connected 2-manifold.
A 2-manifold with boundary is an Hausdorff, first-countable space Y such that each point $y \in Y$ has a neighbourhood homeomorphic with an open set of $\mathbb{R}^2$ or $\mathbb{H}^2$, where $\mathbb{H}^2 = \{ (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 0 \}$. The boundary of Y, denoted with ∂Y, consist of those points y not having a neighbourhood homeomorphic with an open set of $\mathbb{R}^2$. We call *surface with boundary* a connected 2-manifold with boundary.

Definition A.0.2. A compact surface X is *orientable* if $H_2(X) \neq 0$. A compact surface with boundary Y is *orientable* if $H_2(Y, \partial Y) \neq 0$.

Definition A.0.3. Let X_1, X_2 be surfaces, e_i a closed 2-cells in X_i. Let $X_i' = X_i \setminus \mathring{e}_i$ and $h\colon \partial e_1 \to \partial e_2$ a homeomorphism. We define the *connected sum* of X_1 and X_2 as the space $X_1 \# X_2$ obtained from $X_1' \sqcup X_2'$ identifying all point $x \in \partial e_1$ with $h(x)$.

It can be shown that the connected sum is well-defined, *i.e.* it does not depend on any choice.

Theorem A.0.4 (Classification of compact surfaces)**.** *Every compact surface X is homeomorphic with one of the following:*

- *$F_0 = \mathbb{S}^2$ or the connected sum of g tori $F_g = \mathbb{T}^2 \# \ldots \# \mathbb{T}^2$, if X is orientable;*
- *the connected sum of g projective real planes $N_g = \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$, if X is non-orientable.*

The value g is called the genus *of X.*

Lemma A.0.5. *Let X be a surface, $e_1, \ldots, e_k$ a collection of disjoint open 2-cells in X with homeomorphisms $h_i\colon B^2 \to e_i$. Let $\epsilon = 1/2$ and B_ϵ the open 2-ball of radius ϵ. Then $Y = X \setminus \bigcup h_i(B_\epsilon)$ is a surface with boundary and ∂Y has k components. Y is called* X-with-k-holes*. If $k = 1$, Y is called a* punctured surface*.*

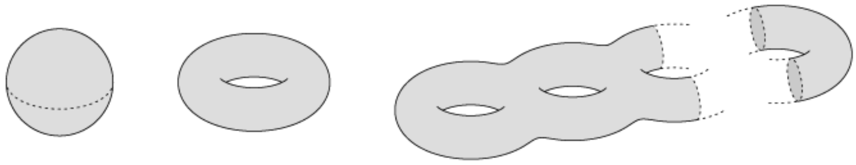


Figure A.1: Compact orientable surfaces.

Theorem A.0.6. *Every compact surface with boundary Y, such that ∂Y has k components, is homeomorphic to a surface of the form X-with-k-holes, where X is a compact surface.*

The theorem allows us to define the genus of Y as the genus of the compact surface X. In the particular case of punctured orientable surfaces, we have the following characterization.

Proposition A.0.7. *Every compact orientable surface with boundary Y with genus g and k boundary components is homeomorphic to a disk with g pairs of handles and $k-1$ single handles as in Figure A.2.*

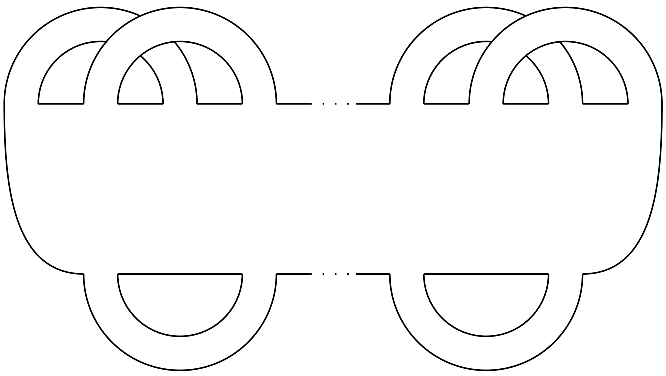


Figure A.2: A compact orientable surface with boundary.

Corollary A.0.8. *If Y is a compact orientable surface with boundary with genus g and k boundary components, then there is a deformation retraction $r\colon \bigvee_1^{2g+k-1} \mathbb{S}^1 \to \mathring{Y}$. Hence,*

$$H_1(\mathring{Y}) \cong \mathbb{Z}^{2g+k-1}.$$

A basis for $H_1(\mathring{Y})$ consists of $2g+k-1$ loops which pass through the disk and one handle.

Appendix B

Convering spaces

In this section, we will give some results about covering spaces, following the line of [Munkres, 2000].

Let us consider B, E topological spaces.

Definition B.0.1. Let $p\colon E \to B$ be a surjective continuous map. An open set $U \in B$ is called *evenly covered* if

$$p^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha,$$

where V_α is an open set in E and $p|_{V_\alpha}$ is an homeomorphism with its image, $\forall \alpha \in J$. The map p is called a *covering map* of B if every $b \in B$ has an evenly covered neighbourhood U of b. We call E the *covering space* and $p^{-1}(b)$ the *fibre* of b.

Definition B.0.2. Let X be a topological space, $p\colon E \to B$ be a covering map and $f\colon X \to B$ continuous. A continuous map $\widetilde{f}\colon X \to E$ such that $f = p \circ \widetilde{f}$ is called a *lift* of f. In other terms, the following diagram commutes.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & B
\end{tikzcd}$$

Lemma B.0.3. *If X is a connected space and $f\colon X \to B$ lifts to two maps $\widetilde{f}, \widetilde{f}'\colon X \to E$, then $\{\, x \in X \mid \widetilde{f}(x) = \widetilde{f}'(x) \,\}$ is either empty or all of X.*

Path lifting lemma B.0.4. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, $f\colon [0,1] \to B$ a path with $f(0) = b_0$. Then there exists a unique lift $\widetilde{f}\colon [0,1] \to E$ of f such that $\widetilde{f}(0) = e_0$.

Homotopy of paths lemma B.0.5. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, $F\colon [0,1]^2 \to B$ a continuous map with $F(0,0) = b_0$. Then there exists a lift $\widetilde{F}\colon\ [0,1]^2 \to\ E$ of F with $\widetilde{F}(0,0) = e_0$. Furthermore, if F is an homotopy of paths, so it is $\widetilde{F}$.

Corollary B.0.6. *Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. Then the induced homomorphism $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.*

This means that $H_0 = p_*(\pi_1(E, e_0))$ is a subgroup of $\pi_1(B, b_0)$. It turns out that H_0, called the *group of covering*, determines the map p, up to a suitable notion of equivalence of coverings and further hypothesis on E and B. In particular, from now on we consider E and B both path-connected and locally path-connected.

Definition B.0.7. Two covering maps $p\colon E \to B$ and $p'\colon E' \to B$ are said to be *equivalent* if there exists a homeomorphism $h\colon E \to E'$ such that $p' \circ h = p$. The homeomorphism h is called *equivalence of covering maps.*

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

General lifting lemma B.0.8. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. Let X be a path-connected, locally path-connected topological space and $f\colon (X, x_0) \to (B, b_0)$ be a continuous map. Then there exists a lift $\widetilde{f} : (X, x_0) \to (E, e_0)$ if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$. Furthermore, if such a lifting exists, it is unique.

The general lifting lemma leads to the following

Theorem B.0.9 (Classification of covering spaces)**.** *Let $p\colon (E, e_0) \to (B, b_0)$ and $p'\colon (E', e_0') \to (B, b_0)$ covering maps. There exists an equivalence of covering maps $h\colon (E, e_0) \to (E', e_0')$ if and only if $H_0 = H_0'$. If h exists, it is unique.*

In addition, neglecting basepoints corresponds to consider conjugate classes of subgroups of $\pi_1(B, b_0)$.

Theorem B.0.10. *The covering maps $p\colon (E, e_0) \to (B, b_0)$ and $p'\colon (E', e_0') \to (B, b_0)$ are equivalent if and only if the subgroups H_0 and H_0' of $\pi_1(B, b_0)$ are conjugate.*

The problem of existence of a covering space can be solved for path-connected, locally path-connected, semi-locally simply connected spaces.

Definition B.0.11. A space X is *semi-locally simply connected* if every $x \in X$ has a neighbourhood U such that the homomorphism induced by the inclusion $i_*\colon \pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Theorem B.0.12. *Let B be a path-connected, locally path-connected, semi-locally simply connected space. Then for every subgroup $G \subset \pi_1(B, b_0)$ there exist a covering $p\colon E \to B$ such that $G = p_*(\pi_1(E, e_0))$, for a suitable choice of $e_0 \in E$.*

An important class of covering spaces is that of regular ones. These coverings are strictly related to the group of covering automorphisms.

Definition B.0.13. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. A homeomorphism $\tau\colon E \to E$ is called *covering automorphism* if $p \circ \tau = p$. The covering automorphisms form a group, denoted $\mathrm{Aut}(E, p)$.

Since a covering automorphism can be seen as a lift of the covering map p, thanks to Lemma B.0.3 it is determined by its action on any point of E.

Definition B.0.14. A covering map $p\colon (E, e_0) \to (B, b_0)$ is said to be *regular* if H_0 is a normal subgroup of $\pi_1(B, b_0)$.

Proposition B.0.15. *A covering map $p\colon E \to B$ is regular if and only if, for any two points e_1, e_2 in the fibre $p^{-1}(b_0)$, there is a covering automorphism $\tau \in \mathrm{Aut}(E, p)$ such that $\tau(e_1) = \tau(e_2)$,* i.e. $\mathrm{Aut}(E, p)$ *acts transitively on E.*

Another important connection between regular coverings and the group of automorphisms is given by the following construction. For every covering map $p\colon E \to B$, one can assign to $[f] \in \pi_1(B, b_0)$ a permutation of the fibre $p^{-1}(b_0)$ as follows: for each $e \in p^{-1}(b_0)$ there is a unique lift $\widetilde{f}$ of f with $\widetilde{f}(0) = e$. Let us call $\sigma_{[f]}(e) = \widetilde{f}(1)$ its terminal point. Then $\sigma_{[f]}\colon p^{-1}(b_0) \to p^{-1}(b_0)$ is a permutation which depends only on the homotopy class of f.

Proposition B.0.16. *Every such permutation $\sigma_{[f]}\colon p^{-1}(b_0) \to p^{-1}(b_0)$ extends to a covering automorphism $\tau_{[f]}\colon E \to E$ if and only if p is regular.*
In addition, the correspondence $\varphi\colon \pi_1(B, b_0) \to \mathrm{Aut}(E, p)$ is a surjective homomorphism, with kernel H_0.

Appendix C

Module presentation

In the following discussion, let us consider R to be a commutative ring with unit 1.

Definition C.0.1. An abelian group M is called a (left unitary) *R-module* if there is a multiplication $R \times M \to M$ such that $\forall a, b \in R$ and $\forall x, y \in M$

$$\begin{aligned} a(x+y) &= ax + ay \\ (a+b)x &= ax + bx \\ (ab)x &= a(bx) \\ 1x &= x. \end{aligned}$$

The multiplication can be equivalently seen as a *ring action* of R on M, *i.e* a ring homomorphism $\rho\colon R \to \mathrm{End}(M)$, writing $\rho_a(x) = ax$ for $a \in R$ and $x \in M$.
A *submodule* is a subgroup of M which is closed under multiplication by elements of R. The *quotient module* by a submodule is the quotient group with multiplication induced.

Definition C.0.2. Given two R-modules M, N, an *R-homomorphism* $\varphi\colon N \to M$ is a homomorphism of abelian groups compatible with the multiplication:

$$\varphi(ax) = a\varphi(x).$$

Definition C.0.3. A *free R-module* on the symbols $x_1, x_2, \dots$ is the set of all finite linear combinations

$$R\langle x_1, x_2, \dots \rangle = \{ \sum_{i=1}^{n} a_i x_i \mid a_i \in R, n \in \mathbb{N} \},$$

with group addition and module multiplication defined by

$$\begin{aligned} (a_1x_1 + a_2x_2 + \dots) + (b_1x_1 + b_2x_2 + \dots) &= (a_1 + b_1)x_1 + (a_2 + b_2)x_2 + \dots \\ \alpha(a_1x_1 + a_2x_2 + \dots) &= (\alpha a_1)x_1 + (\alpha a_2)x_2 + \dots. \end{aligned}$$

The set $\{x_1, x_2, \dots\}$ is called the *basis* of M.

Definition C.0.4. Let $\rho_1, \rho_2, \dots \in R\langle x_1, x_2, \dots \rangle$. An R-module M is said to have an *R-module presentation* of the form $(x_1, x_2, \dots \mid \rho_1, \rho_2, \dots)$ if M is isomorphic to the quotient module

$$M \cong \frac{R\langle x_1, x_2, \dots \rangle}{R\langle \rho_1, \rho_2, \dots \rangle},$$

The elements x_i are called *generators* and the ρ_i *relators*. The equations $\rho_i = 0$ (or an equivalent ones) are called *relations*.
Equivalently, M has a presentation of the form $(x_1, x_2, \dots \,|\, \rho_1, \rho_2, \dots)$ if there exists a short exact sequence

$$0 \longrightarrow R\langle \rho_1, \rho_2, \dots \rangle \xrightarrow{\varphi} R\langle x_1, x_2, \dots \rangle \xrightarrow{\psi} M \longrightarrow 0.$$

M is said to be *finitely presentable* if it has a presentation with a finite number of generators and relators.

Definition C.0.5. Let $M = (x_1, \dots, x_m \,|\, \rho_1, \dots, \rho_n)$ be a finitely presentable R-module. Then each ρ_i is a linear combination of the generators:

$$\rho_i = \sum_{j=1}^{m} a_{ij} x_j \qquad a_{ij} \in R.$$

We define the $n \times m$ matrix $P = (a_{ij})$ as a *presentation matrix* for M, associated to the given presentation. The matrix P can be seen as the matrix which represents the map φ with respect to the bases $\{x_1, \dots, x_m\}$ and $\{\rho_1, \dots, \rho_n\}$.

Since knowing P is the same as knowing the specific presentation, then the presentation matrix determines M, up to R-isomorphisms.

Proposition C.0.6. *If P and P' are two presentation matrices for an R-module M, then these two matrices are related by a finite sequence of the following matrix moves.*

1. *Permutation of rows and/or columns.*
2. *Addition of a scalar multiple of a row (or column) to another row (or column).*
3. *Replacement of a matrix Q with*

$$\begin{pmatrix} & & & 0 \\ & Q & & \vdots \\ & & & 0 \\ * & \cdots & * & 1 \end{pmatrix}$$

   or vice versa.

4. *Adjoin/delete a new row which is an R-linear combination of the other rows.*

A proof can be found in [Zassenhaus, 1949]. The move (1) corresponds to a permutation of generators or relators, (2) to a substitution of generators or relators with linear combination of others, (3) to introducing a generator and a relation defining it in terms of the others or removing a redundant generator, (4) to adding or removing redundant relations.

The properties of the determinant together with the previous result guarantee that the following definition is well-posed.

Definition C.0.7. Let M be a finitely presentable R-module, with an $n \times m$ presentation matrix P. Then the *kth elementary ideal* ϵ_k of M is the ideal of R generated by the $(n-k) \times (m-k)$ minors of P.
Furthermore, if we are dealing with a square $n \times n$ matrix, then ϵ_0 is the principal ideal generated by $\det(P)$. It is called the *order ideal*.

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