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THE J-EQUATION ON KÄHLER MANIFOLDS AND BLOWUPS

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Abstract

In this work the problem of existence of Kähler metrics satisfying the J-equation (introduced by S.K. Donaldson and X.X. Chen) on compact complex manifolds is discussed. In particular, a numerical criterion of M. Lejmi and G. Székelyhidi is studied; this is the natural generalisation of the obstruction to the existence of such metrics found by Donaldson.

A large part of the first two chapters is devoted to the development of the required complex geometry and the basic properties of Kähler manifolds, while in the last chapter we analyse some aspects of the J-equation. One of the main open problems in the study of this equation is to prove that the validity of the numerical criterion is actually sufficient for the existence of critical metrics. The conjecture is related to a circle of ideas in Kähler geometry, relating the existence of special Kähler metrics to algebro-geometric stability conditions. The only cases where the conjecture has been verified are surfaces and toric manifolds.

As main result of this work, we prove that if the numerical criterion holds on a compact Kähler manifold, then the same remains true on the possible blowups at a point, for Kähler classes that make the volume of the exceptional divisor sufficiently small. By applying the result to compact toric manifolds, we obtain the existence of non-trivial solutions on toric blowups. The same arguments extend to the more general inverse σ_m equations without any difficulties.

Sommario

Questo lavoro di tesi ha affrontato il problema dell'esistenza di metriche Kähler che soddisfano la J-equazione, introdotta da S.K. Donaldson e X.X. Chen. In particolare, si è studiato il criterio numerico proposto da M. Lejmi e G. Székelyhidi, una naturale generalizzazione dell'ostruzione all'esistenza di tali metriche trovata da Donaldson.

I primi due capitoli sono dedicati allo sviluppo della necessaria geometria complessa e alla presentazione delle proprietà fondamentali delle varietà Kähler, mentre nel terzo capitolo si analizzano alcuni aspetti della J-equazione. Uno dei problemi più significativi nello studio di tale equazione è la dimostrazione dell'esistenza di metriche critiche, sotto l'ipotesi di validità del criterio numerico. Tale congettura è legata ad un circolo di idee in geometria Kähler che legano l'esistenza di particolari metriche a condizioni geometro-algebriche di stabilità. Gli unici casi dove la congettura è stata verificata sono le superfici e le varietà toriche.

Il risultato principale dell'elaborato è stato quello di dimostrare che, se il criterio numerico vale in una varietà Kähler compatta, allora vale anche nei possibili scoppiamenti in un punto, per classi Kähler che rendono sufficientemente piccolo il volume del divisore eccezionale. Come corollario, si ottiene l'esistenza di soluzioni non banali su scoppiamenti di varietà toriche. Gli stessi risultati si estendono senza difficoltà al caso più generale di equazioni σ_m inverse.

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Introduction

The J-equation was introduced by S.K. Donaldson in [Donaldson, 1999], as the vanishing condition for the moment map of a certain infinite-dimensional Hamiltonian action. At around the same time, X.X. Chen independently discovered the J-equation as the Euler-Lagrange equation of his J-functional (see [Chen, 2000]). He showed that the J-functional is related to the Mabuchi K-energy, which plays a key role in the study of Kähler geometry and stability in the sense of geometric invariant theory.

Explicitly, the J-equation is defined as follows. Let (M, α) be a compact Kähler manifold of dimension n, ω_0 another Kähler metric. Let c be the constant given by

$$c = \frac{\int_M \alpha \wedge \omega_0^{n-1}}{\int_M \omega_0^n}$$

and let $\mathcal{H} = \left\{ \phi \in C^\infty(M, \mathbb{R}) \;\middle|\; \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0 \right\}$ be the space of Kähler potentials. The J-equation defined on $\mathcal{H}$ is given by

$$\alpha \wedge \omega_\phi^{n-1} = c\omega_\phi^n.$$

The equation turns out to be equivalent to $\Lambda_{\omega_\phi}\alpha = nc$, where Λ_{ω_ϕ} is the dual Lefschetz operator associated to the metric ω_ϕ, or to $\Delta_{\omega_\phi}\alpha = 0$, *i.e.* α is a ω_ϕ-harmonic form.

Donaldson in [Donaldson, 1999] asked whether one can find, under the proper assumptions, a solution to the equation in the class $[\omega_0]$. He noted that a necessary condition is $[nc\omega_0 - \alpha]$ being a Kähler class, and conjectured that this condition is also sufficient. Chen confirmed in [Chen, 2000] the conjecture in the case $n = 2$, by observing that the equation reduces to a complex Monge-Ampère equation which can be solved by the well-known result of Yau. In [Lejmi and Székelyhidi, 2015] the authors showed that the Donaldson criterion actually fails in higher dimensions and proposed a new numerical condition, which they conjectured to be equivalent to existence of a solution.

Conjecture (Lejmi and Székelyhidi, 2015)**.** Let (M, α) be a compact Kähler manifold, ω_0 another Kähler metric. There exists a solution of $\Lambda_\omega \alpha = nc$ in $[\omega_0]$ if and only if, for all irreducible analytic subvarieties $V \subset M$ of dimension $k < n$, the following numerical criterion

$$\int_V \left(nc\omega_0^k - k\alpha \wedge \omega_0^{k-1}\right) > 0$$

holds.

It is easy to show that this is indeed a necessary condition. On the other hand one can also naturally arrive at this condition from the point of view of an algebro-geometric stability condition, analogous to K-stability for the constant scalar curvature Kähler (cscK) equation, introduced by M. Lejmi and G. Székelyhidi in the same article. The analogy with cscK metrics is emphasised in Donaldson's original approach: both equations arises as the vanishing condition for moment maps of an infinite-dimensional Hamiltonian action. Together with the aforementioned result of Chen for surfaces, a first step in this direction was given by the proof of the conjecture on toric manifolds (*cf.* [Collins and Székelyhidi, 2014]).

This Master thesis aimed at proving an expected result about the blowup behaviour of the J-equation, analogous to that of C. Arezzo and F. Pacard ([Arezzo and Pacard, 2006, 2009]) for the cscK equation: if the J-equation is solvable on a compact Kähler manifold, then, under certain hypotheses, it can be solved in its possible blowups at a point. Instead of working directly with the equation, we analysed the behaviour of the numerical criterion, proving the following

Theorem. *Let* (M, α) *be a compact Kähler manifold admitting a Kähler class* $[\omega]$ *such that the numerical criterion holds. Consider the blowup* $\sigma\colon \mathrm{Bl}_p(M) \to M$ *at a point* p *and denote by* E *the exceptional divisor. Then there exists* $\epsilon > 0$ *sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\,\mathrm{PD}[E]$$

are positive and the numerical criterion holds on $\mathrm{Bl}_p(M)$ *in the above classes, provided that* $a < \frac{n}{n-1}c$.

In particular, we found the numerical condition $a < \frac{n}{n-1}c$ on the blowup parameter, which reflects the fact the volumes of the exceptional locus with respect to both metrics $\tilde{\omega}$ and $\tilde{\alpha}$ must be of the same order. Combining the above theorem with that of Chen for the J-equation on surfaces and that of T.C. Collins and G. Székelyhidi on toric manifolds, we obtain the following new existence results.

Corollary. *Let* (M, α) *be a compact Kähler surface admitting a solution to the* J*-equation in the Kähler class* $[\omega_0]$*. Then there exists* $\epsilon > 0$ *sufficiently small such that*

$$[\tilde{\omega}_0] = \sigma^*[\omega_0] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\,\mathrm{PD}[E]$$

are positive and there exists a solution to the J*-equation* $\Lambda_{\tilde{\omega}}\tilde{\alpha} = 2\tilde{c}$ *on* $\mathrm{Bl}_p(M)$ *in the Kähler class* $[\tilde{\omega}_0]$*, provided that* $a < 2c$.

Corollary. *Let* (M, ω_0) *be a compact, toric, Kähler manifold,* $p \in M$ *a point invariant under the torus action. Then the blowup* $\mathrm{Bl}_p(M)$ *admits non-trivial solutions to the* J*-equation* $\Lambda_\omega \alpha = nc$ *in the classes*

$$[\omega] = \sigma^*[\omega_0] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\alpha] = \sigma^*[\omega_0] - a\epsilon\,\mathrm{PD}[E]\,,$$

for ϵ *sufficiently small, provided that* $a < \frac{n}{n-1}$.

These results can be simply generalised to a wider class of geometric PDEs, known as inverse σ_m equations.

The thesis is structured as follows.

Chapter 1. From the definition of complex manifolds and analytic subvarieties, we briefly present the concept of almost complex structures. Then we move to differential forms and Doulbeault cohomology groups on complex manifolds. The last part of the chapter is devoted to complex vector bundles, with particular emphasis on Hermitian vector bundles and the connection between holomorphic line bundles and divisors.

Chapter 2. The fundamental Kähler condition with some characterisations is introduced. Further, the basic symmetries of the classical tensors associated to a Kähler metric are presented. The last section is devoted to the introduction of the Lefschetz operators and the related algebraic aspects, followed by the Kähler identities and the theory of harmonic forms. Finally, some topological constraints on compact Kähler manifolds are discussed.

Chapter 3. In this last chapter we introduce the J-equation on compact Kähler manifolds, presenting the original approaches of Donaldson and Chen. We prove a uniqueness statement via a comparison principle, and we briefly review the known result about the dependence on the Kähler classes. Further, the numerical criterion of Lejmi and Székelyhidi is presented, together with the results for surfaces and toric manifolds mentioned above. After the definition of blowup and its basic properties, we show the validity of the numerical criterion on blowups under certain hypotheses. Finally, we briefly introduce the inverse σ_m equations, generalising the above results to this class of equations, and we discuss an application to the blowup of the projective space at a point.

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UniTS, University of Trieste
SISSA, International School for Advanced Studies

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Chapter 1

Complex Geometry

In this first chapter we will focus on complex manifolds, with particular emphasis on vector bundles and divisors. We will develop most of the basic concepts that are needed for the more advanced topics that will be studied in the following. The basic references are [Griffiths and Harris, 1994] and [Huybrechts, 2005].

1.1 Complex manifolds

Complex manifolds are topological spaces that are locally modelled by open subsets of $\mathbb{C}^n$, with holomorphic transition functions. Although they are strictly related to differentiable manifolds, they differ in many aspects. Heuristically speaking, complex manifolds are more rigid structure (*e.g.* the only holomorphic functions on compact complex manifolds are the constant ones), but on the other hand they can often be described in very explicit terms. We firstly recall some basic properties of holomorphic functions of many variables.

Notation. In the following, we will write $z = (z^1, \dots, z^n)$ for a point of $\mathbb{C}^n$, with

$$z^i = x^i + \sqrt{-1} y^i \tag{1.1.1}$$

and $\sqrt{-1}$ the *imaginary unit*. Recall that a smooth map $f \colon U \to V$ between open subsets of $\mathbb{C}^n$ is *holomorphic* if its differential $d_p f$ is a $\mathbb{C}$-linear map for every $p \in U$. This condition is equivalent to the requirement

$$\frac{\partial f^i}{\partial \bar{z}^j} = 0 \qquad \forall i, j = 1, \dots, n, \tag{1.1.2}$$

i.e. f is holomorphic if and only if every component is holomorphic in each variable.

Many properties of holomorphic functions that hold in the one-dimensional case can be generalised to the many-variables setting. We recollect here briefly some of those properties that will be useful in what follows; we refer to [Griffiths and Harris, 1994] for proves of the results.

Definition 1.1.1. Let $z_0 \in \mathbb{C}^n$ be a point, $r_1, \dots, r_n$ be positive real numbers. Set $r = (r_1, \dots, r_n)$. The *polydisc* of centre z_0 and polyradius r is the set

$$\Delta_r(z_0) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z^i - z_0^i| < r_i \ \forall i = 1, \dots, n \right\}. \tag{1.1.3}$$

As in the one-dimensional case, a fundamental tool is the following

Cauchy integral formula. Let Δ be a polydisc in $\mathbb{C}^n$, U an open neighbourhood of $\overline{\Delta}$, and $f\colon U \to \mathbb{C}$ be a continuous map, holomorphic on Δ. Then for any $z \in \Delta$,

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial\Delta} \frac{f(w)}{(w^1 - z^1)\cdots(w^n - z^n)}\, dw^1 \cdots dw^n. \tag{1.1.4}$$

Let us see some of the consequences. Consider an open, connected set $U \subset \mathbb{C}^n$.

Power series expansion. Let $f\colon U \to \mathbb{C}$ be a smooth function. Then f is holomorphic if and only if for all $z_0 \in U$ there exists a polydisk $\Delta_r(z_0) \subset U$ and a collection of complex numbers $\{\, c_{\alpha_1\cdots\alpha_n} \mid \alpha_1, \dots, \alpha_n \in \mathbb{N} \}$ such that for all $z \in \Delta_r(z_0)$ we have

$$f(z) = \sum_{\alpha_1,\dots,\alpha_n \in \mathbb{N}} c_{\alpha_1\cdots\alpha_n} (z^1 - z_0^1)^{\alpha_1} \cdots (z^n - z_0^n)^{\alpha_n}. \tag{1.1.5}$$

Identity theorem. Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ a point such that $f \equiv 0$ in a neighbourhood of z_0. Then $f \equiv 0$ on U.

Maximum principle. Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ a point such that $|f|$ achieves a local maximum at z_0. Then f is constant on U.

Open mapping theorem. Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $V \subset U$ be an open subset. If f is not constant, $f(U)$ is open.

All these results are an immediate generalisation of the well-known properties of holomorphic functions of one variable; however, there are some noticeable differences between the one and many-variables realm. An example is the famous Hartog's theorem, which holds only in dimension at least 2.

Hartogs' theorem. Consider two polyradii $r = (r_1, \dots, r_n)$ and $r' = (r'_1, \dots, r'_n)$ such that $r'_i < r_i$ for all $i = 1, \dots, n$. If $n \geqslant 2$, then any holomorphic function $f\colon \Delta_r(0) \setminus \Delta_{r'}(0) \to \mathbb{C}$ can be uniquely extended to a holomorphic function defined on the whole polydisc $\Delta_r(0)$.

Let us move now to the definition of complex manifold.

Definition 1.1.2. A *holomorphic n-atlas* on a set M is a collection $\{\, (U_\alpha, \varphi_\alpha) \}_{\alpha \in I}$, labelled by an at most countable set of indices I, such that the following conditions hold.

- The sets U_α cover M.
- For any $\alpha \in I$, φ_α is a one-to-one map from U_α to an open domain in the complex space $\mathbb{C}^n$:
$$\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n.$$
- For any pair of intersecting sets $U_\alpha \cap U_\beta \neq \varnothing$, the domains $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{C}^n$ and the one-to-one map
$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is holomorphic. These maps are called *transition functions*. As the condition holds for every pair of indices, we deduce that transition functions are biholomorphisms.

A pair (U, φ) is called a *holomorphic chart*. A subset $U \subset M$ is defined to be open if its intersections with holomorphic charts

$$\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$$

are open for all $\alpha \in I$. This defines a topological structure on M.

Definition 1.1.3. A set M equipped with a holomorphic n-atlas is called a *complex manifold* of dimension n if it is a Hausdorff, second countable topological space.

In the following, we will often say that "M is a complex manifold", assuming that M comes with an assigned holomorphic structure. Similarly, when talking about "holomorphic charts on M", it will be tacitly assumed that these charts belong to the assigned holomorphic structure of M. We will usually denote the complex coordinates as $z^1, \dots, z^n$, which decomposes as $z^i = x^i + \sqrt{-1}y^i$. If not differently stated, the dimension of a complex manifold will be denoted by n. Further, we will tacitly assumed the manifolds to be connected.

Note that a complex manifold M of complex dimension n is automatically a real manifold of real dimension 2n.

Example 1.1.1. The simplest example of complex manifold is any open subset of $\mathbb{C}^n$. Another classical example is the *complex projective space* $\mathbb{CP}^n$ (or for brevity $\mathbb{P}^n$):

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{Z \sim \lambda Z}, \qquad \lambda \in \mathbb{C}^*. \tag{1.1.6}$$

We define an n-atlas on it, defining $U_i = \{[Z] \in \mathbb{P}^n \mid Z^i \neq 0\}$, $i = 0, \dots, n$ and considering the one-to-one maps $\varphi_i \colon U_i \to \mathbb{C}^n$ given by

$$\varphi_i\big([Z]\big) = \left(\frac{Z^0}{Z^i}, \dots, \widehat{\frac{Z^i}{Z^i}}, \dots, \frac{Z^n}{Z^i}\right). \tag{1.1.7}$$

The inverse maps are

$$\varphi_i^{-1}(z^1, \dots, z^n) = \left[z^1 : \dots : z^i : 1 : z^{i+1} : \dots : z^n\right] \tag{1.1.8}$$

and the transition functions (for $j > i$)

$$\varphi_j \circ \varphi_i^{-1}(z^1, \dots, z^n) = \frac{1}{z^j}\left(z^1, \dots, z^i, 1, z^{i+1}, \dots, \widehat{z^i}, \dots, z^n\right) \tag{1.1.9}$$

are clearly holomorphic on $U_i \cap U_j$.

Definition 1.1.4. Let M and N be complex manifolds. A map $f \colon M \to N$ is said to be *holomorphic* if for each pair of holomorphic charts (U, φ) and (V, ψ) on M and N respectively such that $f(U) \subset V$, the map $\psi \circ f \circ \varphi^{-1} \colon \varphi(U) \to \psi(V)$ is holomorphic.

A holomorphic function on M is simply a holomorphic functions $f\colon M \to \mathbb{C}$. These functions form a sheaf $\mathcal{O}_M$ (or simply $\mathcal{O}$) on M, called the *sheaf of holomorphic functions*: for each $U \subset M$ open,

$$\mathcal{O}_M(U) = \{\, f\colon U \to \mathbb{C} \mid f \text{ is holomorphic} \,\}. \tag{1.1.10}$$

The sheaf of holomorphic functions on M is very different from the one of smooth functions on differentiable manifolds, as it is already shown by the following

Lemma 1.1.5. *Let M be a compact complex manifold. If $f\colon M \to \mathbb{C}$ is a holomorphic function, then f is constant.*

Proof. This is a consequence of the maximum principle. By compactness, f assumes maximum at some point $p_0 \in M$. Set

$$A = \{\, p \in M \mid f(p) = f(p_0) \,\}.$$

Then A is not empty and is closed. On the other hand, let $p \in A$ and (U, φ) be a local chart containing p. By the maximum principle, $f \circ \varphi^{-1}$ is constant on $\varphi(U)$, so that $U \subset A$ is a neighbourhood of p in A. As M is connected, $A = M$, *i.e.* f is constant. □

We will see more about the information contained in sheaf of holomorphic functions in section 1.2.2.

Let us analyse now the notion of tangent space. Let M be a complex manifold, $p \in M$ and (z^i) a holomorphic coordinate system around p. Considering M as a real manifold of dimension $2n$, we have the real tangent space at p, denoted by T_pM. It can be realised as the space of $\mathbb{R}$-linear derivations on smooth real-valued functions defined on a neighbourhood of p. In terms of local coordinates,

$$T_pM = \mathbb{R}\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\rangle, \tag{1.1.11}$$

which is a real vector space of real dimension $2n$.
On the other hand, we can define the *complexified tangent space* at p as $T_{\mathbb{C},p}M = T_pM \otimes \mathbb{C}$. It can be realised as the space of $\mathbb{C}$-linear derivations on smooth complex-valued functions defined on a neighbourhood of p. In terms of local coordinates,

$$T_{\mathbb{C},p}M = \mathbb{C}\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\rangle, \tag{1.1.12}$$

which is a complex vector space of complex dimension $2n$. Setting

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \qquad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right), \tag{1.1.13}$$

it can be described as

$$T_{\mathbb{C},p}M = \mathbb{C}\left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i} \right\rangle. \tag{1.1.14}$$

Finally, we have the *holomorphic tangent space* at p

$$T_p^{1,0}M = \mathbb{C}\left\langle \frac{\partial}{\partial z^i} \right\rangle \subset T_{\mathbb{C},p}M. \tag{1.1.15}$$

It can be realised as the space of $\mathbb{C}$-linear derivations vanishing on *antiholomorphic functions*, which clarify the independence on the chosen coordinate system. Analogously, the space

$$T_p^{0,1}M = \mathbb{C}\left\langle \frac{\partial}{\partial \bar{z}^i} \right\rangle \subset T_{\mathbb{C},p}M \tag{1.1.16}$$

is called the the antiholomorphic tangent space at p. Clearly,

$$T_{\mathbb{C},p}M = T_p^{1,0}M \oplus T_p^{0,1}M. \tag{1.1.17}$$

Note that the complex structure on $T_{\mathbb{C},p}M$ induces the operation of *conjugation*, an automorphism of $T_{\mathbb{C},p}M$ sending $\partial/\partial z^i$ to $\partial/\partial \bar{z}^i$. Thus, we have the relation

$$\overline{T_p^{1,0}M} = T_p^{0,1}M. \tag{1.1.18}$$

Further, the projection $T_pM \to T_{\mathbb{C},p}M \to T_p^{1,0}M$ sending

$$\alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial y^i} \longmapsto \left(\alpha^i + \sqrt{-1}\beta^i\right) \frac{\partial}{\partial z^i} \tag{1.1.19}$$

is an $\mathbb{R}$-linear isomorphism of vector space of real dimension $2n$. In the next section, introducing the concept of almost complex structure, we will see that the map is actually a $\mathbb{C}$-linear isomorphism.

We can also bundle these spaces to construct the real, complex, holomorphic and antiholomorphic tangent bundles:

$$\begin{aligned} TM &= \bigsqcup_{p\in M} T_pM, & T_{\mathbb{C}}M &= \bigsqcup_{p\in M} T_{\mathbb{C},p}M, \\ T^{1,0}M &= \bigsqcup_{p\in M} T_p^{1,0}M, & T^{0,1}M &= \bigsqcup_{p\in M} T_p^{0,1}M. \end{aligned} \tag{1.1.20}$$

The first one is a real manifold of dimension $2n$, the second one is a complex manifold of complex dimension $2n$. The holomorphic and antiholomorphic tangent bundles are again complex manifolds of dimension n. The holomorphic one has the further property of being a holomorphic vector bundle (see section 1.3).
The set of global sections of $T_{\mathbb{C}}M$ is denoted by $\mathfrak{X}_{\mathbb{C}}(M)$, while the set of global sections of $T^{1,0}M$ is denoted by $\mathfrak{X}^{1,0}(M)$ and the elements are called *holomorphic vector fields*. Analogous definition holds for the set $\mathfrak{X}^{0,1}(M)$ of antiholomorphic vector fields.

The dual constructions produce the cotangent space at p, the complex one, the holomorphic and antiholomorphic cotangent spaces and the corresponding bundles. In local coordinates,

$$dz^i = dx^i + \sqrt{-1}dy^i, \qquad d\bar{z}^i = dx^i - \sqrt{-1}dy^i \tag{1.1.21}$$

and the spaces are described as

$$\begin{aligned} &T_p^*M = \mathbb{R}\left\langle dx^i, dy^i \right\rangle, && T_{\mathbb{C},p}^*M = \mathbb{C}\left\langle dz^i, d\bar{z}^i \right\rangle, \\ &T_p^{*\,1,0}M = \mathbb{C}\left\langle dz^i \right\rangle, && T_p^{*\,0,1}M = \mathbb{C}\left\langle d\bar{z}^i \right\rangle. \end{aligned} \tag{1.1.22}$$

Note that if M, N are complex manifolds, any smooth function $f\colon M \to N$ induces the linear map

$$d_p f\colon T_pM \to T_{f(p)}N \tag{1.1.23}$$

and hence a map between the complex tangent spaces. However, in general there is no induced map between the holomorphic tangent spaces. Actually, it can be shown that f is holomorphic if and only if df restricts to the holomorphic tangent spaces:

$$d_p f\big(T_p^{1,0}M\big) \subset T_{f(p)}^{1,0}N. \tag{1.1.24}$$

For this particular case, let us analyse the differential map in coordinates. Fix two holomorphic coordinate systems: (z^i) centred at $p \in M$ and (w^j) centred at $q = f(p) \in N$. We have two different notions of Jacobian: the real and the complex one. Writing $z^i = x^i + \sqrt{-1}y^i$, $w^j = u^j + \sqrt{-1}v^j$, the $\mathbb{R}$-linear map df can be expressed in local coordinates $\{\, {\partial}/{\partial x^i}, {\partial}/{\partial y^i} \,\}$ and $\{\, {\partial}/{\partial u^j}, {\partial}/{\partial v^j} \,\}$ as

$$\mathrm{Jac}_{\mathbb{R}}(f) = \left(\begin{array}{c|c} \frac{\partial u^j}{\partial x^i} & \frac{\partial u^j}{\partial y^i} \\ \hline \frac{\partial u^j}{\partial x^i} & \frac{\partial v^j}{\partial y^i} \end{array}\right). \tag{1.1.25}$$

On the other hand, the $\mathbb{C}$-linear extension to the complexified tangent spaces can be represented in local coordinates $\{\, {\partial}/{\partial z^i}, {\partial}/{\partial \bar{z}^i} \,\}$ and $\{\, {\partial}/{\partial w^j}, {\partial}/{\partial \bar{w}^j} \,\}$ as

$$\mathrm{Jac}_{\mathbb{C}}(f) = \begin{pmatrix} \mathrm{Jac}(f) & 0 \\ 0 & \overline{\mathrm{Jac}(f)} \end{pmatrix}, \qquad \mathrm{Jac}(f) = \left(\frac{\partial w^j}{\partial z^i}\right). \tag{1.1.26}$$

In particular, for complex manifolds of the same dimension, we have

$$\mathrm{Jac}_{\mathbb{R}}(f) = A^{-1}\,\mathrm{Jac}_{\mathbb{C}}(f)\,A, \qquad A = \begin{pmatrix} \begin{smallmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{smallmatrix} \end{pmatrix}, \tag{1.1.27}$$

so that

$$\det \mathrm{Jac}_{\mathbb{R}}(f) = \det \mathrm{Jac}(f) \cdot \det \overline{\mathrm{Jac}(f)} = |\det \mathrm{Jac}(f)|^2 \geqslant 0. \tag{1.1.28}$$

As a consequence, holomorphic maps are orientation-preserving. Thus, taking the natural orientation on $\mathbb{C}^n$ given by the $2n$-form

$$dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = \left(\frac{\sqrt{-1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n, \tag{1.1.29}$$

we have a natural orientation on every complex manifold M via the holomorphic transition maps $\varphi_\alpha\colon U_\alpha \to \mathbb{C}^n$.

A consequence of the above discussion is the following

Theorem 1.1.6 (Inverse function theorem). *Let* $U, V \subset \mathbb{C}^n$ *be open sets,* $f\colon U \to V$ *a holomorphic map with* $\mathrm{Jac}(f)$ *nonsingular in* $z_0 \in U$*. Then* f *is a biholomorphism in a neighbourhood of* z_0*.*

Proof. Since $\det \mathrm{Jac}_{\mathbb{R}}(f)(z_0) = |\det \mathrm{Jac}(f)(z_0)|^2 > 0$, by the ordinary inverse function theorem we have a smooth inverse map f^{-1} in a neighbourhood of z_0. Differentiating with respect to $\bar{z}^i$ the identity $f^{-1}(f(z)) = z$, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \bar{z}^i}\left(f^{-1}(f(z))\right) \\ &= \frac{\partial f^{-1}}{\partial z^j}\frac{\partial f^j}{\partial \bar{z}^i} + \frac{\partial f^{-1}}{\partial \bar{z}^j}\frac{\partial \overline{f^j}}{\partial \bar{z}^i} \\ &= \frac{\partial f^{-1}}{\partial \bar{z}^j}\overline{\frac{\partial f^j}{\partial z^i}}. \end{aligned}$$

As $\det \mathrm{Jac}(f)$ is invertible, we have $\frac{\partial f^{-1}}{\partial \bar{z}^j} = 0$ for every j, so f^{-1} is holomorphic. □

In the same spirit, we find the

Theorem 1.1.7 (Implicit function theorem). *Let* $U \subset \mathbb{C}^n$ *be an open set and consider a holomorphic function* $f\colon U \to \mathbb{C}^k$ *with*

$$\det\left(\frac{\partial f^i}{\partial z^j}(z_0)\right)_{1\leqslant i,j\leqslant k} \neq 0 \tag{1.1.30}$$

for a certain $z_0 \in U$*. Then there exist* $g^1, \ldots, g^{n-k}$ *holomorphic functions defined in a neighbourhood of* $(z_0^{k+1}, \ldots, z_0^n)$ *such that*

$$f^1(z) = \cdots = f^k(z) = 0 \quad \Longleftrightarrow \quad z^j = g^j(z^{k+1}, \ldots, z^n) \;\; \forall j = 1, \ldots, k. \tag{1.1.31}$$

A particular consequence of the inverse function theorem, peculiar of holomorphic maps, is the following

Proposition 1.1.8. *Let* $U, V \subset \mathbb{C}^n$ *be open sets,* $f\colon U \to V$ *a holomorphic one-to-one map. Then* f *is a biholomorphism.*

Proof. By the inverse function theorem, it suffices to prove that $\mathrm{Jac}(f)$ is nonsingular. Let us prove it by induction on n. Let (z^i) and (w^j) be holomorphic coordinates on U and V respectively. Let k be the rank of $\mathrm{Jac}(f)(z_0)$. We may assume that

$$\det\left(\frac{\partial f^i}{\partial z^j}(z_0)\right)_{1\leqslant i,j\leqslant k} \neq 0,$$

so that setting

$$\begin{aligned} z'^i &= f^i(z) \qquad && \forall i = 1, \ldots, k \\ z'^j &= z^j && \forall j = k+1, \ldots, n \end{aligned}$$

by the inverse function theorem (z'^i) is a holomorphic coordinate system on a neighbourhood of z_0. Further, the locus $A = \{z'^1 = \cdots = z'^k = 0\}$ is in one-to-one correspondence with the locus $B = \{ w^1 = \cdots = w^k = 0 \}$ via $f|_A$ and the Jacobian of $f|_A$ with respect to $z'^{k+1}, \dots, z'^n$ is singular at $z_0'^{k+1}, \dots, z_0'^n$. By induction, either $k = 0$ or $k = n$. Thus, the Jacobian of f vanishes whenever the Jacobian is singular, so that the connected components of the locus $\{ \det \mathrm{Jac}(f) = 0 \}$ are mapped to single points in V. Since f is one-to-one and the zero locus of the holomorphic function $\det \mathrm{Jac}(f)$ is either empty or positive-dimensional, we find that $\det \mathrm{Jac}(f) \neq 0$ for every point of U. □

Remark 1.1.9. Note that the proposition does not hold in the smooth case: the function $x \mapsto x^3$ is smooth and one-to-one over $\mathbb{R}$, but does not possess a smooth inverse.

The implicit function theorem allows us to give two equivalent definition of complex submanifold.

Definition 1.1.10. A *complex submanifold* S of dimension k of a complex manifold M is a subset locally given either by the zero locus of a holomorphic map $f\colon M \to \mathbb{C}^k$ with rank of $\mathrm{Jac}(f)$ equal to k, or as the image of an open set U of $\mathbb{C}^{n-k}$ under a holomorphic map $g\colon U \to M$ with rank of $\mathrm{Jac}(g)$ equal to $n-k$.

Remark 1.1.11. The definition of submanifold can be reformulated as follows: $S \subset M$ is a complex submanifold of dimension k if there exists an atlas $\{ (U_\lambda, \varphi_\lambda) \}$ of M such that $\varphi_\lambda(U_\lambda \cap S) \cong \varphi_\lambda(U_\lambda) \cap \mathbb{C}^k$ is a biholomorphism. Here $\mathbb{C}^k \subset \mathbb{C}^n$ is interpreted as a subspace.

We can generalise the notion of submanifold, allowing singularities.

Definition 1.1.12. An *analytic subvariety* V of a complex manifold M is a subset locally given by the zero locus of a finite number of holomorphic functions. A point $p \in V$ is called *smooth* or *nonsingular* if it is locally given by holomorphic functions $f^1, \dots, f^k$ with rank of $\mathrm{Jac}(f^1, \dots, f^k)$ equal to k. The locus of smooth points of V will be denoted by V_{sm}. A point is called *singular* if it belongs to the set $V_{\mathrm{sing}} = V \setminus V_{\mathrm{sm}}$, called the singular locus.
An analytic subvariety is called *smooth* or *nonsingular* if $V = V_{\mathrm{sm}}$. In particular, a nonsingular analytic subvariety defined by a constant number of equations, say k, is a complex submanifold of dimension k.

Let us summarise the basic properties of analytic subvarieties. Proves can be found in [Griffiths and Harris, 1994].

Proposition 1.1.13. V_{sing} *is contained in an analytic subvariey of* M*, not equal to* V.

Proposition 1.1.14. *An analytic subvariety* V *is irreducible if and only if* V_{sm} *is connected. In this case, we define the dimension of an irreducible analytic subvariety to be the dimension of the complex manifold* V_{sm}.

For the rest of the thesis, we will always consider subvarieties to be analytic, even when not explicitly stated.

1.2 Differential calculus on complex manifolds

1.2.1 Almost complex structures

The complex structure of a complex manifold is essentially encapsulated in the multiplication by $\sqrt{-1}$, which is naturally inherited by the tangent bundle TM from the holomorphic charts. However, the "multiplication by $\sqrt{-1}$ "can be defined for the more general class of smooth manifolds.

Definition 1.2.1. Let M be a smooth manifold. An *almost complex structure* J on M is a smooth global section of the bundle $\operatorname{End}(TM) \to M$, such that $J^2 = -\operatorname{id}$. A manifold equipped with an almost complex structure is sometimes called an almost complex manifold.

Note that, as $J^2 = -\operatorname{id}$, the tangent spaces must have even dimension. Thus, a smooth manifold with an almost complex structure has even dimension too. For a complex manifold M, the holomorphic structure induces a natural almost complex structure: for every $p \in M$, consider a holomorphic chart (U, φ) containing p and define $J_p \colon T_pM \to T_pM$ as

$$J_p(v) = \left(d_p\varphi^{-1} \circ j \circ d_p\varphi\right)(v), \tag{1.2.1}$$

where $j \colon \mathbb{C}^n \to \mathbb{C}^n$ is the multiplication by $\sqrt{-1}$. Then J_p defines an almost complex structure J on M. In terms of the basis $\{\,{}^{\partial}/_{\partial x^i}, {}^{\partial}/_{\partial y^i}\,\}$, the almost complex structure on a complex manifold reads

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}.$$

In this way, the tangent spaces become complex vector space generated over $\mathbb{C}$ by ${}^{\partial}/_{\partial x^i}$, with

$$(\alpha + \sqrt{-1}\beta)v = \alpha v + \beta J v. \tag{1.2.2}$$

However, the natural setting for the automorphism J is the complexified tangent bundle, where the almost complex structure is defined via complex-linearity extension. In terms of the basis $\{\,{}^{\partial}/_{\partial z^i}, {}^{\partial}/_{\partial \bar{z}^i}\,\}$, we have

$$J\left(\frac{\partial}{\partial z^i}\right) = \sqrt{-1}\frac{\partial}{\partial z^i}, \qquad J\left(\frac{\partial}{\partial \bar{z}^i}\right) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}. \tag{1.2.3}$$

Now, for a general almost complex structure, it is clear that J defined on $T_{\mathbb{C},p}M$ has eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ and the subbundles of $T_{\mathbb{C}}M$ consisting of eigenvectors relative to these eigenvalues, denoted by $T^{1,0}M$ and $T^{0,1}M$, are called the *holomorphic* and *antiholomorphic tangent bundles* respectively:

$$\begin{aligned} T^{1,0}M &= \left\{ v - \sqrt{-1}Jv \;\middle|\; v \in TM \right\}, \\ T^{0,1}M &= \left\{ v + \sqrt{-1}Jv \;\middle|\; v \in TM \right\}. \end{aligned} \tag{1.2.4}$$

On a complex manifold, we deduce from the above representation that these bundles coincides with the ones defined in the previous section. Further, with the complex structure on the real tangent spaces, we find that the $\mathbb{R}$-linear map $T_pM \to T^{1,0}_pM$ is actually $\mathbb{C}$-linear:

$$\alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial y^i} = \left(\alpha^i + \sqrt{-1}\beta^i\right) \frac{\partial}{\partial x^i} \longmapsto \left(\alpha^i + \sqrt{-1}\beta^i\right) \frac{\partial}{\partial z^i}. \tag{1.2.5}$$

Example 1.2.1. As an example, every oriented, Riemannian real surface Σ admits an almost complex structure. Indeed, define for $v \in T_p\Sigma$ the vector $J_p v$ as the $\pi/2$ rotation of v in the counter-clock direction, as in Figure 1.1 (the angle is measured by means of the metric). Then J is a globally well-defined almost complex structure. This result is no longer true in higher dimensions: for instance, a famous result due to Borel and Serre [Borel and Serre, 1953] states that the only even-dimensional spheres which admit an almost complex structure are $\mathbb{S}^2$ and $\mathbb{S}^6$. The first can be realised viewing $\mathbb{S}^2$ as the imaginary quaternions of unitary norm, the latter by means of octonions.

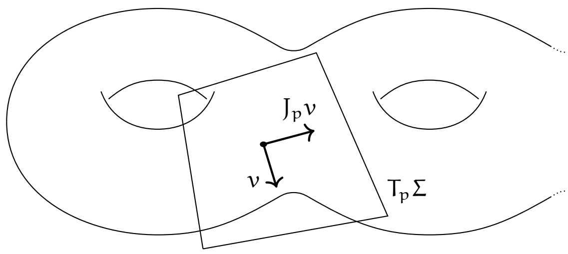


Figure 1.1: Almost complex structures on oriented, Riemannian real surfaces.

An interesting problem is to determine whether an almost complex structure comes from a complex one. Note that the almost complex structure is a purely algebraic object, while the holomorphic structure involves the differential part of the theory. Thus, the difference between an almost complex structure coming from a holomorphic atlas and a generic one must appear from the differential setting. A characterisation is given by this famous result of Newlander and Nirenberg [Newlander and Nirenberg, 1957].

Theorem 1.2.2. *Let* J *be an almost complex structure on a smooth manifold* M*. Then* J *comes from a holomorphic structure on* M *if and only if the distribution* $T^{0,1}M$ *is integrable.*

Note that the integrability condition can be expressed via the Frobenius theorem: the distribution $T^{0,1}M$ is integrable if and only if for all pairs of holomorphic vector fields $X, Y \in \mathfrak{X}^{0,1}(M)$ we have $[X, Y] \in \mathfrak{X}^{0,1}(M)$. An alternative characterisation is given by means of the *Nijenhuis tensor*:

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{1.2.6}$$

Theorem 1.2.3. *Let* J *be an almost complex structure on a smooth manifold* M*. Then* J *comes from a holomorphic structure on* M *if and only if* $N_J \equiv 0$.

We will not spend any more time studying the interplay between real manifolds, holomorphic structures and almost complex structures. We will just assume the manifolds to be complex with the natural almost complex structure, although some constructions can be generalised to generic almost complex manifolds. We refer to [Moroianu, 2007] for further readings.

1.2.2 Differential forms and cohomology

The decomposition of the space of vector fields of a complex manifold can be naturally extended to differential forms. This fact will allows us to introduce the del and del-bar operators and the Dolbeault cohomology groups, which contains information about the holomorphic structure of the manifold. We will principally follow [Griffiths and Harris, 1994] and [Huybrechts, 2005] as references. See [Voisin, 2002] for more details about sheaf cohomology on complex manifolds.

Let M be a complex manifold. The decomposition of the cotangent bundle into holomorphic and antiholomorphic bundles can be extended to the wedge products:

$$\Lambda^k T^*_{\mathbb{C}} M = \bigoplus_{p+q=k} \left(\Lambda^p T^{*\,1,0} M \otimes \Lambda^q T^{*\,0,1} M \right). \tag{1.2.7}$$

Let $\mathcal{A}^k_{\mathbb{C}}$ denote the sheaf of complex-valued differential forms of degree k, *i.e.* the sheaf of sections of the bundle $\Lambda^k T^*_{\mathbb{C}} M \to M$. The above decomposition leads to the corresponding one a the level of differential forms:

$$\mathcal{A}^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M), \tag{1.2.8}$$

where $\mathcal{A}^{p,q}(M)$ is the space of global sections of $\Lambda^p T^{*\,1,0} M \otimes \Lambda^q T^{*\,0,1} M$. Note that $\mathcal{A}^{p,q}$ is a sheaf too. We will say that a *differential form* $\alpha \in \mathcal{A}^{p,q}(M)$ is of *type* (p, q). In local holomorphic coordinates (z^i), an element $\alpha \in \mathcal{A}^{p,q}(M)$ is of the form

$$\alpha = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ}\, dz^I \wedge d\bar{z}^J, \tag{1.2.9}$$

where we used the multiindex notation

$$dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}, \qquad d\bar{z}^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}. \tag{1.2.10}$$

The type decomposition comes with the projection operators

$$\pi^{p,q} \colon \mathcal{A}^\bullet_{\mathbb{C}}(M) \longrightarrow \mathcal{A}^{p,q}(M). \tag{1.2.11}$$

The exterior derivative $d \colon \mathcal{A}^k_{\mathbb{C}}(M) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M)$, defined as the $\mathbb{C}$-linear extension of the standard one, can be decomposed as follows: noting that for a form α of type (p, q), its differential decomposes as

$$d\alpha \in \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M), \tag{1.2.12}$$

we define the *del* and *del-bar operators*

$$\begin{aligned} \partial\colon \mathcal{A}^{p,q}(M) &\to \mathcal{A}^{p+1,q}(M) \\ \bar{\partial}\colon \mathcal{A}^{p,q}(M) &\to \mathcal{A}^{p,q+1}(M) \end{aligned} \tag{1.2.13}$$

by

$$\partial = \pi^{p+1,q} \circ d, \qquad \bar{\partial} = \pi^{p,q+1} \circ d. \tag{1.2.14}$$

As a consequence, we find $d = \partial + \bar{\partial}$. In local holomorphic coordinates (z^i), for a k-differential form $\alpha = \alpha_{IJ}\, dz^I \wedge d\bar{z}^J$ we have

$$\begin{aligned} \partial\alpha &= \frac{\partial \alpha_{IJ}}{\partial z^i}\, dz^i \wedge dz^I \wedge d\bar{z}^J \\ \bar{\partial}\alpha &= \frac{\partial \alpha_{IJ}}{\partial \bar{z}^j}\, d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J \end{aligned} \tag{1.2.15}$$

Note that for a holomorphic map $f\colon M \to N$,

$$f^*\big(\mathcal{A}^{p,q}(N)\big) \subset \mathcal{A}^{p,q}(M) \tag{1.2.16}$$

and $f^* \circ \bar{\partial} = \bar{\partial} \circ f^*$.

Let us see the basic properties of the del and del-bar operators.

Lemma 1.2.4. *The following relations hold:*

$$\partial^2 = \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{1.2.17}$$

Proof. In local holomorphic coordinates,

$$\partial^2\alpha = \frac{\partial^2 \alpha_{IJ}}{\partial z^k \partial z^l}\, dz^k \wedge dz^l \wedge dz^I \wedge d\bar{z}^J,$$

which is zero since the first term is symmetric and the second skew-symmetric in (k, l). Analogously for $\bar{\partial}^2 = 0$. On the other hand,

$$\begin{aligned} \partial\bar{\partial}\alpha &= \frac{\partial^2 \alpha_{IJ}}{\partial z^i \partial \bar{z}^j}\, dz^i \wedge d\bar{z}^j \wedge dz^I \wedge d\bar{z}^J \\ &= -\frac{\partial^2 \alpha_{IJ}}{\partial z^i \partial \bar{z}^j}\, d\bar{z}^j \wedge dz^i \wedge dz^I \wedge d\bar{z}^J \\ &= -\bar{\partial}\partial\alpha. \end{aligned}$$

□

Another operation which comes from the complex structure is the conjugation, an automorphism of $\mathcal{A}^k_{\mathbb{C}}(M)$ sending $\mathcal{A}^{p,q}(M)$ to $\mathcal{A}^{q,p}(M)$. In local holomorphic coordinates, for $\alpha = \alpha_{IJ}\, dz^I \wedge d\bar{z}^J$ we have

$$\bar{\alpha} = \bar{\alpha}_{IJ}\, d\bar{z}^I \wedge dz^J. \tag{1.2.18}$$

We are ready now to introduce the Dolbeault cohomology groups. Recall the definition of the (complexified) de Rham cohomology: let $\mathcal{Z}^k(M, \mathbb{C})$ be the space of closed differential k-forms and $\mathcal{B}^k(M,\mathbb{C})$ be the space of exact differential k-forms. Define the de Rham cohomology groups as

$$H^k_{dR}(M,\mathbb{C}) = \frac{\mathcal{Z}^k(M,\mathbb{C})}{\mathcal{B}^k(M,\mathbb{C})}. \tag{1.2.19}$$

In the same spirit, let $\mathcal{Z}^{p,q}_{\bar\partial}(M)$ be the space of $\bar\partial$-closed differential forms of type (p,q) and $\mathcal{B}^{p,q}_{\bar\partial}(M)$ be the space of $\bar\partial$-exact differential forms of type (p,q). Since $\bar\partial^2 = 0$, $\mathcal{B}^{p,q}_{\bar\partial}(M) \subset \mathcal{Z}^{p,q}_{\bar\partial}(M)$ and we can define the *Dolbeault cohomology groups*

$$H^{p,q}_{\bar\partial}(M) = \frac{\mathcal{Z}^{p,q}_{\bar\partial}(M)}{\mathcal{B}^{p,q}_{\bar\partial}(M)}. \tag{1.2.20}$$

Note that if $f\colon M \to N$ is holomorphic, it induces a group homomorphism

$$f^*\colon H^{p,q}_{\bar\partial}(N) \to H^{p,q}_{\bar\partial}(M). \tag{1.2.21}$$

In the same way, one could define the groups $H^{p,q}_{\partial}(M)$. In general, they are not isomorphic groups, but we will see in the next sections that for compact Kähler manifolds an isomorphism between ∂ and $\bar\partial$-cohomology holds, together with many other symmetries for the Dolbeault cohomology.

Analogously to the real case, we can prove the $\bar\partial$-Poicaré lemma: every $\bar\partial$-closed differential form is locally $\bar\partial$-exact. The result is due to Grothendieck.

Lemma 1.2.5 ($\bar\partial$-Poicaré lemma in one variable)**.** *Let $\Delta \subset \mathbb{C}$ be an open disk and U be a neighbourhood of the closure: $\overline{\Delta} \subset U$. Let $\alpha = f\,d\bar z \in \mathcal{A}^{0,1}(U)$. The function*

$$g(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{f(w)}{w-z} dw \wedge d\bar w \tag{1.2.22}$$

satisfies $\bar\partial g = \alpha$ on Δ.

Proof. Firstly, let us prove that g is well-defined. Fix $z_0 \in \Delta$ in a neighbourhood V of z_0 in compactly contained in Δ. Choose a smooth function $\psi\colon \Delta \to \mathbb{R}$ with compact support in Δ and $\psi|_V \equiv 1$. Set $f_1 = \psi f$, $f_2 = (1-\psi) f$, so that $f = f_1 + f_2$. We define

$$g_i(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{f_i(w)}{w-z} dw \wedge d\bar w.$$

Note that for $z \in V$, g_2 is well-defined, since $f_2|_V \equiv 0$ and the singularity at the denominator is avoided. On the other hand, as f_1 has compact support in Δ, we rewrite g_1 as

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f_1(w)}{w-z} dw \wedge d\bar w \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f_1(z+\zeta)}{\zeta} d\zeta \wedge d\bar\zeta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f_1\big(z+re^{i\theta}\big)}{re^{i\theta}} \big(-2\sqrt{-1}r\big) dr \wedge d\theta \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} f_1\big(z+re^{i\theta}\big) e^{-i\theta}\, dr \wedge d\theta. \end{aligned}$$

The last integral is clearly well-defined, so that $g = g_1 + g_2$ is well-defined too in the open set V. With the same procedure in every point of the disk, we obtain that g is well-defined in the whole Δ. Let us now compute $\bar{\partial} g(z)$ for $z \in V$ by computing $\bar{\partial} g_i$. Note that $1/(w-z)$ is holomorphic in $\Delta \setminus V$, so that

$$\frac{\partial g_2}{\partial \bar{z}}(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta f_2(w) \frac{\partial}{\partial \bar{z}} \left(\frac{1}{w-z} \right) dw \wedge d\bar{w} = 0.$$

On the other hand, using polar coordinates again,

$$\begin{aligned} \frac{\partial g_1}{\partial \bar{z}}(z) &= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar{z}} (z + re^{i\theta}) e^{-i\theta} \, dr \wedge d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial f_1}{\partial \bar{z}}(w) \frac{dw \wedge d\bar{w}}{w-z}. \end{aligned}$$

Introducing a ball of small radius ϵ centred at z and using the fact that $1/(w-z)$ is holomorphic in $\Delta \setminus B_\epsilon(z)$, Stokes's theorem and $\mathrm{supp}(f_1) \subset \Delta$, we find

$$\begin{aligned} \frac{\partial g_1}{\partial \bar{z}}(z) &= \frac{1}{2\pi\sqrt{-1}} \lim_{\epsilon \to 0} \int_{\Delta \setminus B_\epsilon(z)} \frac{\partial f_1}{\partial \bar{z}}(w) \frac{dw \wedge d\bar{w}}{w-z} \\ &= -\frac{1}{2\pi\sqrt{-1}} \lim_{\epsilon \to 0} \int_{\Delta \setminus B_\epsilon(z)} d\left(\frac{f_1(w)}{w-z} dw \right) \\ &= \frac{1}{2\pi\sqrt{-1}} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z)} \frac{f_1(w)}{w-z} dw \\ &= \lim_{\epsilon \to 0} \int_0^{2\pi} f_1(z + \epsilon e^{i\omega}) d\omega = f_1(z). \end{aligned}$$

Finally, $\bar{\partial} g(z) = \bar{\partial} g_1(z) + \bar{\partial} g_2(z) = f_1(z) = f(z)$. Since this holds in V, with the same procedure in every point of the disk, we have the thesis. □

Theorem 1.2.6 ($\bar{\partial}$-Poicaré lemma). *Let $\Delta = \{ z \in \mathbb{C}^n \mid |z^i - z_0^i| < r_i \}$ be a polydisc in $\mathbb{C}^n$. Then*

$$H^{p,q}_{\bar{\partial}}(\Delta) = 0 \qquad \forall p, \forall q \geqslant 1. \tag{1.2.23}$$

Proof. Firstly, we can restrict ourselves to the case $H^{0,q}_{\bar{\partial}}(\Delta)$. In fact, observe that for $\alpha = \alpha_{IJ} dz^I \wedge d\bar{z}^J \in \mathcal{A}^{p,q}(M)$, setting $\alpha_I = \alpha_{IJ} d\bar{z}^J \in \mathcal{A}^{0,q}(M)$ we find

$$\bar{\partial}\alpha = 0 \qquad \Longleftrightarrow \qquad \bar{\partial}\alpha_I = 0 \ \forall I$$

and

$$\alpha = \bar{\partial}\big(\beta_{IK} dz^I \wedge d\bar{z}^K\big) \qquad \Longleftrightarrow \qquad \alpha_I = \bar{\partial}\big(\beta_{IK} d\bar{z}^K\big) \ \forall I.$$

So we can assume $\alpha = f_I d\bar{z}^I \in \mathcal{Z}^{p,q}_{\bar{\partial}}(\Delta)$. Let k be the greatest integer such that $d\bar{z}^i$ does not appear in α for any $i > k$. Then we can write

$$\alpha = \alpha_1 \wedge d\bar{z}^k + \alpha_2,$$

where α_2 does not contain $d\bar{z}^k$. Let us set

$$\bar{\partial}_i = \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i \wedge .$$

From $\bar{\partial}\alpha = 0$ and the previous assumptions, we obtain

$$\bar{\partial}_i \alpha_1 = \bar{\partial}_i \alpha_2 = 0 \qquad \forall i > k.$$

Thus, f_I are holomorphic functions in $z^{k+1}, \ldots, z^n$. We now set for every $z \in \Delta$,

$$g_I(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\{|w|<r_k\}} \frac{f_I(z^1, \ldots, z^{k-1}, w, z^{k+1}, \ldots, z^n)}{w - z^k} dw \wedge d\bar{w}.$$

The function is smooth in every variable, holomorphic in $z^{k+1}, \ldots, z^n$ and, by the previous lemma, $\frac{\partial g_I}{\partial \bar{z}^k} = f_I$. By setting

$$\gamma = g_{i_1 \cdots i_q} d\bar{z}^{i_1} \wedge \cdots \wedge \widehat{d\bar{z}^k} \wedge \cdots \wedge d\bar{z}^{i_q},$$

we find $\bar{\partial}_i \gamma = 0$ for $i > k$ and $\bar{\partial}_k \gamma = (-1)^{q-1} \alpha_1 \wedge d\bar{z}^k$. Thus, $\alpha + (-1)^q \bar{\partial}\gamma$ is a $\bar{\partial}$-closed $(0, q)$-form without $d\bar{z}^i$ for all $i \geqslant k$. We conclude by induction on k. □

A well-known results asserts that the de Rham cohomology on a real manifold can be realised as the sheaf cohomology of the real constant sheaf:

$$H^k(M, \mathbb{R}) \cong H^k_{dR}(M, \mathbb{R}). \tag{1.2.24}$$

This is the *de Rham isomorphism*. Analogously, the Dolbeault cohomology of a complex manifold can be realised as the sheaf cohomology of a holomorphic sheaf. We will say that α of type (p,0) is a *holomorphic* if $\bar{\partial}\alpha = 0$. In local coordinates, a holomorphic form can be written as $\alpha = \alpha_I \, dz^I$ with α_I holomorphic functions. The holomorphic p-forms define a sheaf Ω^p, which is nothing but $\mathcal{Z}^{p,0}_{\bar{\partial}}$, the sheaf of $\bar{\partial}$-closed forms of type (p,0).

Lemma 1.2.7. *For a complex manifold* M,

$$H^k(M, \mathcal{A}^{p,q}) = 0 \tag{1.2.25}$$

for all $k > 0$ *and for all* p, q.

Proof. The thesis readily follows by showing that the sheaf $\mathcal{A}^{p,q}$ is fine, but let us prove it directly by means of Čech cohomology. The proof is based on the existence of a partition of unity: given a locally finite cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of M, consider a partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\mathfrak{U}$. Take Čech cocycle $\alpha \in \check{Z}^k(\mathfrak{U}, \mathcal{A}^{p,q})$ and define $\beta \in \check{C}^{k-1}(\mathfrak{U}, \mathcal{A}^{p,q})$ as

$$\beta_{i_0 \cdots i_{k-1}} = \sum_{j \in I} \rho_j \, \alpha_{j i_0 \cdots i_{k-1}}.$$

Note that the definition is well-posed, since $\mathcal{A}^{p,q}$ is a sheaf of C^∞-modules and the sections $\rho_j \, \alpha_{j i_0 \cdots i_{k-1}}$ are extended to $U_{i_0} \cap \cdots \cap U_{i_{k-1}}$ by zero. It can be shown by direct calculations that $\check{\delta}\beta = \alpha$. In the particular case of $k = 1$, we have $\beta_U = \sum_{V \in \mathfrak{U}} \rho_V \alpha_{VU}$ and

$$\begin{aligned}(\check{\delta}\beta)_{UV} &= \beta_V - \beta_U \\ &= \sum_{W \in \mathfrak{U}} \rho_W(\alpha_{WV} - \alpha_{WU}) \\ &= \sum_{W \in \mathfrak{U}} \rho_W \alpha_{UV} = \alpha_{UV}.\end{aligned}$$

In the last two steps we used the cocycle condition $\alpha_{UV} - \alpha_{WV} + \alpha_{WU} = 0$ and the fact that $\sum_W \rho_W = 1$. □

Theorem 1.2.8 (Dolbeault theorem). *For a complex manifold* M,

$$H^q(M, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(M). \tag{1.2.26}$$

Proof. By the $\bar{\partial}$-Poincaré lemma, the following sequences are exact.

$$\begin{gathered} 0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,1}_{\bar{\partial}} \to 0 \\ \vdots \\ 0 \to \mathcal{Z}^{p,q}_{\bar{\partial}} \to \mathcal{A}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,q+1}_{\bar{\partial}} \to 0 \end{gathered}$$

By the previous lemma, that is $H^k(M, \mathcal{A}^{p,q}) = 0$, the associated long exact sequences in cohomology simplify in the isomorphisms

$$\begin{aligned} H^{k+1}(M, \mathcal{Z}^{p,q}_{\bar{\partial}}) &\cong H^k(M, \mathcal{Z}^{p,q+1}_{\bar{\partial}}) \qquad \forall k > 0, \\ H^1(M, \mathcal{Z}^{p,q}_{\bar{\partial}}) &\cong \frac{H^0(M, \mathcal{Z}^{p,q+1}_{\bar{\partial}})}{\bar{\partial}^* H^0(M, \mathcal{A}^{p,q})}. \end{aligned}$$

for all p, q. Thus, by induction,

$$H^q(M, \Omega^p) \cong H^1(M, \mathcal{Z}^{p,q-1}_{\bar{\partial}}) \cong \frac{H^0(M, \mathcal{Z}^{p,q}_{\bar{\partial}})}{\bar{\partial}^* H^0(M, \mathcal{A}^{p,q-1})} = H^{p,q}_{\bar{\partial}}(M)$$

and the statement is proved. □

Let us finally discuss integration of forms and Poincaré duality.

Definition 1.2.9. Let V be a (possibly singular) k-subvariety of a complex manifold M. We define the integral of a $2k$-form with compact support in M to be the integral over the smooth locus of V.

Theorem 1.2.10 (Stokes' theorem). *For a complex manifold* M, *a* k*-subvariety* V *and a* $(2k-1)$*-form* α *with compact support in* M

$$\int_V d\alpha = 0. \tag{1.2.27}$$

A proof of the above statement can be found in [Griffiths and Harris, 1994]. Stokes' theorem illustrates the fact that singularities of analytic subvarieties occur only in real codimension 2, so that integration over subvarieties behaves like integration over submanifolds. More importantly, it will allow us to associate to any analytic subvariety a homology class in $H_\bullet(M, \mathbb{R})$.

We will firstly state the Poincaré duality for real manifolds and then we will apply it to the complex case, defining in particular the Poincaré dual of an irreducible analytic subvariety.

Theorem 1.2.11 (Poincaré duality)**.** *Let M be a compact, orientable manifold of (real) dimension m. Then the kth homology group of M is isomorphic to the $(m-k)$th cohomology group of M for all integers k:*

$$H_k(M, \mathbb{R}) \xrightarrow{\text{PD}} H^{m-k}(M, \mathbb{R}) \cong H^{m-k}_{dR}(M, \mathbb{R}). \tag{1.2.28}$$

For $\gamma \in H_k(M, \mathbb{R})$, the de Rham cohomology class $\text{PD}(\gamma)$ is characterised by the relation

$$\int_\gamma a = \int_M \text{PD}(\gamma) \smile a \qquad \forall a \in H^k_{dR}(M, \mathbb{R}). \tag{1.2.29}$$

Here $\smile$ is the cup product, induced by the wedge product in cohomology as

$$[\alpha] \smile [\beta] = [\alpha \wedge \beta]. \tag{1.2.30}$$

Let us consider now a complex, compact manifold M of dimension n. Any irreducible analytic k-subvariety V, thanks to Stokes' theorem, induces a well-defined functional on $H^{2k}_{dR}(M, \mathbb{R})$ as

$$\alpha \longmapsto \int_V \alpha. \tag{1.2.31}$$

Thus, we have that V determines an element $[V] \in H_{2k}(M, \mathbb{R})$, called the *fundamental class* of V. By Poincaré duality, the linear functional determines also a cohomology class $\text{PD}[V] \in H^{2n-2k}_{dR}(M, \mathbb{R})$, called the *Poincaré dual* of V. The Poincaré duality theorem asserts that this cohomology class is determined by the property that for any $(2n-2k)$ cohomology class a,

$$\int_V a = \int_M \text{PD}[V] \smile a. \tag{1.2.32}$$

This fact will be useful in the following. A more complete discussion about Poicaré duality can be found in [Griffiths and Harris, 1994], [Hatcher, 2002] and [Bott and Tu, 1982].

1.3 Complex vector bundles

Let us recall the definition of vector bundle. Intuitively, a vector bundle over a smooth manifold M may be regarded as a family of vector spaces parametrized by points of the manifold, satisfying a local triviality condition.

Definition 1.3.1. Let M be a smooth manifold. A *complex vector bundle of rank* r over M is a smooth submersion $\pi\colon E \to M$ of smooth manifolds satisfying the following properties.

- For each $p \in M$, the fibre $E_p = \pi^{-1}(p)$ has the structure of a complex vector space of dimension r.
- For each $p \in M$, there exists a neighbourhood U of p and a diffeomorphism $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that the following diagram commute.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times \mathbb{C}^r \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

  The restrictions of ϕ over the fibres are required to be vector space isomorphisms between $\{p\} \times \mathbb{C}^r$ and E_p, for all $p \in U$.

The manifold E is called the *total space*, while M is called the *base space*. A rank 1 vector bundle is simply called a *line bundle*. A map satisfying the above properties is called a *local trivialization over* U. From the definition it follows that there is an open covering $\{U_\alpha\}_{\alpha \in I}$ of M such that for every $\alpha \in I$ there is a local trivialization ϕ_α defined over U_α. In turn, any such family of local trivializations determines a family of smooth maps, called *transition functions*: setting $U_{\alpha\beta} = U_\alpha \cap U_\beta$, these are

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{C}), \tag{1.3.1}$$

uniquely determined by

$$\left(\phi_\alpha \circ \phi_\beta^{-1}\right)(p, v) = (p, g_{\alpha\beta}(p)v) \tag{1.3.2}$$

on $U_{\alpha\beta} \times \mathbb{C}^r$. A simple computation shows that

$$g_{\alpha\beta}^{-1} = g_{\beta\alpha}, \tag{1.3.3}$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{id}, \tag{1.3.4}$$

whenever the compositions make sense. Equation 1.3.4 is called the *cocycle condition*. We will call a family of smooth maps $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{C})$ satisfying these relations a *family of* $\mathrm{GL}(r, \mathbb{C})$*-cocycles* over M. It is a well-known fact that any family of $\mathrm{GL}(r, \mathbb{C})$-cocycles determines a complex vector bundle over M up to isomorphism, see for example [Moroianu, 2007].

In the following, we will denote for an open set $U \subset M$ the preimage $\pi^{-1}(U)$ as $E|_U$, called the restriction of E at U. It is clear that the restriction is a vector bundle over U. A *local section* is a smooth map $\sigma\colon U \to E|_U$ such that $\pi \circ \sigma = \mathrm{id}_U$. The local sections form a locally free sheaf of C^∞-modules (here C^∞ is the sheaf of complex-valued smooth functions):

$$\Gamma(U, E) = \left\{\sigma\colon U \to E|_U \text{ smooth} \;\middle|\; \pi \circ \sigma = \mathrm{id}_U\right\}. \tag{1.3.5}$$

The sheaf will be sometimes denoted by $H^0(\cdot, E)$, in accordance with the derived functor approach to the cohomology with values in E.

A morphism of complex vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ is a smooth map $E \to E'$ such that the following diagram commute.

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & M & \end{array}$$

As a general rule, every operation between vector spaces can be defined on vector bundles: for E and F vector bundles over M, we can make sense for the direct sum bundle $E \oplus F$, the tensor product $E \otimes F$, $\mathrm{Hom}(E, F)$, the dual E^* or the exterior powers $\Lambda^p E$. Another example is the determinant bundle: for a rank r vector bundle, we set $\det(E) = \Lambda^r E$. These vector bundles can be explicitly constructed via transition functions: if $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are the transition functions over $U_{\alpha\beta}$ for E and F respectively, then the maps

$$g_{\alpha\beta} \oplus h_{\alpha\beta},\ g_{\alpha\beta} \otimes h_{\alpha\beta},\ {}^t\big(g_{\alpha\beta}^{-1}\big),\ \det\big(g_{\alpha\beta}\big) \tag{1.3.6}$$

will be the transition function of $E \oplus F$, $E \otimes F$, E^* and $\det(E)$ respectively. Another operation with vector bundles is the pull-back: consider a smooth map $f\colon M \to N$ and a complex vector bundle $\pi\colon E \to N$. We define the *pull-back bundle* $\pi'\colon f^*E \to M$ as follows:

$$f^*E = \{\, (p, v) \in M \times E \mid f(p) = \pi(v)\,\} \tag{1.3.7}$$

and π' as the projection onto the first component. It can be shown that this construction defines a complex vector bundle on M. From a section $\sigma \in \Gamma(U, E)$, we define the pull-back section $f^*\sigma \in \Gamma(f^{-1}(U), f^*E)$

$$\big(f^*\sigma\big)(p) = \big(p, \sigma(p)\big). \tag{1.3.8}$$

A *subbundle* of a complex vector bundle $\pi\colon E \to M$ is a submanifold F of E such that $\pi|_F\colon F \to M$ is a complex vector bundle and the fibres $F_p = \pi|_F^{-1}(p)$ are vector subspaces of E_p (here the map is assumed to be of constant rank, so that every F_p are vector spaces of the same dimension). The condition is equivalent to the existence of local trivializations $\phi\colon E|_U \to U \times \mathbb{C}^r$ such that

$$\phi|_{F|_U}\colon F|_U \to U \times \mathbb{C}^s \subset U \times \mathbb{C}^r \tag{1.3.9}$$

are local trivializations for F. In this case, the transition functions for E are of the form

$$g_{\alpha\beta} = \left(\begin{array}{c|c} h_{\alpha\beta} & * \\ \hline 0 & k_{\alpha\beta} \end{array}\right), \tag{1.3.10}$$

where $h_{\alpha\beta}$ are the transition functions for F. In this notation, we can define the *quotient bundle* E/F via the transition functions $k_{\alpha\beta}$.

If $\pi\colon E \to M$ is a complex vector bundle of rank r, then every local trivialization $\phi\colon E|_U \to U \times \mathbb{C}^r$ induces r local sections of E: let $e_1, \dots, e_r$ be the standard basis of $\mathbb{C}^r$, and set

$$\begin{aligned} \sigma_a\colon U &\longrightarrow E \\ p &\longmapsto \phi^{-1}(p, e_a). \end{aligned} \tag{1.3.11}$$

Then $\sigma_1, \dots, \sigma_r$ are sections over U, and for any other local section $\sigma \in \Gamma(U, E)$ there are smooth complex-valued functions $f^1, \dots, f^r$ defined on U such that

$$\sigma = f^a \sigma_a. \tag{1.3.12}$$

This is nothing but the definition of locally free sheaf of C^∞-modules of rank r. Any set of local sections with this property is called a *local frame* for E. In particular, if $\sigma = (\sigma_a)$ and $\tau = (\tau_a)$ are local frames, on the overlap we have

$$\tau_a = \sum_{b=1}^{r} g_{ab} \sigma_b. \tag{1.3.13}$$

If σ and τ are local frames defined in terms of a trivialization $\mathfrak{U}$, then the functions g_{ab} are nothing but the transition functions for the trivialization $\mathfrak{U}$.

We can analogously give the definition of real vector bundle over a smooth manifold. Just for complex manifolds, we can also give the definition of *holomorphic vector bundle*, where the total space is a complex manifold and every map is considered to be holomorphic. As in the complex case, a holomorphic vector bundle can be reconstructed from holomorphic transitions functions. In particular, a local section is holomorphic if the map $\sigma\colon U \to E|_U$ is holomorphic. A local frame (σ_a) is holomorphic if each σ_a is. Further, in terms of a holomorphic frame (σ_a), a smooth local section σ is holomorphic if and only if

$$\sigma = f^a \sigma_a \tag{1.3.14}$$

for holomorphic functions f^a.

One of the particular features of a holomorphic vector bundle over a complex manifold is the existence of a natural derivation, namely the $\bar\partial$-operator on E-valued differential forms. For $E \to M$ holomorphic, take a holomorphic local frame $\sigma_1, \dots, \sigma_r$ and write $\zeta \in \mathcal{A}^{p,q}(U, E)$ as

$$\zeta = \sum_a \alpha_a \otimes \sigma_a, \qquad \alpha_a \in \mathcal{A}^{p,q}(U) \tag{1.3.15}$$

Then set

$$\bar\partial\zeta = \sum_a \bar\partial\alpha_a \otimes \sigma_a. \tag{1.3.16}$$

Let us show that the definition does not depend on the choice of the local holomorphic frame. If $\tau_1, \dots, \tau_r$ is another frame, write $\sigma_a = \sum_b g_{ab}\tau_b$ with g_{ab} holomorphic functions. Then $\zeta = \sum_{a,b}(g_{ab}\alpha_a) \otimes \tau_b$ and

$$\bar\partial\zeta = \sum_{a,b} \bar\partial(g_{ab}\alpha_a) \otimes \tau_b = \sum_{a,b} \left(g_{ab}\bar\partial\alpha_a\right) \otimes \tau_b = \sum_a \bar\partial\alpha_a \otimes \tau_a. \tag{1.3.17}$$

Thus, we have a well-defined map $\bar{\partial}\colon \mathcal{A}^{p,q}(M,E) \to \mathcal{A}^{p,q+1}(M,E)$. The del-bar operator on E-valued forms actually encodes the holomorphic structure of E.

Example 1.3.1. Let M be a complex manifold.

- The complexified tangent bundle is an example of complex vector bundle (this is also true for a general manifold, not necessarily complex). The real tangent bundle is a complex bundle thanks to the natural almost complex structure, while the holomorphic tangent and cotangent bundles are examples of holomorphic vector bundles.
- The exterior powers $\Lambda^p T^{*1,0}M$ are holomorphic. In particular, if M has complex dimension n, then $K_M = \Lambda^n T^{*1,0}M$ is called the *canonical line bundle* of M. Its dual is sometimes called the *anticanonical line bundle*.
- Consider a complex submanifold S of dimension k. We have the subbundle $T^{1,0}S \subset T^{1,0}M|_S$, so that we can define the quotient bundle over S. It is called the *(holomorphic) normal bundle* $N_{S/M}$ of S in M.

Example 1.3.2. Another classical example of holomorphic line bundle is the *tautological line bundle* over the complex projective space $\mathbb{P}^n$: consider the subbundle of the trivial one, defined as

$$\mathcal{O}(-1) = \left\{ (\ell, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \;\middle|\; z \in \ell \right\}, \tag{1.3.18}$$

where ℓ is a line through the origin in $\mathbb{C}^{n+1}$. We can find local trivializations of $\mathcal{O}(-1)$ over each U_i, where U_i is defined as in Example 1.1.1, as follows:

$$\phi_i\big([Z], z\big) = \big([Z], \lambda\big) \in U_i \times \mathbb{C}, \tag{1.3.19}$$

where λ is uniquely determined by $z = \lambda \frac{Z}{Z^i}$. The inverse map is given by

$$\phi_i^{-1}\big([Z], \lambda\big) = \left([Z], \lambda \frac{Z}{Z^i}\right). \tag{1.3.20}$$

We can compute the transition functions on the overlaps by means of

$$\big(\phi_i \circ \phi_j^{-1}\big)\big([Z], \lambda\big) = \left([Z], \frac{Z^i}{Z^j}\lambda\right), \tag{1.3.21}$$

so that

$$g_{ij}\big([Z]\big) = \frac{Z^i}{Z^j}. \tag{1.3.22}$$

The dual line bundle $\mathcal{O}(1)$ is called the *hyperplane line bundle* and by taking tensor powers, we find the line bundles $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, whose transition functions are

$$g_{ij}^{(d)}\big([Z]\big) = \left(\frac{Z^j}{Z^i}\right)^d. \tag{1.3.23}$$

In local holomorphic coordinates over U_i, they simply reads $g_{ij}^{(d)}(z) = \big(z^j\big)^d$, which are clearly holomorphic. It can be shown that the canonical line bundle of $\mathbb{P}^n$ is isomorphic to the $(n+1)$th power of the tautological line bundle:

$$K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1). \tag{1.3.24}$$

A useful construction for holomorphic vector bundles is the analogue of the projectivisation of $\mathbb{C}^{n+1}$ to obtain the complex projective space. The result is a holomorphic bundle, with fibres given by complex projective spaces (the definition of holomorphic bundle is analogue to that of holomorphic vector bundle, where the fibres are given by a complex manifold and all the involved maps are holomorphic).

Proposition 1.3.2. *Let $\pi\colon E \to M$ be a holomorphic vector bundle of rank r. Define the manifold $\mathbb{P}(E)$ as the quotient of E minus the zero section by the natural $\mathbb{C}^*$-action. It has the structure of a complex manifold and that of a holomorphic bundle over M, with fibres given by $\mathbb{P}^{r-1}$. It is called the* projective bundle *associated to E.*

Proof. The complex structure comes from the fact that the $\mathbb{C}^*$-action is proper and free (see [Huybrechts, 2005] for the construction of complex manifolds via holomorphic quotients). On the other hand, the holomorphic map $\hat{\pi}\colon \mathbb{P}(E) \to M$ can be obtained by π passing to the quotient. Then $\hat{\pi}$ realises $\mathbb{P}(E)$ as a holomorphic bundle over M: if $\{ U_\alpha \}$ is a trivialization of E, then we have the biholomorphisms

$$\hat{\pi}^{-1}(U_\alpha) \xrightarrow[\cong]{\hat{\phi}_\alpha} U_\alpha \times \mathbb{P}^{r-1}.$$

The compositions $\hat{\phi}_\alpha \circ \hat{\phi}_\beta^{-1}$ determines the holomorphic maps $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to PGL(r, \mathbb{C})$ on the overlaps $U_{\alpha\beta}$ by

$$(\hat{\phi}_\alpha \circ \hat{\phi}_\beta^{-1})(p, [Z]) = \big(p, \hat{g}_{\alpha\beta}(p)([Z])\big).$$

Here $\hat{g}_{\alpha\beta}(p)$ is the projection of $g_{\alpha\beta}(p)$ onto the projective linear group. □

1.3.1 Connections and Hermitian vector bundles

As for the particular case of the cotangent bundle, for a complex vector bundle E over a manifold M we can consider E-*valued* k-*forms*: they are local sections of the complex vector bundle $\Lambda^k T^*_{\mathbb{C}} M \otimes E$. We will denote the associated spaces of sections over U as $\mathcal{A}^k_{\mathbb{C}}(U, E)$. If the base space is a complex manifold, we can define the E-*valued forms of type* (p, q) as local sections of the bundle $\Lambda^p T^{*\,1,0} M \otimes \Lambda^q T^{*\,0,1} M \otimes E$ and denote the space of sections as $\mathcal{A}^{p,q}(U, E)$. We are ready now to discuss connections on complex vector bundles, following the exposition of [Griffiths, 1984].

Definition 1.3.3. A connection on a complex vector bundle $E \to M$ is a map

$$\nabla\colon \Gamma(M, E) \to \mathcal{A}^1_{\mathbb{C}}(M, E) \tag{1.3.25}$$

satisfying the Leibniz rule:

$$\nabla(f \cdot \sigma) = df \otimes \sigma + f \cdot \nabla\sigma \tag{1.3.26}$$

for all $f \in C^\infty(M)$, $\sigma \in \Gamma(M, E)$.

Let $\sigma_1, \dots, \sigma_r$ be a frame for E over U. Given a connection ∇, we can decompose $\nabla\sigma_a$ in components:

$$\nabla\sigma_a = \theta_a^b \otimes \sigma_b. \tag{1.3.27}$$

and we call the 1-form valued matrix θ the *connection matrix* of ∇ relative to the frame $\sigma_1, \dots, \sigma_r$. Note that for a general local section $\sigma = f^a\sigma_a$ over U, applying Leibniz rule we obtain

$$\nabla\sigma = \left(df^b + f^a \cdot \theta_a^b\right) \otimes \sigma_b. \tag{1.3.28}$$

A connection ∇ on E can be extended to a map $\nabla \colon \mathcal{A}^k_{\mathbb{C}}(M, E) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M, E)$ imposing the graded Leibniz rule: for an element $\zeta = \alpha \otimes \sigma$, where $\alpha \in \mathcal{A}^k_{\mathbb{C}}(M)$ and $\sigma \in \Gamma(M, E)$, the following relation holds.

$$\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^k \alpha \wedge \nabla\sigma. \tag{1.3.29}$$

In particular, we can define the *curvature of* ∇ as the map $\nabla^2 \colon \Gamma(M, E) \to \mathcal{A}^2_{\mathbb{C}}(M, E)$. Note that ∇^2 is C^∞-linear:

$$\begin{aligned} \nabla^2(f \cdot \sigma) &= \nabla(df \otimes \sigma + f \cdot \nabla\sigma) \\ &= d^2 f \otimes \sigma - df \wedge \nabla\sigma + df \cdot \nabla\sigma + f \cdot \nabla^2\sigma \\ &= f \cdot \nabla^2\sigma. \end{aligned}$$

Thus, ∇^2 is induced by a global section Ω of the bundle $\Lambda^2 T^*_{\mathbb{C}} M \otimes E^* \otimes E$. If $\sigma_1, \dots, \sigma_r$ is a local frame for E, we will have

$$\nabla^2\sigma_a = \Omega_a^b \otimes \sigma_b. \tag{1.3.30}$$

The 2-forms valued matrix Ω_a^b is called the *curvature matrix* of ∇ relative to the frame $\sigma_1, \dots, \sigma_r$. It can be expressed in terms of the connection matrix as follows:

$$\begin{aligned} \nabla^2\sigma_a = \nabla(\theta_a^b \otimes \sigma_b) &= \left(d\theta_a^c - \theta_a^b \wedge \theta_b^c\right) \otimes \sigma_c \\ &= \left(d\theta_a^c + \theta_b^c \wedge \theta_a^b\right) \otimes \sigma_c. \end{aligned}$$

In matrix notation, $\Omega = d\theta + \theta \wedge \theta$. This is the so-called *Cartan equation*. Exterior differentiating the equation gives the *Bianchi identity*:

$$d\Omega + \theta \wedge \Omega - \Omega \wedge \theta = 0. \tag{1.3.31}$$

Remark 1.3.4. Note that for a local frame $\sigma = (\sigma_a)$, we have the form-valued $r \times r$ matrices θ_σ and Ω_σ. If $\tau = (\tau_a)$ is another local frame and $\tau_a = \sum_b g_{ab}\sigma_b$, then on the overlap

$$\nabla\tau = (dg + g \cdot \theta) \otimes \sigma, \tag{1.3.32}$$

so that

$$\begin{aligned} \theta_\tau &= dg \cdot g^{-1} + g \cdot \theta_\sigma \cdot g^{-1}, \\ \Omega_\tau &= g \cdot \Omega_\sigma \cdot g^{-1}. \end{aligned} \tag{1.3.33}$$

Here we have used matrix notation. In particular, if E is a line bundle, then for every point p in the trivialization $g(p) \in GL(1, \mathbb{C})$ which is abelian, so that θ and Ω are well-defined forms on M. Further, the Bianchi identity shows that the curvature form is actually closed, so that it defines a cohomology class on M. We will see the interpretation of this characteristic class in section 1.3.2.

Let us define now the concept of Hermitian metric on a complex vector bundle: intuitively, it is a Hermitian inner product on each fibre, smoothly varying on the base space.

Definition 1.3.5. Let $E \to M$ be a complex vector bundle. A *Hermitian metric* is a smooth section h of the bundle $E^* \otimes E^*$ such that for every point $p \in M$, h_p is a Hermitian inner product on E_p, *i.e.* $h_p \colon E_p \times E_p \to \mathbb{C}$ satisfies the following properties.

1) $h_p(\lambda_1\sigma_1 + \lambda_2\sigma_2, \tau) = \lambda_1 h_p(\sigma_1, \tau) + \lambda_2 h_p(\sigma_2, \tau)$ for all $\lambda_i \in \mathbb{C}$, $\sigma_i, \tau \in E_p$.

2) $h_p(\sigma, \tau) = \overline{h_p(\tau, \sigma)}$ for all $\sigma, \tau \in E_p$.

3) $h_p(\sigma, \sigma) \geqslant 0$ for all $\sigma \in E_p$ and equality holds if and only if $\sigma = 0$.

A complex vector bundle equipped with a Hermitian metric is called a *Hermitian vector bundle*.

For a frame $\sigma_1, \dots, \sigma_r$ we define the smooth functions

$$h_{ab}(p) = h_p\big(\sigma_a(p), \sigma_b(p)\big). \tag{1.3.34}$$

A frame $\sigma_1, \dots, \sigma_r$ on U is called unitary if $\sigma_1(p), \dots, \sigma_r(p)$ is an orthonormal frame in E_p for every point $p \in U$. Locally, unitary frames always exists, as we can apply the Gram-Schmidt process to a generic frame of E.

In general, there is no natural connection on a complex vector bundle E. However, as in the case of the Levi-Civita connection for a Riemannian manifold, if M is complex and E is a holomorphic vector bundle with a Hermitian metric we can require compatibility conditions that determine a canonical choice of connection: the Chern connection.

Definition 1.3.6. Let $E \to M$ be a holomorphic vector bundle with connection ∇. From the decomposition $T^*M = {T^*}^{1,0}M \oplus {T^*}^{0,1}M$, we can write $\nabla = \nabla^{1,0} + \nabla^{0,1}$, with

$$\nabla^{1,0} \colon \Gamma(M, E) \to \mathcal{A}^{1,0}(M, E), \qquad \nabla^{0,1} \colon \Gamma(M, E) \to \mathcal{A}^{0,1}(M, E). \tag{1.3.35}$$

We say that ∇ is *compatible with the holomorphic structure* if $\nabla^{0,1} = \bar{\partial}$.

Definition 1.3.7. Let $E \to M$ be a Hermitian vector bundle with metric h and a connection ∇. We say that ∇ is *compatible with the metric* if

$$d\big(h(\sigma, \tau)\big) = h(\nabla\sigma, \tau) + h(\sigma, \nabla\tau). \tag{1.3.36}$$

Theorem 1.3.8. *Let $E \to M$ be a holomorphic Hermitian vector bundle with metric h. Then there exists a unique connection ∇ on E compatible with both the holomorphic structure and the metric. It is called the* Chern connection *associated to the metric h.*

Proof. Let $\sigma_1, \dots, \sigma_r$ be a local holomorphic frame, $h_{ab} = h(\sigma_a, \sigma_b)$. Suppose such a connection exists. Firstly, let us prove that the holomorphic compatibility implies that the connection matrix θ_a^b is composed by holomorphic forms. Note that $\nabla\sigma_a = \nabla^{1,0}\sigma_a$, since the frame is holomorphic. On the other hand, consider $X \in \mathfrak{X}^{0,1}(M)$ an antiholomorphic vector field. Then

$$\theta_a^b(X) \otimes \sigma_b = (\nabla\sigma_a)(X) = (\nabla^{1,0}\sigma_a)(X) = 0,$$

as X is an antiholomorphic vector field, while $\nabla^{1,0}\sigma_a \in \mathcal{A}^{1,0}(U, E)$. Thus, θ_a^b is composed by holomorphic forms. Now, from the metric compatibility condition,

$$\begin{aligned} dh_{ab} &= h(\nabla e_a, e_b) + h(e_a, \nabla e_b) \\ &= \underbrace{\theta_a^c\, h_{cb}}_{\in \mathcal{A}^{1,0}} + \underbrace{\overline{\theta}_b^c\, h_{ac}}_{\in \mathcal{A}^{0,1}}. \end{aligned}$$

On the other hand, $dh_{ab} = \partial h_{ab} + \bar\partial h_{ab}$, so that comparing types we obtain the equations

$$\partial h = {}^t\theta\, h, \qquad \bar\partial h = h\, \overline{\theta}.$$

Using the fact that h is a Hermitian matrix, we find that the two equations are equivalent. The unique solution is ${}^t\theta = \partial h \cdot h^{-1}$. Since the conditions determine the connection, we have the uniqueness. For the existence, it is easy to check that the local definition ${}^t\theta = \partial h \cdot h^{-1}$ defines a matrix of global 1-forms satisfying the above requirements. □

We can say a little more about the curvature associated to Chern connections. Let us write $\nabla = \nabla^{1,0} + \bar\partial$. Then from $\bar\partial^2 = 0$, we find

$$\nabla^2 = \left(\nabla^{1,0}\right)^2 + \left(\nabla^{1,0} \circ \bar\partial + \bar\partial \circ \nabla^{1,0}\right), \tag{1.3.37}$$

so that in the decomposition $\Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$, we have $\Omega^{0,2} = 0$. On the other hand, if we consider a unitary frame for a Hermitian vector bundle, then the compatibility condition

$$dh_{ab} = \theta_a^c\, h_{cb} + \overline{\theta}_b^c\, h_{ac} \tag{1.3.38}$$

implies that θ is a skew-Hermitian matrix: ${}^t\theta + \overline{\theta} = 0$. From the Cartan equation, it follows that Ω is also skew-Hermitian, and $\Omega^{2,0} = -\overline{\Omega^{0,2}}$. Since the type of the curvature does not depend on the chosen frame, we obtain that the curvature associated to a Chern connection is purely of type (1,1):

$$\Omega = \Omega^{(1,1)}. \tag{1.3.39}$$

Example 1.3.3. Let us construct an example of Hermitian metric on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{P}^n$. The fibres are naturally endowed with a Hermitian structure, being linear subspaces of $\mathbb{C}^{n+1}$:

$$\mathcal{O}(-1)\big|_\ell = \ell \subset \mathbb{C}^{n+1}. \tag{1.3.40}$$

Thus, the Hermitian metric on $\mathcal{O}(-1)$ is defined as the restriction of the standard one in $\mathbb{C}^{n+1}$ on each fibre. Recall now the definition of the standard open cover of $\mathbb{P}^n$ (see Example 1.1.1). Choosing on U_i the local frame

$$\sigma_i([Z]) = \left([Z], \frac{Z}{Z^i}\right), \tag{1.3.41}$$

which in coordinates $z \mapsto [z^1 : \cdots : 1 : \cdots : z^n]$ is given by

$$\sigma_i(z) = \left([z^1 : \cdots : 1 : \cdots : z^n], (z^1, \ldots, 1, \ldots, z^n)\right), \tag{1.3.42}$$

the Hermitian metric will be be

$$h_z(\sigma_i, \sigma_i) = \left\|(z^1, \ldots, 1, \ldots, z^n)\right\|^2_{\mathbb{C}^{n+1}} = 1 + |z|^2. \tag{1.3.43}$$

The connection matrix (which is actually a form of type $(1,0)$) will be

$$\theta = \partial h \cdot h^{-1} = \sum_{i=1}^{n} \frac{\bar{z}^i}{1+|z|^2} dz^i \tag{1.3.44}$$

and finally, by Cartan equation, the curvature will be

$$\begin{aligned}
\Omega &= d\theta + \theta \wedge \theta = d\theta \\
&= \sum_{i,j} \frac{\partial}{\partial z^j}\left(\frac{\bar{z}^i}{1+|z|^2}\right) dz^j \wedge dz^i + \sum_{i,j} \frac{\partial}{\partial \bar{z}^j}\left(\frac{\bar{z}^i}{1+|z|^2}\right) d\bar{z}^j \wedge dz^i + \\
&\qquad + \sum_{i,j} \frac{\bar{z}^i \bar{z}^j}{(1+|z|^2)^2} dz^i \wedge dz^j \\
&= -\sum_{i,j} \frac{\bar{z}^i \bar{z}^j}{(1+|z|^2)^2} dz^j \wedge dz^i + \sum_{i,j} \frac{\delta^{ij}(1+|z|^2) - \bar{z}^i z^j}{(1+|z|^2)^2} d\bar{z}^j \wedge dz^i + \\
&\qquad + \sum_{i,j} \frac{\bar{z}^i \bar{z}^j}{(1+|z|^2)^2} dz^i \wedge dz^j \\
&= -\sum_{i,j} \frac{\delta^{ij}(1+|z|^2) - \bar{z}^i z^j}{(1+|z|^2)^2} dz^i \wedge d\bar{z}^j.
\end{aligned}$$

The local expression holds on U_i. In the following section, we will construct a Kähler metric on $\mathbb{P}^n$, the Fubini-Study metric, whose metric form will be proportional the above curvature form.

1.3.2 Divisors

Recall that an analytic hypersurface of of a complex manifold M is a subset $V \subset M$ such that, for every $p \in V$, there exists a neighbourhood U and a holomorphic function f defined on U such that $V \cap U$ is the zero set of f. Such an f is called a *local defining function* for V near p. Note that the quotient of any two local defining functions around p is a non-vanishing holomorphic function around p (this is a consequence of the stalk properties of the sheaf $\mathcal{O}$, see for instance [Griffiths, 1984]).

Definition 1.3.9. A *divisor* D of a complex manifold is a locally finite formal linear combination

$$D = \sum_i a_i V_i, \qquad a_i \in \mathbb{Z}, \tag{1.3.45}$$

of irreducible analytic hypersurfaces of M. Here "locally finite" means that for every point $p \in M$, there exists a neighbourhood of p that meets a finite number of V_i's appearing in D. The set of divisors is an abelian group, denoted with $\mathrm{Div}(M)$. A divisor D is called effective if $a_i \geqslant 0$ for all i. Further, an analytic hypersurfaces will be identified with the divisor $\sum_i V_i$, where V_i are its irreducible components.

It turns out that the divisor group can be described in a sheaf theoretic way, leading to the important relation between divisors and line bundles. To show it, we have to introduce the concept meromorphic function, which will encodes the local data defining a divisor.

Definition 1.3.10. A *meromorphic function* on a open subset U of a complex manifold M is an equivalence class of collections $(U_\alpha, g_\alpha, h_\alpha)_\alpha$, where $\{U_\alpha\}$ is an open cover of U, and g_α, h_α are holomorphic functions defined on U_α such that

$$g_\alpha h_\beta = g_\beta h_\alpha \qquad \text{on } U_\alpha \cap U_\beta. \tag{1.3.46}$$

Two such collections $(U_\alpha, g_\alpha, h_\alpha)_{\alpha\in I}$ and $(U'_\beta, g'_\beta, h'_\beta)_{\beta\in J}$ are equivalent if

$$g_\alpha h'_\beta = g'_\beta h_\alpha \qquad \text{on } U_\alpha \cap U'_\beta \text{ for all } \alpha \in I, \beta \in J. \tag{1.3.47}$$

The meromorphic functions form a sheaf on M, denoted by $\mathcal{M}$.

We will say that a meromorphic function is written locally as $\frac{g_\alpha}{h_\alpha}$ on U_α. The equivalence relation is due to the fact that the pair (g_α, h_α) is not uniquely defined: we can "multiply numerator and denominator" to obtain the same meromorphic function.

Consider now an irreducible analytic hypersurface V, with local defining function f around some $p \in V$. Then for every holomorphic function g around p, the *order of* f *along* V *at* p is defined to be the largest positive integer a such that $\frac{g}{f^a}$ is holomorphic around p. It can be shown that the order of g is a well-defined positive integer, which does not depend on p (see for instance [Griffiths, 1984]), and is denoted by $\mathrm{ord}_V(g)$. For g, h holomorphic functions,

$$\mathrm{ord}_V(gh) = \mathrm{ord}_V(g) + \mathrm{ord}_V(h) \tag{1.3.48}$$

This additivity property suggests the following definition. For a meromorphic function φ on M, defined locally as $\frac{g}{h}$, and an irreducible hypersurface V, define

$$\mathrm{ord}_V(\varphi) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h). \tag{1.3.49}$$

The definition does not depend on the local representation of the meromorphic function: if $(U_\alpha, g_\alpha, h_\alpha)$ and $(U_\beta, g'_\beta, h'_\beta)$ are local descriptions, then Equation (1.3.47) and the additivity property imply $\mathrm{ord}_V(g_\alpha) + \mathrm{ord}_V(h'_\beta) = \mathrm{ord}_V(g'_\beta) + \mathrm{ord}_V(h_\alpha)$, so that

$$\mathrm{ord}_V(g_\alpha) - \mathrm{ord}_V(h_\alpha) = \mathrm{ord}_V(g'_\beta) - \mathrm{ord}_V(h'_\beta).$$

We will usually say that φ has a "zero of order a" if $a = \mathrm{ord}_V(\varphi) > 0$ and that φ has a "pole of order a" if $-a = \mathrm{ord}_V(\varphi) < 0$. A meromorphic function naturally defines a divisor (φ) as

$$(\varphi) = \sum_V \mathrm{ord}_V(\varphi)\, V, \tag{1.3.50}$$

where the sum runs over all irreducible hypersurfaces of M. The above sum is locally finite, since for every open set U where φ is represented by $\frac{g}{h}$, there are only finitely many irreducible analytic hypersurfaces along which g of h have non-vanishing order. The divisors associated to a meromorphic function will play an important role in the next section for the relation between divisors and line bundle.

The sheaf of meromorphic function allows us to characterise the group of divisors as follows.

Proposition 1.3.11. *There is a group isomorphism*

$$H^0\left(M, {}^{\mathcal{M}^*}\!/_{\mathcal{O}^*}\right) \cong \mathrm{Div}(M). \tag{1.3.51}$$

Here $\mathcal{M}^*$ is the multiplicative sheaf of meromorphic functions on M not identically zero and $\mathcal{O}^*$ is the subsheaf of non-zero holomorphic functions.

Proof. Consider a global section φ of $\mathcal{M}^*/\mathcal{O}^*$. For a covering $\{U_\alpha\}$ of M, this is given by meromorphic functions $\{\varphi_\alpha\}$ such that

$$\frac{\varphi_\alpha}{\varphi_\beta} \in \mathcal{O}^*(U_{\alpha\beta}).$$

As a consequence $\mathrm{ord}_V(\varphi_\alpha) = \mathrm{ord}_V(\varphi_\beta)$ and we can define

$$D_\varphi = \sum_V \mathrm{ord}_V(\varphi_\alpha)\, V.$$

Here for every irreducible analytic subvariety V of M, we choose α such that $V \cap U_\alpha \neq \varnothing$. Note that the additivity property implies that the map $\varphi \mapsto D_\varphi$ is a group homomorphism. The local meromorphic functions $\{\varphi_\alpha\}$ will be called the local defining data for the divisor D_φ.
Consider now a divisor $D = \sum_i a_i V_i$ and a covering $\{U_\alpha\}$ such that, for every α, we can find a local defining function $f_{i,\alpha}$ for V on U_α. This is always possible, as the sum is locally finite (we just need to intersect the finite number of neighbourhoods of a point where we have local defining functions). Set on U_α

$$\varphi_\alpha = \prod_i f_{i,\alpha}^{a_i},$$

which define a global section φ_D in $\mathcal{M}^*/\mathcal{O}^*$. This is a consequence of the fact that the quotient of any two local defining functions around p is a non-vanishing holomorphic function around p. The map $D \mapsto \varphi_D$ is clearly a group homomorphism and it can be shown that it is the inverse morphism of $\varphi \mapsto D_\varphi$. □

Line bundles and divisors

Let us explain now the connection between divisors and holomorphic line bundles (from now on, the line bundles will be tacitly assumed to be holomorphic). Recall that tensor products and duals of a line bundles are still line bundles. Considering the line bundles modulo isomorphisms, it can be simply shown that they form a group, with multiplication given by tensor product, inverses by dual bundles, and neutral element by the isomorphism class of the trivial bundle. The group is called the *Picard group* of M an is denoted with Pic(M).

Now, for a line bundle $L \to M$, we can find local trivializations $\{ U_\alpha, \phi_\alpha \}$, which determine holomorphic transition maps $g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathbb{C}^*$. They satisfy the conditions

$$\begin{aligned} g_{\alpha\beta}^{-1} &= g_{\beta\alpha}, \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= 1, \end{aligned} \tag{1.3.52}$$

which imply that $g = \{ g_{\alpha\beta} \}$ is a closed Čech 1-cocycle with coefficients in $\mathcal{O}^*$, *i.e.* $[g] \in H^1(M, \mathcal{O}^*)$. On the other hand, for a line bundle L' isomorphic to L with local trivialization $\{ V_\mu, \psi_\mu \}$ and transition functions $h_{\mu\nu}$, we can find a refinement of both $\{ U_\alpha \}$ and $\{ V_\mu \}$, say $\{ W_\rho \}$, such that

$$\psi_\rho = f_\rho \phi_\rho \tag{1.3.53}$$

for non-zero holomorphic functions $f_\rho \in \mathcal{O}^*(W_\rho)$. Thus,

$$h_{\rho\sigma} = \frac{f_\rho}{f_\sigma} g_{\rho\sigma}, \qquad \frac{f_\rho}{f_\sigma} \in \mathcal{O}^*(W_{\rho\sigma}). \tag{1.3.54}$$

As a consequence, the cocycles $\{ g_{\rho\sigma} \}$ and $\{ h_{\rho\sigma} \}$ differ by the cocycle $\left\{ \frac{f_\rho}{f_\sigma} \right\}$, which is the coboundary of $f = \{ f_\alpha \}$, and they define the same element in cohomology. In particular, we have a well-defined map $\mathrm{Pic}(M) \to H^1(M, \mathcal{O}^*)$. On the other hand, from a closed Čech 1-cocycle with coefficients in $\mathcal{O}^*$ we can construct a line bundle L. It can be simply shown that cohomologous cocycles give rise to isomorphic line bundles, so that we actually have a group isomorphism

$$\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}^*). \tag{1.3.55}$$

We will omit new notation for the isomorphism class of a line bundle L, denoting it with the same symbol. The correspondence between divisors and line bundles is given by the following

Proposition 1.3.12. *There is a group isomorphism*

$$\mathrm{Pic}(M) \cong {}^{\mathrm{Div}(M)}\!/\!_{\sim}, \tag{1.3.56}$$

where the equivalence relation $\sim$ *on* Div(M) *is given by*

$$D \sim D' \quad \textit{if and only if} \quad D - D' = (\varphi) \tag{1.3.57}$$

for a meromorphic function φ *on* M. *The group* $\mathrm{Div}(M)/\sim$ *is called the* divisor class group *on* M *and is denoted by* Cl(M).

Proof. The isomorphism follows from the sheaf exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to {}^{\mathcal{M}^*}\!\diagup_{\mathcal{O}^*} \to 0,$$

that in cohomology gives us the isomorphism

$$\frac{H^0\left(M, {}^{\mathcal{M}^*}\!\diagup_{\mathcal{O}^*}\right)}{H^0(M, \mathcal{M}^*)} \cong H^1(M, \mathcal{O}^*).$$

The subgroup $H^0(M, \mathcal{M}^*)$ is exactly the subgroup of $\mathrm{Div}(M)$ given by the divisors associated to global meromorphic functions. However, the isomorphism can be realised explicitly as follows. Consider the divisor D, locally given by the datum φ_α on U_α. Set

$$g_{\alpha\beta} = \frac{\varphi_\alpha}{\varphi_\beta},$$

which is an element of $\mathcal{O}^*(U_{\alpha\beta})$. Certainly $g_{\alpha\beta}^{-1} = \frac{\varphi_\beta}{\varphi_\alpha} = g_{\beta\alpha}$ and

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \frac{\varphi_\alpha}{\varphi_\beta} \cdot \frac{\varphi_\beta}{\varphi_\gamma} \cdot \frac{\varphi_\gamma}{\varphi_\alpha} = 1.$$

The line bundle given by the transition functions $g_{\alpha\beta}$ is denoted by $\mathcal{O}(D)$, and is called the *line bundle associated to the divisor* D. The map $D \mapsto \mathcal{O}(D)$ is clearly a surjective. The multiplication is respected, as D is given by meromorphic functions φ_α and D' by φ'_α (we can assume the open coverings to coincide) which give us the transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$, then the sum $D - D'$ is locally given by $\varphi_\alpha \cdot (\varphi'_\alpha)^{-1}$, whose associated transition functions becomes

$$\frac{\varphi_\alpha \cdot (\varphi'_\alpha)^{-1}}{\varphi_\beta \cdot (\varphi'_\beta)^{-1}} = \frac{\varphi_\alpha}{\varphi_\beta} \cdot \left(\frac{\varphi'_\alpha}{\varphi'_\alpha}\right)^{-1} = g_{\alpha\beta} \cdot (g'_{\alpha\beta})^{-1}.$$

These transition functions are exactly those of $\mathcal{O}(D) \otimes \mathcal{O}(D')^*$. Thus, $D \mapsto \mathcal{O}(D)$ is a group epimorphism. Finally, let us show that $\mathcal{O}(D)$ is trivial if and only if $D = (\varphi)$ for a meromorphic function φ. We can assume the local data for the divisor $D = (\varphi)$ to be given by the restrictions of the global meromorphic function: $\varphi_\alpha = \varphi|_{U_\alpha}$, so that $\frac{\varphi_\alpha}{\varphi_\beta} = 1$ and $\mathcal{O}(D)$ has trivial transition functions. Thus, $\mathcal{O}(D)$ is the trivial bundle. On the other hand, if D is given by local data φ_α and $\mathcal{O}(D)$ is trivial, then there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

In particular, we can define the global meromorphic function ψ, whose local expression is $\varphi_\alpha \cdot f_\alpha^{-1}$. It is well defined, as

$$\varphi_\alpha \cdot f_\alpha^{-1} = \varphi_\beta \cdot f_\beta^{-1}.$$

Then $D = (\psi)$ and the theorem is proved. □

Chern classes

In this section we are going to introduce the notion of Chern class for line bundles, following [Griffiths, 1984]. Chern classes are part of a more general theory of characteristic classes, see for instance [Milnor and Stasheff, 1974].

Definition 1.3.13. Let M be a complex manifold and consider the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e^{2\pi\sqrt{-1}}} \mathcal{O}^* \longrightarrow 0. \tag{1.3.58}$$

Here $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions. The long exact sequence in cohomology gives us a coboundary map between the Picard group $\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}^*)$ and the second integral cohomology group of M:

$$c_1 \colon H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}). \tag{1.3.59}$$

We define the *first Chern class* of a holomorphic line bundle L on M to be the element $c_1(L) \in H^2(M, \mathbb{Z})$. We will sometimes write $c_1(L) \in H^2_{dR}(M, \mathbb{R})$ for the image of the first Chern class under the map $H^2(M, \mathbb{Z}) \to H^2_{dR}(M, \mathbb{R})$.

As the coboundary map is a group homomorphism, we immediately have that

$$\begin{aligned} c_1(L \otimes L') &= c_1(L) + c_1(L') \\ c_1(L^*) &= -c_1(L). \end{aligned} \tag{1.3.60}$$

Further, the naturality of the long exact sequence in cohomology implies that for any holomorphic map $f \colon M \to N$ between complex manifolds and a line bundle $L \in \mathrm{Pic}(N)$, we have

$$c_1(f^*L) = f^*c_1(L). \tag{1.3.61}$$

We want to give now two alternative interpretations of the first Chern class of a holomorphic line bundle L: as the class of the curvature form of any connection on L or as the Poincaré dual of the divisor D associated to L.

Proposition 1.3.14. *For any line bundle L on M and curvature form Ω,*

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Omega\right] \in H^2_{dR}(M, \mathbb{R}). \tag{1.3.62}$$

Proof. Let us work in Čech cohomology. Consider a line bundle L with transition functions $g_{\alpha\beta}$ relative to the trivialization $\mathfrak{U} = \{U_\alpha\}$. The cocycle $g = \{g_{\alpha\beta}\}$ represents L and we can take $f \in \check{C}^1(\mathfrak{U}, \mathcal{O})$ such that $\exp\left(2\pi\sqrt{-1}[f]\right) = [g]$. If the U_α are simply connected, this is the cocycle given by

$$f_{\alpha\beta} = \frac{1}{2\pi\sqrt{-1}} \log g_{\alpha\beta}.$$

Since at the level of sections the map $\mathbb{Z} \to \mathcal{O}$ is just an injection, the element representing the image of $[g]$ under the coboundary map is just the Čech differential $z = \check{\delta}(f) \in \check{C}^2(\mathfrak{U}, \mathbb{Z})$:

$$\begin{aligned} z_{\alpha\beta\gamma} &= f_{\beta\gamma} - f_{\alpha\gamma} + f_{\beta\gamma} \\ &= \frac{1}{2\pi\sqrt{-1}} \left(\log g_{\beta\gamma} - \log g_{\alpha\gamma} + \log g_{\beta\gamma}\right). \end{aligned}$$

Thus, $c_1(L) = [z]$. On the other hand, as L is a line bundle, the curvature form associated to a connection ∇ on L is a well-defined global 2-form $\Omega \in H^2_{dR}(M, \mathbb{R})$. Locally, for a fixed trivialization $\mathfrak{U}$ of L with associated local frames σ_α, it is given by

$$\Omega = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha.$$

Here we have used the fact that θ_α is just a 1-form. On the overlaps, we simply have $\theta_\alpha = \theta_\beta + dg_{\alpha\beta} \cdot g^{-1}_{\alpha\beta}$, so that

$$\theta_\beta - \theta_\alpha = -d\left(\log g_{\alpha\beta}\right).$$

Note that Ω is given by a closed 2-form, while $c_1(L)$ is a Čech cocycle, so to compare them me must explicitly write the de Rham isomorphism. Let us set $\mathcal{A}^k$ and $\mathcal{Z}^k$ for the sheaf of real k-forms and the sheaf of real, closed k-forms respectively. The de Rham isomorphism $H^2_{dR}(M, \mathbb{R}) \cong H^2(M, \mathbb{R})$ descends from the exact sequences

$$0 \to \mathbb{R} \to C^\infty \xrightarrow{d} \mathcal{Z}^1 \to 0 \qquad 0 \to \mathcal{Z}^1 \to \mathcal{A}^1 \xrightarrow{d} \mathcal{Z}^2 \to 0$$

as the composition of the coboundary isomorphisms in cohomology

$$H^2_{dR}(M, \mathbb{R}) = \frac{H^0(M, \mathcal{Z}^2)}{dH^0(M, \mathcal{A}^1)} \xrightarrow{\delta_1} H^1(M, \mathcal{Z}^1) \qquad H^1(M, \mathcal{Z}^1) \xrightarrow{\delta_2} H^2(M, \mathbb{R}).$$

In particular, we have that $\delta_1([\Omega])$ is the Čech coboundary of an element in $\check{C}^0(\mathfrak{U}, \mathcal{Z}^1)$ representing the locally exact form Ω. As locally $\Omega = d\theta_\alpha$, we have

$$\delta_1([\Omega]) = \left[\check{\delta}(\{\theta_\alpha\})\right] = \left[\{\theta_\beta - \theta_\alpha\}\right] \in H^1(M, \mathcal{Z}^1).$$

Similarly, as $\theta_\beta - \theta_\alpha = -d\left(\log g_{\alpha\beta}\right)$, we have

$$\delta_2\big(\delta_1([\Omega])\big) = \left[\check{\delta}(\{-\log g_{\alpha\beta}\})\right] = \left[-\log g_{\alpha\beta} + \log g_{\alpha\gamma} - \log g_{\beta\gamma}\right] \in H^2(M, \mathbb{R}).$$

Thus, $c_1(L) = \frac{\sqrt{-1}}{2\pi}\delta_2\big(\delta_1([\Omega])\big)$. □

For the next proposition, it will be useful to introduce the differential operator $d^c = -\sqrt{-1}(\partial - \bar{\partial})$. A simple computation shows that $dd^c = -2\sqrt{-1}\bar{\partial}\partial$. This relation will be useful in the following proof. Further, recall that to any irreducible analytic k-subvariety V of a compact complex manifold M, we can associate its Poicaré dual $PD[V] \in H^{2n-2k}_{dR}(M, \mathbb{R})$. By linearity, we can define the Poicaré dual of a divisor $D = \sum_i a_i V_i$ as

$$PD[D] = \sum_i a_i \, PD[V_i] \in H^2_{dR}(M, \mathbb{R}). \tag{1.3.63}$$

Proposition 1.3.15. *If* $L = \mathcal{O}(D)$ *for some divisor* $D \in \mathrm{Div}(M)$*,*

$$c_1(L) = PD[D] \in H^2_{dR}(M, \mathbb{R}). \tag{1.3.64}$$

Sketch of the proof. Thanks to the previous proposition, we can consider a curvature form Ω on L associated to a Hermitian metric h and we just need to show that for any real, closed form $\omega \in \mathcal{A}^{2n-2}(M)$, having set $D = \sum_i a_i V_i$,

$$\frac{\sqrt{-1}}{2\pi}\int_M \Omega \wedge \omega = \sum_i a_i \int_{V_i} \omega.$$

As the Chern class and the Poicaré duality are both group homomorphisms, we can suppose $D = V$ for an irreducible hypersurface V. Fix a local trivialization and a non-zero holomorphic section e. Set

$$h(p) = |e(p)|^2,$$

where the norm is given by the metric h. Then for any local holomorphic section s, write $s = \lambda e$ for a local holomorphic function λ. We have

$$\begin{aligned} d\big(|s|^2\big) &= h\big(\nabla s, s\big) + h\big(s, \nabla s\big) \\ &= h\big((d\lambda + \lambda\theta) \otimes e, \lambda e\big) + h\big(\lambda e, (d\lambda + \lambda\theta) \otimes e\big) \\ &= \bar{\lambda} h \cdot d\lambda + \lambda h \cdot d\bar{\lambda} + |\lambda|^2 h \cdot \big(\theta + \bar{\theta}\big). \end{aligned}$$

On the other hand,

$$d\big(|s|^2\big) = d\big(\lambda\bar{\lambda}h\big) = \bar{\lambda}h \cdot d\lambda + \lambda h \cdot d\bar{\lambda} + |\lambda|^2 \cdot dh,$$

so that we obtain $\theta + \bar{\theta} = \frac{dh}{h} = d\log(h)$. Remember that, in terms of the holomorphic frame e, the connection matrix is purely of type (1,0), so that we can simply write $\theta = \partial \log(h)$. From the equation $\Omega = d\theta$, we find

$$\Omega = \bar{\partial}\partial \log(h) = \frac{\sqrt{-1}}{2} dd^c \log(h).$$

This is the local expression for the curvature form. Consider now the local defining functions $\{ f_\alpha \}$ for the hypersurface V, associated to the cover $\{ U_\alpha \}$ of M. They determine a section $s \in H^0(M, L)$, vanishing exactly on V. For $\epsilon > 0$, set

$$D_\epsilon = \{ p \in M \mid h_p(s(p), s(p)) < \epsilon \},$$

which for small ϵ is a tubular neighbourhood of V. Fix a real, closed form $\omega \in \mathcal{A}^{2n-2}(M)$. Then

$$\frac{\sqrt{-1}}{2\pi}\int_M \Omega \wedge \omega = \lim_{\epsilon \to 0} -\frac{1}{4\pi}\int_{M \setminus D_\epsilon} dd^c \log|s|^2 \wedge \omega = \lim_{\epsilon \to 0} \frac{1}{4\pi}\int_{\partial D_\epsilon} d^c \log|s|^2 \wedge \omega.$$

Here $\Omega = \frac{1}{2\sqrt{-1}} dd^c \log|s|^2$ on $M \setminus D_\epsilon$, as s does not vanish on it. Now, on U_α we have $|s^2| = f_\alpha \bar{f}_\alpha h_\alpha$, so that

$$d^c \log|s|^2 = -\sqrt{-1}\big(\partial \log(f_\alpha) - \bar{\partial} \log(\bar{f}_\alpha)\big) + d^c \log(h_\alpha).$$

It can be shown that the integral involving $d^c \log(h_\alpha)$ tend to zero as $\epsilon \to 0$. Further, $\bar\partial \log(\bar f_\alpha) = \overline{\partial \log(f_\alpha)}$, so that $\partial \log(f_\alpha) - \bar\partial \log(\bar f_\alpha) = 2\sqrt{-1}\,\mathrm{Im}(\partial \log(f_\alpha))$ and the above integral reduces to

$$\frac{\sqrt{-1}}{2\pi}\int_M \Omega \wedge \omega = \lim_{\epsilon\to 0}\frac{1}{2\pi}\,\mathrm{Im}\int_{\partial D_\epsilon} \partial \log(f_\alpha)\wedge\omega.$$

Note that in the neighbourhood of any smooth point $z_0 \in V \cap U_\alpha$, we can find a holomorphic coordinate system (w^i) such that $w^1 = f_\alpha$. Split the differential form as $\omega = \omega_0 dw' \wedge d\bar w' + \eta$, where $w' = (w^2, \dots, w^n)$ and every term of η contains $dw^1 \wedge d\bar w^1$ (it must contain both terms by reality assumptions). Then for a sufficiently small polydisk Δ around z_0

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial D_\epsilon\cap\Delta} \partial\log(f_\alpha)\wedge\omega &= \lim_{\epsilon\to 0}\int_{\{w^1=\epsilon\}\cap\Delta} \frac{dw^1}{w^1}\wedge\omega\\ &= \lim_{\epsilon\to 0}\int_{\{w^1=\epsilon\}\cap\Delta} \omega_0\frac{dw^1}{w^1}\wedge dw'\wedge d\bar w'.\end{aligned}$$

For the integral in w^1 we can use Cauchy integral formula, obtaining $2\pi\sqrt{-1}\omega_0(0,w')$. Thus, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial D_\epsilon\cap\Delta} \partial\log(f_\alpha)\wedge\omega &= 2\pi\sqrt{-1}\int_{\{w^1=0\}\cap\Delta} \omega_0(0,w')dw'\wedge d\bar w'\\ &= 2\pi\sqrt{-1}\int_{V\cap\Delta}\omega.\end{aligned}$$

Finally, we find

$$\frac{\sqrt{-1}}{2\pi}\int_M \Omega\wedge\omega = \frac{1}{2\pi}\,\mathrm{Im}\left(2\pi\sqrt{-1}\int_V \omega\right) = \int_V \omega.$$

□

The first Chern class is of fundamental importance in many applications, as the Kodaira embedding theorem which characterises the compact complex manifolds which can embedded into the projective space. This property comes from a particular feature of the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$: its first Chern class is positive, where positivity has to be intended as follows.

Definition 1.3.16. Let ω be a real form of type (1,1) on a complex manifold M. We will say that ω is a *positive form* if the symmetric tensor $\omega(\cdot, J\cdot)$ is positive definite. A real cohomology class, that is an element $a \in H^2_{dR}(M,\mathbb{R}) \cap H^{1,1}_{\bar\partial}(M)$, is said to be positive if it can be represented by a positive form. In this case, we will often write $a > 0$. An analogous definition can be given for real, negative (1,1)-forms and negative cohomology classes.

Alternatively, the real $(1,1)$-form $\omega = \sqrt{-1}\omega_{i\bar j} dz^i \wedge d\bar z^j$ is positive when the Hermitian matrix $\omega_{i\bar j}$ is positive definite.

Example 1.3.4. In the case of $\mathcal{O}(1)$, we have that $c_1(\mathcal{O}(1)) = -\left[\frac{\sqrt{-1}}{2\pi}\Omega\right]$, with

$$-\frac{\sqrt{-1}}{2\pi}\Omega = \sqrt{-1}\underbrace{\left(\frac{1}{2\pi}\sum_{i,j}\frac{\delta^{ij}(1+|z|^2) - \bar{z}^i z^j}{(1+|z|^2)^2}\right)}_{=\omega_{i\bar{j}}} dz^i \wedge d\bar{z}^j. \tag{1.3.65}$$

Since Ω is invariant under the transitive action of $U(n+1)$ on $\mathbb{P}^n$ and the positivity is a pointwise condition, we can restrict ourself to the point $p = [1 : 0 : \cdots : 0] \in U_0$. Thus, in p we find $\omega_{i\bar{j}}(p) = \frac{1}{2\pi}\mathrm{Id}_{\mathbb{C}^n}$, which is clearly positive definite. In particular, $c_1(\mathcal{O}(d)) = -\left[\frac{d\sqrt{-1}}{2\pi}\Omega\right]$ has the same sign of d.

The notion of positivity can be extended to line bundle as follows.

Definition 1.3.17. Consider $L \in \mathrm{Pic}(M)$. Then L is called a *positive line bundle* if $c_1(L) > 0$.

Theorem 1.3.18 (Kodaira embedding theorem)**.** *Let M be a compact complex manifold, $L \to M$ a line bundle. Then L is positive if and only if there exists an integer k such that the map*

$$\begin{aligned} i_{L^{\otimes k}} \colon M &\longrightarrow \mathbb{P}(H^0(M, L^{\otimes k}))^* \\ p &\longmapsto \left\{ s \in H^0(M, L^{\otimes k}) \;\middle|\; s(p) = 0 \right\} \end{aligned} \tag{1.3.66}$$

is a well-defined holomorphic embedding. In this case, L is a said to be ample. *Here $\mathbb{P}(V)$ is the projective space associated to the complex vector space V and $i_{L^{\otimes k}}(p)$ is an hyperplane of $H^0(M, L^{\otimes k})$.*

It can be proved that if L is a *very ample line bundle* over M, that is the above integer can be chosen to be equal to 1, then L is isomorphic to the pull-back $i_L^*\mathcal{O}(1)$ of the hyperplane bundle of $\mathbb{P}(H^0(M, L^{\otimes k}))^*$. A standard reference for this result is [Griffiths, 1984].

Chapter 2

Kähler manifolds

The main topic of this section will be Kähler manifolds and their properties. In the first section we present the Kähler condition and some characterisations, while in the second section some rich aspects of these structures are shown via Hodge theory. The basic references are [Voisin, 2002], [Székelyhidi, 2014] and [Huybrechts, 2005].

2.1 Kähler metrics

In this section we introduce the important notion of Kähler manifold, following the presentation of [Voisin, 2002] and [Székelyhidi, 2014]. These are complex manifolds with a Hermitian metric on the real tangent bundle, viewed as a complex vector bundle via the natural almost complex structure J, with the a further condition of compatibility between the Hermitian metric and the almost complex structure. More precisely, we will show that a Hermitian metric h on the real tangent space can be decomposed as

$$h = g - \sqrt{-1}\omega,$$

where g is a Riemannian metric compatible with J and ω is a real form of type $(1,1)$. A Kähler metric satisfies the further condition $d\omega = 0$. This is equivalent to the condition for J to be covariantly constant with respect to the Levi-Civita connection associated to g, or to the equality of the Levi-Civita and Chern connections associated to g and h respectively. Another equivalent condition is the existence of holomorphic normal coordinates.

While it is easy to show that every complex manifold admits a Hermitian metric by using partition of unity, the Kähler conditions is very restrictive. However, this fact produces an extraordinary amount of symmetries, as we will see in the section devoted to Hodge theory.

Lemma 2.1.1. *Let M be a complex manifold, with natural almost complex structure J on the real tangent bundle TM. There is a natural bijection between*

1) *symmetric, real-valued $\binom{0}{2}$ tensors g compatible with J,* i.e. $g(X,Y) = g(Y,X) = g(JX,JY)$ *for every $X, Y \in \mathfrak{X}(M)$,*

2) *Hermitian, complex-valued* $\binom{0}{2}$ *tensors* h, i.e. h *satisfies the condition* $h(X,Y) = \overline{h(Y,X)}$ *for every* $X, Y \in \mathfrak{X}(M)$,

3) *real-valued* 2*-forms* ω *of type* $(1,1)$, i.e. *if* ω *is extended to* $T_{\mathbb{C}}M$ *by complex linearity, then* $\omega(U,V) = 0$ *for every* $U, V \in \mathfrak{X}^{1,0}(M)$ *and for every* $U, V \in \mathfrak{X}^{0,1}(M)$.

The correspondence is given by

$$h(X,Y) = g(X,Y) - \sqrt{-1}\omega(X,Y). \tag{2.1.1}$$

If g *is positive definite (or equivalently* h *or* ω*), then* M *is called a* Hermitian manifold.

Proof. Fix a symmetric, real-valued $\binom{0}{2}$ tensors g compatible with J. Define

$$h(X,Y) = g(X,Y) - \sqrt{-1}g(JX,Y).$$

Clearly, h is a complex-valued $\binom{0}{2}$ tensor. Further,

$$\begin{aligned}
h(X,Y) &= g(X,Y) - \sqrt{-1}g(JX,Y) \\
&= g(Y,X) - \sqrt{-1}g(Y,JX) && \text{since } g \text{ is symmetric} \\
&= g(Y,X) - \sqrt{-1}g(JY,J^2X) && \text{from the compatibility condition} \\
&= g(Y,X) + \sqrt{-1}g(JY,X) && \text{since } J^2 = -\operatorname{id} \\
&= \overline{h(Y,X)},
\end{aligned}$$

so that h is Hermitian. Fix now a Hermitian, complex-valued $\binom{0}{2}$ tensor h. Define ω as (minus) the imaginary part of h: $\omega = -\operatorname{Im} h$. Note that

$$\begin{aligned}
\omega(X,Y) &= -\operatorname{Im} h(X,Y) \\
&= -\operatorname{Im} \overline{h(Y,X)} && \text{since } h \text{ is Hermitian} \\
&= +\operatorname{Im} h(Y,X) \\
&= -\omega(Y,X).
\end{aligned}$$

Thus, ω is a real-valued 2-form. Take now two holomorphic vector fields on M: $U = X - \sqrt{-1}JX$, $V = Y - \sqrt{-1}JY$. Then

$$\omega(U,V) = \omega(X,Y) - \omega(JX,JY) - \sqrt{-1}\big(\omega(X,JY) + \omega(JX,Y)\big).$$

On the other hand, since h is Hermitian, we find that $h(JX,JY) = h(X,Y)$ and $h(JX,Y) = -h(X,JY)$, so that the same holds for ω and

$$\omega(U,V) = \omega(X,Y) - \omega(X,Y) - \sqrt{-1}\big(\omega(X,JY) - \omega(X,JY)\big) = 0.$$

The same argument hold for antiholomorphic vector fields, so that ω is of type $(1,1)$. Finally, fix a real-valued 2-forms ω of type $(1,1)$. Define

$$g(X,Y) = -\omega(JX,Y),$$

which is a real-valued $\binom{0}{2}$ tensor. On the other hand, the above calculations show that being of type (1,1) implies that $\omega(JX, JY) = \omega(X, Y)$ and $\omega(JX, Y) = -\omega(X, JY)$, so that

$$\begin{aligned} g(X, Y) &= -\omega(JX, Y) = \omega(X, JY) \\ &= -\omega(JY, X) = g(Y, X), \end{aligned}$$

and similarly

$$g(JX, JY) = -\omega(J^2X, JY) = -\omega(JX, Y) = g(X, Y).$$

Thus, g is symmetric and compatible with J.
It can be simply shown that the condition of being positive definite for g, h and ω are equivalent to each other. □

Remark 2.1.2. Note, that a Hermitian manifold is both Riemannian and Poisson, with the further condition on the 2-form ω of being *positive definite*. This form is called the *metric form* associated to g or h. It can be shown that every complex manifold admits a Hermitian metric, by arguments analogue to the Riemannian case using a partition of unity.

Let us analyse the above constructions in local holomorphic coordinates (z^i), starting from a Riemann metric g compatible with J. Extending the metric to the complexified tangent space, we define

$$\begin{aligned} g_{ij} &= g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right), \quad g_{\bar{i}\bar{j}} = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right), \\ g_{\bar{i}j} &= g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}\right), \quad g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right). \end{aligned} \tag{2.1.2}$$

The compatibility condition implies that $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Indeed,

$$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}\right) = g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \sqrt{-1}\frac{\partial}{\partial z^j}\right) = -g_{ij}.$$

Similarly for $g_{\bar{i}\bar{j}} = 0$. Thus, we can write $g = g_{i\bar{j}}\, dz^i d\bar{z}^j$, where the sum over i and j is intended. More generally, the compatibility condition for g implies that vector fields belonging to the same J-eigenspace are orthogonal:

$$g\big(\mathfrak{X}^{1,0}(M), \mathfrak{X}^{1,0}(M)\big) = 0, \qquad g\big(\mathfrak{X}^{0,1}(M), \mathfrak{X}^{0,1}(M)\big) = 0. \tag{2.1.3}$$

With analogous notation, we have from $\omega(X, Y) = g(JX, Y)$ that

$$\omega_{ij} = \omega_{\bar{i}\bar{j}} = 0, \qquad \omega_{i\bar{j}} = \sqrt{-1}g_{i\bar{j}}.$$

Thus, in coordinates we can write

$$\omega = \sqrt{-1}g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j. \tag{2.1.4}$$

Note that, as a consequence of the reality condition, $\overline{g_{i\bar{j}}} = g_{\bar{i}j}$.

Let us discuss now the Kähler condition.

Theorem 2.1.3. *Consider a Hermitian manifold* (M, g). *We say that* g *is a* Kähler metric *if one of the following equivalent conditions holds:*

1) $\nabla^g J = 0$, *where* ∇^g *is the Levi-Civita connection associated to* g,

2) *the metric form* ω *is closed:* $d\omega = 0$.

In this case, ω *is called a* Kähler form. *A complex manifold* M *together with a Kähler metric on it is called a* Kähler manifold.

Proof. Let us simply set $\nabla = \nabla^g$. We know that for every $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} d\omega(X, Y, Z) = \partial_X \omega(Y, Z) + \partial_Y \omega(Z, X) + \partial_Z \omega(X, Y) \\ - \omega\big([X, Y], Z\big) - \omega\big([Y, Z], X\big) - \omega\big([Z, Y], X\big). \end{aligned}$$

On the other hand, as ∇ is compatible with g,

$$\begin{aligned} \partial_X \omega(Y, Z) = \partial_X g(JY, Z) &= g\big(\nabla_X(JY), Z\big) + g\big(JY, \nabla_X Z\big) \\ &= g\big((\nabla_X J)Y, Z\big) + g\big(J(\nabla_X Y), Z\big) + g\big(JY, \nabla_X Z\big) \\ &= g\big((\nabla_X J)Y, Z\big) + \omega\big(\nabla_X Y, Z\big) + \omega\big(Y, \nabla_X Z\big). \end{aligned}$$

Similarly for the other terms, so that

$$\begin{aligned} d\omega(X, Y, Z) = g\big((\nabla_X J)Y, Z\big) + g\big((\nabla_Y J)Z, X\big) + g\big((\nabla_Z J)X, Y\big) \\ + \omega\big(\nabla_X Y, Z\big) + \omega\big(\nabla_Y Z, X\big) + \omega\big(\nabla_Z X, Y\big) \\ + \omega\big(Y, \nabla_X Z\big) + \omega\big(Z, \nabla_Y X\big) + \omega\big(X, \nabla_Z Y\big) \\ - \omega\big([X, Y], Z\big) - \omega\big([Y, Z], X\big) - \omega\big([Z, Y], X\big) \end{aligned}$$

and, from the fact that ∇ is torsion-free and ω skew-symmetric, the above expression reduces to

$$d\omega(X, Y, Z) = g\big((\nabla_X J)Y, Z\big) + g\big((\nabla_Y J)Z, X\big) + g\big((\nabla_Z J)X, Y\big).$$

Now it is clear that if $\nabla J \equiv 0$, then ω is closed. On the other hand, consider the expression

$$\begin{aligned} d\omega(X, Y, Z) - d\omega(JX, JY, Z) = g\big((\nabla_X J)Y, Z\big) - g\big((\nabla_{JX} J)JY, Z\big) \quad (1) \\ + g\big((\nabla_Y J)Z, X\big) - g\big((\nabla_{JY} J)Z, JX\big) \quad (2) \\ + g\big((\nabla_Z J)X, Y\big) - g\big((\nabla_Z J)JX, JY\big). \quad (3) \end{aligned}$$

Let us rewrite (3) in a more convenient way. Note that $(\nabla_Z J)JX = -\nabla_Z X - J(\nabla_Z JX)$, so that

$$\begin{aligned} (3) &= g\big((\nabla_Z J)X, Y\big) - g(\nabla_Z X, JY) - g\big(J(\nabla_Z JX), JY\big) \\ &= g\big((\nabla_Z J)X, Y\big) - g(\nabla_Z X, JY) - g\big((\nabla_Z JX), Y\big) \\ &= g\big((\nabla_Z J)X, Y\big) - g(\nabla_Z X, JY) - g\big((\nabla_Z J)X, Y\big) + g\big(J(\nabla_Z X), Y\big) \\ &= 2g\big((\nabla_Z J)X, Y\big). \end{aligned}$$

On the other hand,

$$\begin{aligned}(1) &= g\big(\nabla_X JY, Z\big) - g\big(J(\nabla_X Y), Z\big) + g\big(\nabla_{JX} Y, Z\big) + g\big(J(\nabla_{JX} JY), Z\big)\\ &= g\big(\nabla_X JY, Z\big) + g\big(\nabla_X Y, JZ\big) + g\big(\nabla_{JX} Y, Z\big) - g\big(\nabla_{JX} JY, JZ\big),\end{aligned}$$

$$\begin{aligned}(2) &= g\big(\nabla_Y JZ, X\big) - g\big(J(\nabla_Y Z), X\big) - g\big(\nabla_{JY} JZ, JX\big) + g\big(J(\nabla_{JY} Z), JX\big)\\ &= g\big(\nabla_Y JZ, X\big) + g\big(\nabla_Y Z, JX\big) - g\big(\nabla_{JY} JZ, JX\big) + g\big(\nabla_{JY} Z, X\big)\\ &= -g\big(JZ, \nabla_Y X\big) - g\big(Z, \nabla_Y JX\big) + g\big(JZ, \nabla_{JY} JX\big) - g\big(Z, \nabla_{JY} X\big)\\ &\qquad + \partial_Y g(JZ, X) + \partial_Y g(Z, JX) - \partial_{JY} g(JZ, JX) + \partial_{JY} g(Z, X)\\ &= -g\big(\nabla_Y X, JZ\big) - g\big(\nabla_Y JX, Z\big) + g\big(\nabla_{JY} JX, JZ\big) - g\big(\nabla_{JY} X, Z\big).\end{aligned}$$

As a consequence,

$$\begin{aligned}(1) + (2) &= g\big([X, JY], Z\big) + g\big([X, Y], JZ\big) + g\big([JX, Y], Z\big) - g\big([JX, JY], JZ\big)\\ &= g\big(N_J(X, Y), JZ\big) = 0,\end{aligned}$$

as for integrable almost complex structures, the Nijenhius tensor vanishes. Thus, we finally have

$$d\omega(X, Y, Z) - d\omega(JX, JY, Z) = 2g\big((\nabla_Z J)X, Y\big),$$

so that $d\omega = 0$ implies $\nabla J = 0$. ☐

Remark 2.1.4. In local holomorphic coordinates (z^i), the Kähler condition reads

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}. \tag{2.1.5}$$

Note that the closedness condition for ω implies that every Kähler manifold is actually symplectic. In particular, we have the volume form

$$\mathrm{Vol}_\omega = \frac{\omega^n}{n!} \in H^{2n}_{dR}(M, \mathbb{R}). \tag{2.1.6}$$

A topological consequence is that every compact Kähler manifold (or more generally, every compact symplectic manifold) has non-trivial second cohomology group: if by contradiction $\omega = d\alpha$, then

$$0 < n!\,\mathrm{Vol}_\omega(M) = \int_M (d\alpha)^n = \int_M d\big(\alpha \wedge (d\alpha)^{n-1}\big) = 0.$$

As a consequence, $\mathbb{S}^6$ can not admit a Kähler structure (apart from $\mathbb{S}^2$, it was the only sphere with an almost complex structure by the result of Serre and Borel), while in the next example we will see that $\mathbb{S}^2 \cong \mathbb{CP}^1$ is actually Kähler. In section 2.2 we will analyse more topological constraints on compact Kähler manifold.

In the following we will talk about "Kähler metric" for both the Riemannian metric or the Kähler form without ambiguities, since in the Kähler world one determines the other. Let us see some examples.

Example 2.1.1. The complex space $\mathbb{C}^n$ is obviously Kähler, with the *Euclidean metric* defined by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i. \tag{2.1.7}$$

Another example is a complex curve: since every complex manifold admits a Hermitian metric and, for complex curves, the third cohomology group vanishes, then every Hermitian metric is actually Kähler. Thus, in dimension 1 being Kähler is a topological property.

Let us construct a Kähler form on the complex projective space $\mathbb{P}^n$. Consider the projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and, for a sufficiently small open set U, fix a local holomorphic section

$$s\colon U \to \mathbb{C}^{n+1} \setminus \{0\}, \qquad \pi \circ s = \mathrm{id}_U\,. \tag{2.1.8}$$

A section of this type always exists, take for example on the open sets U_i of the standard cover the sections

$$s_i\big([Z]\big) = \frac{Z}{Z^i}. \tag{2.1.9}$$

Define now the local 2-form

$$\omega|_U = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|s\|^2, \tag{2.1.10}$$

where $\|\cdot\|$ is the standard norm in $\mathbb{C}^{n+1}$. Comparing the formula with the curvature of the natural metric on the tautological bundle $\mathcal{O}(-1)$ defined in Example 1.3.3 and the formula for the curvature of a line bundle (see Proposition 1.3.15), we immediately have that ω is a Kähler metric. However, let us explicitly check the needed properties. Firstly, note that the definition is well-posed, since s has values in $\mathbb{C}^{n+1} \setminus \{0\}$. In local coordinates, we have

$$\omega|_U = \frac{\sqrt{-1}}{2\pi} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \|s\|^2 \, dz^i \wedge d\bar{z}^j, \tag{2.1.11}$$

so that $\omega|_U$ is a real form of type (1,1) on each U. Let us show that the form does not depends on the choice of the open set and the section. For a different section s' on U', write $s' = fs$ with f a holomorphic function on the overlap $U \cap U'$. Then

$$\omega'|_{U'} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|fs\|^2 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|f\|^2 + \omega|_U. \tag{2.1.12}$$

On the other hand, writing $\|f\|^2 = f \cdot \bar{f}$,

$$\partial\bar{\partial} \log \|fs\|^2 = \partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f} = 0, \tag{2.1.13}$$

as $\bar{\partial} \log f = 0$ for f holomorphic and $\partial \log \bar{f} = 0$ for $\log \bar{f}$ antiholomorphic. Thus, $\omega'|_{U'} = \omega|_U$ and we have a global real form of type (1,1). Note that it is closed, as it is locally exact:

$$\omega|_U = d\left(\frac{\sqrt{-1}}{2\pi} \bar{\partial} \log \|s\|^2\right). \tag{2.1.14}$$

The positivity has been check in Example 1.3.4. The Kähler metric so obtained is called the *Fubini-Study metric* on $\mathbb{P}^n$.
It is interesting to note that the quaternionic projective spaces

$$\mathbb{HP}^n = \frac{\mathbb{H}^{n+1} \setminus \{0\}}{Q \sim \lambda Q}, \qquad \lambda \in \mathbb{H}^* \tag{2.1.15}$$

are not Kähler, since the second cohomology group vanishes, in contradiction with the previous remark about compact Kähler manifolds.

We can show now another useful characterisation of the Kähler condition: the existence of *holomorphic normal coordinates*.

Theorem 2.1.5. *Let* M *be a complex manifold and consider a Riemannian metric* g *compatible with the complex structure. Then* g *is Kähler if and only if for every point* $p \in M$ *there exist local holomorphic coordinates* (z^i) *centred at* p*, such that*

$$g_{i\bar{j}} = \delta_{ij} + O\big(|z|^2\big). \tag{2.1.16}$$

Proof. Clearly, if there exists such coordinate systems, then $d\omega|_p = 0$ for every point $p \in M$, so that the metric is Kähler. Conversely, suppose that ω is Kähler. With a holomorphic coordinate system centred at p, we can perform a holomorphic change of coordinate such that $g_{i\bar{j}}(0) = \delta_{ij}$. Thus, we can write

$$\omega = \sqrt{-1}\Big(\delta_{ij} + a_{ijk}z^k + a_{ij\bar{k}}\bar{z}^k + O\big(|z|^2\big)\Big)dz^i \wedge d\bar{z}^j.$$

Note that the reality and Kähler conditions imply that

$$a_{ij\bar{k}} = \overline{a_{jik}}, \qquad a_{ijk} = a_{kji}.$$

We perform now a quadratic change of variable:

$$z^k = w^k - \frac{1}{2}b_{krs}w^r w^s, \qquad b_{krs} = b_{ksr}.$$

A simple computation shows that

$$\omega = \sqrt{-1}\big(\delta_{rs} + (a_{rsk} - b_{skr})w^k + (a_{rs\bar{k}} - \overline{b_{rks}})\bar{w}^k + O\big(|w|^2\big)\big)dw^r \wedge d\bar{w}^s.$$

Let us choose $b_{skr} = a_{rsk}$. Then the symmetry requirement on b is ensured by the Kähler condition on a, while thanks to the reality condition $\overline{b_{rks}} = \overline{a_{srk}} = a_{rs\bar{k}}$. Thus, we find

$$\omega = \sqrt{-1}\Big(\delta_{rs} + O\big(|w|^2\big)\Big)dw^r \wedge d\bar{w}^s.$$

□

Remark 2.1.6. Note that also in Riemannian geometry we can always find local normal coordinates. The crucial point in the Kähler case is that such normal coordinate system is also holomorphic. The above theorem shows that the holomorphic condition on the normal coordinates is actually equivalent to the Kähler condition.

An immediate corollary of the existence of normal holomorphic coordinates is the equality between the symplectic volume form induced by ω and the Riemannian volume form induced by g.

Corollary 2.1.7. *Let* (M, ω) *be a Kähler manifold. Then the volume form* $\mathrm{Vol}_\omega = \frac{\omega^n}{n!}$ *coincides with the Riemannian volume form.*

Proof. We can check the statement pointwise. Fix a point $p \in M$ and holomorphic normal coordinates centred at p. Then the Kähler form in p is $\omega = \sqrt{-1}\sum_i dz^i \wedge d\bar{z}^i$, so that

$$\begin{aligned}\mathrm{Vol}_\omega &= (\sqrt{-1})^n\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= 2^n\, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n,\end{aligned}$$

as $dz^i \wedge d\bar{z}^i = -2\sqrt{-1} dx^i \wedge dy^i$. On the other hand, from $g_{i\bar{j}} = \delta_{ij}$ and $g_{ij} = g_{\bar{i}\bar{j}} = 0$, a simple computation shows that

$$\begin{aligned}g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\delta_{ij}, \\ g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) &= 0.\end{aligned}$$

Thus, the matrix of g associated to the basis $\{\, \partial/\partial x^i, \partial/\partial y^i \,\}$ is $2\mathrm{Id}_{\mathbb{R}^{2n}}$. In particular, the square root of its determinant is 2^n, so that Vol_ω is the Riemannian volume form. □

Remark 2.1.8. Note that in local holomorphic coordinates (z^i), we can write

$$\mathrm{Vol}_\omega = (\sqrt{-1})^n \det(g)\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n, \tag{2.1.17}$$

where $\det(g) = \det(g_{i\bar{j}})_{1 \leqslant i,j \leqslant n}$.

The last characterisation of the Kähler we want to introduce is the relation between the Levi-Civita connection of g and the Chern connection of h.

Theorem 2.1.9. *Let* M *be a Hermitian manifold, with Riemannian metric* g *and associated Hermitian metric* h*. Then* M *is Kähler if and only if the Levi-Civita connection* ∇^g *associated to* g *coincides with the Chern connection* ∇^h *associated to* h*.*

Proof. Suppose that g is Kähler. We firstly note that the complex extensions of the Levi-Civita and Chern connections coincides in the case of the standard Euclidean metric on $\mathbb{C}^n$, since in both cases they are determined by vanishing connection matrices with respect to the natural trivialization of the complex tangent bundle of $\mathbb{C}^n$. Further, in both cases the connection matrices at a point depends just on the the local expression of the metric up to first order. From the previous lemma, we can choose local coordinates around each point such that the Kähler metric is the standard Euclidean metric up to first order. Thus, the matrices of the the connections coincide at each point as they do for the the metric on $\mathbb{C}^n$, proving that $\nabla^g = \nabla^h$.

Suppose now $\nabla^g = \nabla^h$. We know that, by definition, J is covariantly constant with respect to the Chern connection:

$$\nabla^h JX = J\nabla^h X \qquad \forall X \in \mathfrak{X}(M),$$

as it must be $\mathbb{C}$-linear. Thus, J is covariantly constant with respect to the Levi-Civita connection, so it is Kähler. □

The Kähler condition carries many symmetries with it. Let us see, for example, the simplifications on the Christoffel symbols. Set

$$\begin{aligned} \partial_i &= \frac{\partial}{\partial z^i}, & \partial_{\bar{\imath}} &= \frac{\partial}{\partial \bar{z}^i}, \\ \nabla_i &= \nabla_{\partial_i}, & \nabla_{\bar{\imath}} &= \nabla_{\partial_{\bar{\imath}}}, \end{aligned} \tag{2.1.18}$$

for ∇ the Levi-Civita connection extended to $T_{\mathbb{C}}M$ by complex linearity. Define the Christoffel symbols

$$\nabla_i \partial_j = \Gamma^k_{ij}\,\partial_k + \Gamma^{\bar{k}}_{ij}\,\partial_{\bar{k}} \tag{2.1.19}$$

and similarly for the other combinations.

Lemma 2.1.10. *The only non-vanishing Christoffel symbols are the non-mixed ones: Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}}$. Further,*

$$\Gamma^k_{ij} = g^{k\bar{r}}\partial_i g_{j\bar{r}}, \qquad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = \overline{\Gamma^k_{ij}}. \tag{2.1.20}$$

Proof. We have that

$$\begin{aligned} \Gamma^k_{i\bar{\jmath}} &= \frac{1}{2} g^{kr}\left(\partial_i g_{\bar{\jmath}r} + \partial_{\bar{\jmath}} g_{ri} - \partial_r g_{i\bar{\jmath}}\right) + \frac{1}{2} g^{k\bar{r}}\left(\partial_i g_{\bar{\jmath}\bar{r}} + \partial_{\bar{\jmath}} g_{\bar{r}i} - \partial_{\bar{r}} g_{i\bar{\jmath}}\right) \\ &= \frac{1}{2} g^{k\bar{r}}\left(\partial_{\bar{\jmath}} g_{\bar{r}i} - \partial_{\bar{r}} g_{i\bar{\jmath}}\right) = 0, \end{aligned}$$

where in the last two steps we have used the fact that mixed metric elements vanishes and the Kähler conditions. Similarly for the other mixed symbols. With the same procedure, we find

$$\Gamma^k_{ij} = g^{k\bar{r}}\partial_i g_{j\bar{r}}, \qquad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = g^{\bar{k}r}\partial_{\bar{\imath}} g_{\bar{\jmath}r}.$$

Finally, thanks to the reality condition,

$$\overline{\Gamma^k_{ij}} = \overline{g^{k\bar{r}}}\,\overline{\partial_i g_{j\bar{r}}} = g^{\bar{k}r}\partial_{\bar{\imath}} g_{\bar{\jmath}r} = \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}}.$$

□

Other simplifications for a Kähler metric can be shown for the Riemann curvature tensor.

Definition 2.1.11. Let (M, g) be Kähler. Define the *Riemann curvature tensor* as $R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(\mathrm{End}(TM))$,

$$R(X, Y) = \left[\nabla_X, \nabla_Y\right] - \nabla_{[X,Y]}. \tag{2.1.21}$$

We extend R to $T_{\mathbb{C}}M$ by complex linearity.

Lemma 2.1.12. *Let* (M, g) *be Kähler. Then*

$$g\big(R(X,Y)Z, W\big) = g\big(R(X,Y)JZ, JW\big). \tag{2.1.22}$$

Further, if Z *and* W *belongs to the same* J*-eigenspace, then* $g\big(R(X,Y)JZ, JW\big) = 0$.

Proof. As J is covariantly constant, $R(X,Y)JZ = JR(X,Y)Z$. Thus,

$$g\big(R(X,Y)JZ, JW\big) = g\big(JR(X,Y)Z, JW\big) = g\big(R(X,Y)Z, W\big).$$

The second statement follows from a previous remark about the orthogonality of J-eigenspaces. □

For a fixed holomorphic set of local coordinates, set

$$R_{i\bar{j}k\bar{l}} = g\big(R(\partial_k, \partial_{\bar{l}})\partial_i, \partial_{\bar{j}}\big)$$

and similarly for the other combinations.

Corollary 2.1.13. *Let* (M, g) *be Kähler. The only non-vanishing elements of the Riemann curvature tensor are* $R_{i\bar{j}k\bar{l}}$ *and those obtained from* $R_{i\bar{j}k\bar{l}}$ *by symmetries of* R*. Further,*

$$R_{i\bar{j}k\bar{l}} = g^{r\bar{s}}(\partial_k g_{i\bar{s}})(\partial_{\bar{l}} g_{r\bar{j}}) - \partial_k \partial_{\bar{l}} g_{i\bar{j}}. \tag{2.1.23}$$

Proof. From the previous lemma and the symmetries of the Riemann curvature tensor, we obtain the first statement. Using the fact that mixed Christoffel symbols vanish, we find the expression in local coordinates:

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= g\big((\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k)\partial_i, \partial_{\bar{j}}\big) \\ &= -g\big(\nabla_{\bar{l}}(\Gamma^r_{ik}\partial_r), \partial_{\bar{j}}\big) = -g\big((\partial_{\bar{l}}\Gamma^r_{ik})\partial_r, \partial_{\bar{j}}\big) \\ &= -(\partial_{\bar{l}}\Gamma^r_{ik})g_{r\bar{j}}. \end{aligned}$$

Recall the expression $\Gamma^r_{ik} = g^{r\bar{s}}\partial_i g_{k\bar{s}}$, so that

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= -(\partial_{\bar{l}} g^{r\bar{s}})(\partial_i g_{k\bar{s}})g_{r\bar{j}} - g^{r\bar{s}}(\partial_{\bar{l}}\partial_i g_{k\bar{s}})g_{r\bar{j}} \\ &= g^{r\bar{s}}(\partial_i g_{k\bar{s}})(\partial_{\bar{l}} g_{r\bar{j}}) - \partial_{\bar{l}}\partial_i g_{k\bar{j}} \\ &= g^{r\bar{s}}(\partial_k g_{i\bar{s}})(\partial_{\bar{l}} g_{r\bar{j}}) - \partial_k \partial_{\bar{l}} g_{i\bar{j}}, \end{aligned}$$

where in the last step we used the Kähler condition. □

Definition 2.1.14. Let (M, g) be a Kähler manifold. For a fixed local holomorphic chart (z^i), define the *Ricci form*

$$\mathrm{Ric} = \sqrt{-1} R_{k\bar{l}}\, dz^k \wedge d\bar{z}^l, \tag{2.1.24}$$

where $R_{k\bar{l}} = g^{i\bar{j}} R_{i\bar{j}k\bar{l}}$.

Lemma 2.1.15. *The Ricci form is a closed, real* 2*-form, with*

$$R_{k\bar{l}} = -\partial_k \partial_{\bar{l}} \log(\det g), \tag{2.1.25}$$

where $\det g = \det(g_{i\bar{j}})_{1 \leqslant i,j \leqslant n}$.

Proof. Assuming the expression for $R_{k\bar{l}}$, we immediately have that Ric is real and closed, since locally exact:

$$\mathrm{Ric} = -\sqrt{-1}\partial\bar{\partial}\log(\det g) = -\sqrt{-1}d\bar{\partial}\log(\det g).$$

To prove the local expression, firstly note that we can rewrite $R_{k\bar{l}}$ as follows:

$$\begin{aligned} R_{k\bar{l}} &= g^{i\bar{j}}g^{r\bar{s}}(\partial_k g_{i\bar{s}})(\partial_{\bar{l}} g_{r\bar{j}}) - g^{i\bar{j}}\partial_k\partial_{\bar{l}} g_{i\bar{j}} \\ &= -g^{i\bar{j}}(\partial_k g^{r\bar{s}})g_{i\bar{s}}(\partial_{\bar{l}} g_{r\bar{j}}) - \partial_k(g^{i\bar{j}}\partial_{\bar{l}} g_{i\bar{j}}) + (\partial_k g^{i\bar{j}})(\partial_{\bar{l}} g_{i\bar{j}}) \\ &= -\partial_k(g^{i\bar{j}}\partial_{\bar{l}} g_{i\bar{j}}). \end{aligned}$$

So we just need to prove that $\partial_{\bar{l}}\log(\det g) = g^{i\bar{j}}\partial_{\bar{l}} g_{i\bar{j}}$. Observe that

$$\partial_{\bar{l}}\log(\det g) = \frac{1}{\det g}(\partial_{\bar{l}} g_{i\bar{j}})\frac{\partial \det g}{\partial g_{i\bar{j}}}.$$

With the Laplace expansion along the $\bar{j}$th column, we find

$$\begin{aligned} \frac{\partial \det g}{\partial g_{i\bar{j}}} &= \frac{\partial}{\partial g_{i\bar{j}}}\sum_{k=1}^{n}(-1)^{k+j} g_{k\bar{j}} G_{k\bar{j}} \\ &= (-1)^{i+j} G_{i\bar{j}} = (\det g) g^{\bar{j}i} = (\det g) g^{i\bar{j}}. \end{aligned}$$

Here $G_{k\bar{j}}$ is the $(k, \bar{j})$-minor of g, which does not depend on $g_{i\bar{j}}$. Thus, we have the formula we were looking for. □

It turns out that the cohomology class of the Ricci form does not depend on the metric, so that it defines an invariant of the complex manifold M: the first Chern class.

Proposition 2.1.16. *Let (M, g) be a Kähler manifold. The cohomology class of the associated Ricci form is an invariant of the complex manifold M, called the* first Chern class*:*

$$c_1(M) = \frac{1}{2\pi}\left[\mathrm{Ric}(g)\right] \in H^2_{dR}(M, \mathbb{R}). \tag{2.1.26}$$

Proof. Let h be another Kähler metric on M. Then

$$\mathrm{Ric}(g) - \mathrm{Ric}(h) = -\sqrt{-1}\partial\bar{\partial}\log\frac{\det g}{\det h}.$$

Although the determinant of a metric is not a globally defined function, the ratio is a positive, globally defined function. Indeed, for a change of coordinates $z^i \mapsto z'^i$, we have

$$g_{i\bar{j}} = \frac{\partial z'^k}{\partial z^i}\frac{\partial \bar{z}'^l}{\partial \bar{z}^j} g'_{k\bar{l}},$$

so that $\det g = \left|\det\frac{\partial z'}{\partial z}\right|^2 \det g'$. The same holds for h, so that the ratio is well-defined. As a consequence, the difference between Ricci forms associated to different Kähler metrics is a global exact form:

$$\mathrm{Ric}(g) - \mathrm{Ric}(h) = d\left(-\sqrt{-1}\bar{\partial}\log\frac{\det g}{\det h}\right).$$

□

The normalization factor is due to the fact that $c_1(M)$ is actually an integral cohomology class, which coincides with the first Chern class of the anticanonical line bundle of M, defined in section 1.3.2:

$$c_1(M) = -c_1(K_M). \tag{2.1.27}$$

This is a consequence of the fact that a Hermitian metric g on M induces the Hermitian metric $\det(g)$ on the line bundle $K_M^* = \Lambda^n T^{1,0}M$, whose curvature form is given by $\Omega = -\partial\bar{\partial}\log(\det g) = -\sqrt{-1}\,\mathrm{Ric}$. Thus,

$$c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}] = \frac{\sqrt{-1}}{2\pi}[\Omega] = c_1(K_M^*) = -c_1(K_M).$$

The first Chern class is of fundamental importance in the famous Calabi-Yau theorem, conjectured by Calabi in [Calabi, 1954] and definitely proved by Yau in [Yau, 1978].

Theorem 2.1.17 (Calabi-Yau theorem). *Let (M, ω_0) be a compact Kähler manifold, α a closed, real form of type $(1,1)$ representing $c_1(M)$. Then there exists a unique Kähler form ω such that $[\omega] = [\omega_0]$ and*

$$\mathrm{Ric}(\omega) = 2\pi\alpha. \tag{2.1.28}$$

The theorem can be derived by Yau's theorem on the complex Monge-Ampére equation: for a compact Kähler manifold (M, ω), consider a smooth function $F\colon M \to \mathbb{R}$. In terms of the "Kähler potential" ϕ for ω (see the $\partial\bar{\partial}$-lemma 2.2.18), the equation reads

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^n = e^F\omega^n.$$

Yau's theorem shows that a solution to the complex Monge-Ampère equation always exists. More precisely:

Theorem 2.1.18 (Yau theorem). *Let (M, ω) be a compact Kähler manifold, consider a smooth function $F\colon M \to \mathbb{R}$, normalized as*

$$\int_M e^F\omega^n = \int_M \omega^n. \tag{2.1.29}$$

Then there exists a smooth function $\phi\colon M \to \mathbb{R}$, unique up to an additive constant, such that $\omega + \sqrt{-1}\partial\bar{\partial}\phi$ is positive and satisfies the complex Monge-Ampère equation

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^n = e^F\omega^n. \tag{2.1.30}$$

A consequence of Calabi-Yau theorem is that on a compact *Calabi-Yau manifold*, that is a Kähler manifold with vanishing first Chern class, every Kähler class contains a unique Ricci-flat metric. This is a particular example of *Kähler-Einstein metric*, *i.e.* a Kähler metric ω whose Ricci tensor is proportional to the Kähler form:

$$\mathrm{Ric}(\omega) = \lambda\omega \qquad \text{for some } \lambda \in \mathbb{R}. \tag{2.1.31}$$

Note that a necessary condition for the existence of a Kähler-Einstein metric is that the first Chern class is either definite for $\lambda \neq 0$, or vanishes for $\lambda = 0$. The case

$\lambda = 0$ on compact manifolds is a corollary of the Calabi-Yau theorem, so that the above necessary condition is also sufficient on Calabi-Yau manifolds. In the case $c_1(M) < 0$ the above condition is again sufficient, accordingly to the following theorem due to Aubin and Yau (see [Aubin, 1978; Yau, 1978]).

Theorem 2.1.19 (Aubin-Yau theorem). *A compact Kähler manifold with negative first Chern class admits a unique Kähler metric $\omega \in -2\pi c_1(M)$ such that $\mathrm{Ric}(\omega) = -\omega$.*

The case of *Fano manifolds*, that is Kähler manifolds with positive first Chern class, is the most complicated one. Here algebro-geometric obstructions occur and the existence of Kähler-Einstein metrics on Fano manifolds in relation to stability in the sense of geometric invariant theory was conjectured by Yau, Tian and Donaldson. The conjecture was recently solved by Chen, Donaldson and Song in the series of papers [Chen, Donaldson, and Song, 2015a,b,c]. We refer to [Székelyhidi, 2014] for further readings.

Another important object is the scalar curvature associated to a Kähler metric.

Definition 2.1.20. Let (M, g) be a Kähler manifold. For a fixed local holomorphic chart (z^i), define the *scalar curvature*

$$S = g^{i\bar{j}} R_{i\bar{j}}. \tag{2.1.32}$$

It is a globally well-defined smooth function on M.

Example 2.1.2. Let us compute the Ricci form of the Fubini-Study metric on $\mathbb{P}^n$. From Example 2.1.1, we have that on U_i the metric can be written as

$$g_{i\bar{j}} = \frac{1}{2\pi}\partial_i \partial_{\bar{j}} \log\left(1 + |z|^2\right) = \frac{1}{2\pi}\frac{\delta_{ij}\left(1 + |z|^2\right) - \bar{z}^i z^j}{(1 + |z|^2)^2}. \tag{2.1.33}$$

Thus, from the matrix determinant lemma[1], we find

$$\det g = \frac{1}{(2\pi)^n}\frac{1}{(1 + |z|^2)^{2n}}\left(1 - \frac{\bar{z}^i \delta_{ij} z^j}{1 + |z|^2}\right)\left(1 + |z|^2\right)^n = \frac{1}{(2\pi)^n}\frac{1}{(1 + |z|^2)^{n+1}}. \tag{2.1.34}$$

As a consequence,

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\left(\frac{1}{(1 + |z|^2)^{n+1}}\right) = (n+1)\partial_i \partial_{\bar{j}} \log\left(1 + |z|^2\right) = 2\pi(n+1) g_{i\bar{j}}. \tag{2.1.35}$$

Further, we find that the Fubini-Study metric on $\mathbb{P}^n$ has constant scalar curvature $S_{FS} = 2\pi n(n+1)$. Note that the metric form satisfies the Kähler-Einstein equation $\mathrm{Ric}_{FS} = 2\pi(n+1)\omega_{FS}$. Thus, the projective spaces are Fano manifolds and the

[1] The matrix determinant lemma states that for a invertible matrix A, a column vector v and a row vector u, we have

$$\det(A + vu) = (1 + uA^{-1}v)\det(A).$$

Fubini-Study metric is an example of Kähler-Einstein metric.
We can also compute the volume of $\mathbb{P}^n$ as follows:

$$\begin{aligned}\mathrm{Vol}_{FS}(\mathbb{P}^n) &= \int_{\mathbb{C}^n} (\sqrt{-1})^n \det(g) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \int_{\mathbb{R}^{2n}} \frac{1}{\pi^n} \frac{1}{(1+|(x,y)|^2)^{n+1}} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \\ &= \frac{2}{(n-1)!} \int_0^{+\infty} \frac{r^{2m-1}}{(1+r^2)^{n+1}} dr.\end{aligned}$$

A simple computation shows that the integral equals $\frac{1}{2n}$, so that

$$\mathrm{Vol}_{FS}(\mathbb{P}^n) = \frac{1}{n!}. \tag{2.1.36}$$

In particular, $\int_{\mathbb{P}^n} [\omega_{FS}]^n = 1$.

2.2 Hodge theory

By the de Rham theorem, we know that the de Rham cohomology groups of a smooth manifold are a topological invariant. However, if a compact manifold is equipped with a Riemannian metric, we can exhibit natural representatives: the harmonic forms. In this section we will develop the theory of harmonic forms on compact Kähler manifolds, following [Huybrechts, 2005] and [Voisin, 2002]. Thanks to the Kähler identities, Hodge theory on Kähler manifolds presents an extraordinary amount of symmetries. This fact will allows us to prove the Hard Lefschetz theorem and the Lefschetz decomposition formula in cohomology, leading to topological constraints that a compact Kähler manifold must satisfy.

Consider M be a complex manifold. Let us gather some of the linear operators defined on the fibres of the bundles $\Lambda^k T^*_{\mathbb{C}} M$ and $\Lambda^p T^{*\,1,0} M \otimes \Lambda^q T^{*\,0,1} M$. We will call a generic fibre of these bundles as $\Lambda^k_{\mathbb{C}}$ and $\Lambda^{p,q}$ respectively. We will have

$$\Lambda^\bullet_{\mathbb{C}} = \bigoplus_{k=0}^{2n} \Lambda^k_{\mathbb{C}}, \qquad \Lambda^k_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q} \tag{2.2.1}$$

and analogously for the real counterpart $\Lambda^\bullet$. The following operations can obviously be extended to smooth sections of proper bundles.

Conjugation. The complex structure defines the conjugation

$$\begin{aligned}\Lambda^{p,q} &\longrightarrow \Lambda^{q,p} \\ \alpha &\longmapsto \overline{\alpha}.\end{aligned} \tag{2.2.2}$$

Projection operators. We have already defined the projection operators

$$\begin{aligned}\pi^k \colon \Lambda^\bullet_{\mathbb{C}} &\longrightarrow \Lambda^k_{\mathbb{C}} \\ \pi^{p,q} \colon \Lambda^\bullet_{\mathbb{C}} &\longrightarrow \Lambda^{p,q}.\end{aligned} \tag{2.2.3}$$

They satisfy the relations $\overline{\pi^k} = \pi^k$ and $\overline{\pi^{p,q}} = \pi^{q,p}$.

Almost complex structure. The natural almost complex structure on $T_{\mathbb{C},p}M$ can be extended to the wedge products as

$$J\colon \Lambda^k_{\mathbb{C}} \longrightarrow \Lambda^k_{\mathbb{C}}, \tag{2.2.4}$$

where $(J\alpha)(X_1, \dots, X_k) = \alpha(JX_1, \dots, JX_k)$. Note that $J^2 = (-1)^k \,\mathrm{id}$. On elements of type (p, q), we have the expression

$$J\alpha = (\sqrt{-1})^{p-q}\alpha. \tag{2.2.5}$$

Type operator. Define the *type operator* $H\colon \Lambda^\bullet_{\mathbb{C}} \to \Lambda^\bullet_{\mathbb{C}}$ as

$$H = \sum_{k=0}^{2n} (k-n)\pi^k. \tag{2.2.6}$$

Note that $\overline{H} = H$.

Hodge dual. A Hermitian metric on $T_{\mathbb{C},p}M$ defines a Hermitian non-degenerate paring $\langle \cdot, \cdot \rangle$ on $\Lambda^{p,q} \times \Lambda^{p,q}$ as

$$\langle \alpha, \beta \rangle = g^{IK} g^{JL} \alpha_{IJ} \overline{\beta_{KL}}. \tag{2.2.7}$$

where we have set

$$\begin{aligned} \alpha_{IJ} &= \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_p}, & \beta_{IJ} &= \beta_{k_1 \cdots k_q \bar{l}_1 \cdots \bar{l}_p}, \\ g^{IK} &= g^{i_1 \bar{k}_1} \cdots g^{i_p \bar{k}_p}, & g^{JL} &= g^{\bar{j}_1 l_1} \cdots g^{\bar{j}_q l_q}. \end{aligned} \tag{2.2.8}$$

On the other hand, we have the non-degenerate pairing $\Lambda^{p,q} \times \Lambda^{n-p,n-q} \to \mathbb{C}$ given by the wedge product. Thus, we define for $\beta \in \Lambda^{p,q}$ its *Hodge dual* as the unique element $*\beta \in \Lambda^{n-p,n-q}$ such that

$$\alpha \wedge *\beta = \langle \alpha, \bar{\beta} \rangle \,\mathrm{Vol} \qquad \forall \alpha \in \Lambda^{p,q}, \tag{2.2.9}$$

where Vol is the metric volume form restricted to the fibre. For a compact Hermitian manifold, we obtain a Hermitian product on $\mathcal{A}^k_{\mathbb{C}}(M)$ integrating over M:

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\bar{\beta}. \tag{2.2.10}$$

The induced norm will be denoted by

$$\|\alpha\| = \left(\int_M \alpha \wedge *\bar{\alpha} \right)^{\frac{1}{2}}. \tag{2.2.11}$$

Note that the Hodge dual is real and, up to a sign, is self-adjoint and involutive. More precisely, $\overline{*\alpha} = *\bar{\alpha}$ and acting on $\Lambda^k_{\mathbb{C}}$ we have

$$\langle *\alpha, \beta \rangle = (-1)^k \langle \alpha, *\beta \rangle, \qquad *^2 = (-1)^k \,\mathrm{id}. \tag{2.2.12}$$

We will see that the product $\langle \cdot, \cdot \rangle$ can be used to define some useful adjoint operators on compact Kähler manifolds, namely the dual Lefschetz and the dual del and del-bar operators.

2.2.1 Lefschetz operators and primitive forms

In this section, we will present the Lefschetz operator, defined specifically on Kähler manifolds (which for simplicity are assumed to be compact) and the Lefschetz decomposition theorem. These are just algebraic constructions, so we will still proceed working on fibres.

Definition 2.2.1. Let (M, ω) be a compact Kähler manifold. With the above notation, we define the *Lefschetz operator* as the twisting by the Kähler form:

$$\begin{aligned} L\colon \Lambda^{p,q} &\longrightarrow \Lambda^{p+1,q+1} \\ \alpha &\longmapsto \omega \wedge \alpha. \end{aligned} \tag{2.2.13}$$

The formal adjoint of L is called the *dual Lefschetz operator*:

$$\Lambda\colon \Lambda^{p,q} \longrightarrow \Lambda^{p-1,q-1}. \tag{2.2.14}$$

Note that both operators are real: $\bar{L} = L$ and $\bar{\Lambda} = \Lambda$.

Let us find a closed expression for the dual Lefschetz.

Lemma 2.2.2. *The following equality holds:*

$$\Lambda = *^{-1} L * . \tag{2.2.15}$$

Proof. We have that

$$\langle \alpha, L\beta \rangle \operatorname{Vol} = \langle L\beta, \alpha \rangle \operatorname{Vol} = \omega \wedge \beta \wedge * \bar{\alpha}.$$

As ω is a 2-form, $\omega \wedge \beta = \beta \wedge \omega$. Thus, using the fact that ω, L and $*$ are real,

$$\begin{aligned} \langle \alpha, L\beta \rangle \operatorname{Vol} &= \beta \wedge * \overline{\left(*^{-1} \omega \wedge * \alpha \right)} \\ &= \left\langle \beta, *^{-1} L * \alpha \right\rangle \operatorname{Vol} \\ &= \left\langle *^{-1} L * \alpha, \beta \right\rangle \operatorname{Vol}. \end{aligned}$$

□

We are ready to show that the Lefschetz operators defined on $\Lambda^\bullet_{\mathbb{C}}$ or $\Lambda^\bullet$, together with the type operator H, form a Lie algebra representation of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{R})$ respectively.

Theorem 2.2.3. *The maps* $\mathfrak{sl}(2, \mathbb{C}) \to \operatorname{End}\left(\Lambda^\bullet_{\mathbb{C}}\right)$ *and* $\mathfrak{sl}(2, \mathbb{R}) \to \operatorname{End}\left(\Lambda^\bullet\right)$ *defined by*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto L, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \Lambda, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H, \tag{2.2.16}$$

are Lie algebra morphisms, for both the complex and real cases.

Proof. Let us focus on the complex case, as the real one is completely analogue. We just need to check that

$$[H, L] = 2L, \qquad [H, \Lambda] = -2\Lambda, \qquad [L, \Lambda] = H.$$

For $\alpha \in \Lambda^k_{\mathbb{C}}$, we have

$$[H, L]\alpha = (k + 2 - n)L\alpha - (k - n)L\alpha = 2L\alpha$$

and similarly for $[H, \Lambda] = 2\Lambda$. The interesting part is the computation of $[L, \Lambda]$. We can perform induction on the dimension n. For $n = 1$, we have the decomposition $\Lambda^\bullet_{\mathbb{C}} = \Lambda^0_{\mathbb{C}} \oplus \Lambda^1_{\mathbb{C}} \oplus \Lambda^2_{\mathbb{C}}$. Further,

$$\begin{aligned} &L|_{\Lambda^0_{\mathbb{C}}} : 1 \mapsto \omega && \Lambda|_{\Lambda^0_{\mathbb{C}}} = \Lambda|_{\Lambda^1_{\mathbb{C}}} = 0 \\ &L|_{\Lambda^1_{\mathbb{C}}} = L|_{\Lambda^2_{\mathbb{C}}} = 0 && \Lambda|_{\Lambda^2_{\mathbb{C}}} : \omega \mapsto 1. \end{aligned}$$

Thus, we find

$$\begin{aligned} &[L, \Lambda]|_{\Lambda^0_{\mathbb{C}}} = -\Lambda L|_{\Lambda^0} = -\operatorname{id} = (0 - 1)\operatorname{id} = H|_{\Lambda^0_{\mathbb{C}}} \\ &[L, \Lambda]|_{\Lambda^1_{\mathbb{C}}} = 0 = (1 - 1)\operatorname{id} = H|_{\Lambda^1_{\mathbb{C}}} \\ &[L, \Lambda]|_{\Lambda^2_{\mathbb{C}}} = L\Lambda|_{\Lambda^2} = \operatorname{id} = (2 - 1)\operatorname{id} = H|_{\Lambda^2_{\mathbb{C}}}. \end{aligned}$$

This proves the theorem for $n = 1$. For $n > 1$, choose a non-trivial g_p-orthogonal decomposition of $T_{\mathbb{C},p}M = W_1 \oplus W_2$ with $J(W_i) \subset W_i$. Such decomposition always exists: take for example the holomorphic tangent space and its orthogonal complement. Then

$$\Lambda^\bullet_{\mathbb{C}} = \left(\Lambda^\bullet W_1^*\right) \otimes \left(\Lambda^\bullet W_2^*\right)$$

and $\Lambda^\bullet_{\mathbb{C}}$ will be generated by split forms $\alpha_1 \otimes \alpha_2$, with $\alpha_i \in \Lambda^\bullet W_i^*$. Let us prove that ω splits:

$$\omega = \omega_1 \oplus \omega_2, \qquad \omega_i \in \Lambda^2 W_i^*.$$

Indeed, the orthogonality of W_1 and W_2 implies that $g_p = g_1 \oplus g_2$, with g_i a Hermitian product on W_i, while the requirement $J(W_i) \subset W_i$ implies that $J = J_1 \oplus J_2$ with $J_i \in \operatorname{End}(W_i)$. Since $\omega(X, Y) = g(JX, Y)$, we have the decomposition for ω. As a consequence,

$$L = L_1 \otimes \operatorname{id} + \operatorname{id} \otimes L_2.$$

Further, as $\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle = \langle \alpha_1, \beta_1 \rangle \langle \alpha_2, \beta_2 \rangle$, we obtain that

$$\Lambda = \Lambda_1 \otimes \operatorname{id} + \operatorname{id} \otimes \Lambda_2,$$

with Λ_i the dual Lefschetz of L_i. By induction, we have the thesis. □

Corollary 2.2.4. *On $\Lambda^k_{\mathbb{C}}$ or Λ^k, we have*

$$[L^r, \Lambda] = r(k - n + r - 1)L^{r-1}. \tag{2.2.17}$$

Proof. We proceed by induction on r. For $r = 1$, it follows from the previous theorem. Suppose now that the formula holds for $r - 1$. Then for $\alpha \in \Lambda^k_{\mathbb{C}}$ or Λ^k, we have

$$\begin{aligned}
[L^r, \Lambda]\alpha &= L\big(L^{r-1}\Lambda\alpha\big) - \big(\Lambda L\big)L^{r-1}\alpha \\
&= L\big(L^{r-1}\Lambda\big) - L\big(\Lambda L^{r-1}\alpha\big) + \big(L\Lambda\big)L^{r-1}\alpha - \big(\Lambda L\big)L^{r-1}\alpha \\
&= L[L^{r-1}, \Lambda]\alpha + [L, \Lambda]L^{r-1}\alpha \\
&= \big((r-1)(k-n+r-2) + (k+2(r-1)-n)\big)L^{r-1}\alpha \\
&= r(k-n+r-1)L^{r-1}\alpha.
\end{aligned}$$

Here we used the fact that $L^{r-1}\alpha \in \Lambda^{k+2(r-1)}_{\mathbb{C}}$. □

Corollary 2.2.5. *Let $\alpha \in \Lambda^k_{\mathbb{C}}$ such that $\Lambda\alpha = 0$. Then the linear space*

$$\operatorname{span}\left\{\, \alpha, L\alpha, \dots, L^{\mathrm{top}}\alpha \,\right\} \subset \Lambda^{\bullet}_{\mathbb{C}} \tag{2.2.18}$$

is a subrepresentation of $\mathfrak{sl}(2, \mathbb{C})$. The same holds for the real case.

Proof. Certainly, the linear space is preserved by H, since its generators are homogeneous elements in the graded algebra $\Lambda^{\bullet}_{\mathbb{C}}$. For L there is nothing to prove. For the dual Lefschetz,

$$\Lambda L^r\alpha = L^r \underbrace{\Lambda\alpha}_{=0} - [L^r, \Lambda]\alpha = -r(k-n+r-1)L^{r-1}\alpha.$$

□

The previous corollary suggests the importance of elements in the kernel of Λ. Indeed we will see that, together with the Lefschetz operator, they determine the whole algebra $\Lambda^{\bullet}_{\mathbb{C}}$.

Definition 2.2.6. Let us set

$$P^k_{\mathbb{C},p}M = \ker\big(\Lambda\colon \Lambda^k T^*_{\mathbb{C},p}M \to \Lambda^{k-2}T^*_{\mathbb{C},p}M\big). \tag{2.2.19}$$

As Λ has constant rank, $P^k_{\mathbb{C},p}M$ actually forms a bundle, denoted by $P^k_{\mathbb{C}}M$. We set

$$P^{\bullet}_{\mathbb{C}}M = \bigoplus_{k\geqslant 0} P^k_{\mathbb{C}}M. \tag{2.2.20}$$

A section of $P^k_{\mathbb{C}}M$, *i.e.* an element $\alpha \in \mathcal{A}^k_{\mathbb{C}}(M)$ such that $\Lambda\alpha = 0$, is called a *primitive* k*-form*. Similar definitions can be given for the real case.

Denote with $P^k_{\mathbb{C}}$ a generic fibre of $P^k_{\mathbb{C}}M$. We are ready now to state the bundle Lefschetz decomposition theorem, which still holds at the level of fibres and consequently at the level of sections. The theorem will allow us to prove the useful Weyl's formula, which relates the Lefschetz operator and the Hodge dual.

Theorem 2.2.7 (Bundle Lefschetz decomposition theorem)**.** *The following orthogonal decomposition holds:*

$$\Lambda^k_{\mathbb{C}} = \bigoplus_{r \geqslant 0} L^r P^{k-2r}_{\mathbb{C}}. \tag{2.2.21}$$

Further,

1) $P^k_{\mathbb{C}} = 0$ *for* $k > n$,
2) *the map* $L^{n-k} \colon P^k_{\mathbb{C}} \to \Lambda^{2n-k}_{\mathbb{C}}$ *is into for* $k \leqslant n$,
3) *the map* $L^{n-k} \colon \Lambda^k_{\mathbb{C}} \to \Lambda^{2n-k}_{\mathbb{C}}$ *is an isomorphism for* $k \leqslant n$,
4) *an element* $\alpha \in \Lambda^k_{\mathbb{C}}$ *for* $k \leqslant n$ *is primitive if and only if* $L^{n-k+1}\alpha = 0$.

The same results hold in the real case.

Proof. We know that $\mathfrak{sl}(2, \mathbb{C})$ is reductive: every representation of $\mathfrak{sl}(2, \mathbb{C})$ is the direct sum of irreducible subrepresentations. Firstly, let us show that for $\alpha \in P^k_{\mathbb{C}}$, the space

$$\operatorname{span}\left\{ \alpha, L\alpha, \dots, L^{\mathrm{top}}\alpha \right\} \subset \Lambda^{\bullet}_{\mathbb{C}}$$

is an irreducible subrepresentation (note that, considering $\beta, \Lambda\beta, \Lambda^2\beta, \dots$ we can always find a primitive element by dimensional arguments). Indeed, consider a non-trivial subrepresentation U of the linear span. Then $L^r\alpha \in U$ for certain $r \geqslant 0$. If $r = 0$, then U coincides with the linear span; if $r > 0$, then $L^s\alpha \in U$ for $s \geqslant r$ and

$$L^{r-1}\alpha = \frac{1}{r(k-n+r-1)}[L^r, \Lambda]\alpha$$

proves that $L^{r-1}\alpha \in U$ and, by induction U coincides with the linear span. Since every such space is an irreducible representation, we have the decomposition. We need to check the orthogonality. Fix $r > 0$; we want to prove that $\langle L^r\alpha, L^s\beta \rangle = 0$ for any s. We perform induction: for $\alpha \in P^{k-2r}_{\mathbb{C}}$ and $\beta \in P^k_{\mathbb{C}}$,

$$\langle L^r\alpha, \beta \rangle = \langle L^{r-1}\alpha, \Lambda\beta \rangle = 0$$

since β is primitive. Suppose that we have the orthogonality for $s - 1$. Then, for $\beta \in P^{k-2s}_{\mathbb{C}}$,

$$\begin{aligned} \langle L^r\alpha, L^s\beta \rangle &= \langle L^{r-1}\alpha, \Lambda L^s\beta \rangle \\ &= \langle L^{r-1}\alpha, L^s\Lambda\beta \rangle - s(k-n-s-1)\,\langle L^{r-1}\alpha, L^{s-1}\beta \rangle = 0. \end{aligned}$$

This proves the first statement.
1) Fix now $\alpha \in P^k_{\mathbb{C}}$ with $k > n$. Choose r minimal such that $L^r\alpha = 0$. Then

$$[L^r, \Lambda]\alpha = r(k-n+r-1)\underbrace{L^{r-1}\alpha}_{\neq 0},$$

while $[L^r, \Lambda]\alpha = 0$ since both operators annihilate α. Note that if $r > 0$, then $k - n + r - 1 > 0$, which is a contradiction. Thus, $r = 0$, that is $\alpha = 0$.

2) Let us prove now that L^{n-k} restricted to $P^k_{\mathbb{C}}$ is into. Fix $\alpha \in P^k_{\mathbb{C}}$, $\alpha \neq 0$, with $L^{n-k}\alpha = 0$. Choose r minimal such that $L^r\alpha = 0$ (note that $r > 0$). Then, as before,

$$0 = [L^r, \Lambda]\alpha = r(k - n + r - 1)L^{r-1}\alpha,$$

so that $k-n+r-1 = 0$, *i.e.* $r = n-k+1$. By minimality of the exponent, $L^{n-k}\alpha \neq 0$, contradiction. This also proves that if $\alpha \in P^k_{\mathbb{C}}$, then $L^{n-k+1}\alpha = 0$.

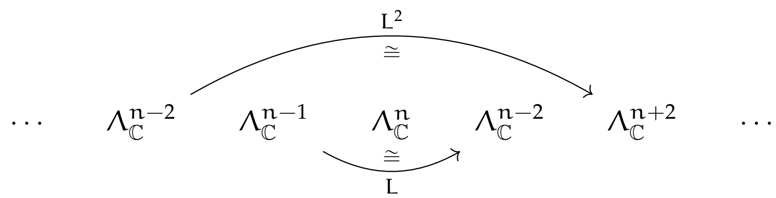
3) The isomorphism given by L^{n-k} follows from the decomposition and the fact that there are no primitive elements for $k > n$.

4) We just need to prove that if an element $\alpha \in \Lambda^k_{\mathbb{C}}$ is such that $L^{n-k+1}\alpha = 0$, then α is primitive. Note that

$$\begin{aligned} L^{n-k+2}\Lambda\alpha &= L^{n-k+2}\Lambda\alpha - \Lambda L^{n-k+2}\alpha && \text{since } L^{n-k+1}\alpha = 0, \\ &= [L^{n-k+2}, \Lambda]\alpha = cL^{n-k+1}\alpha = 0. \end{aligned}$$

But $\Lambda\alpha$ is in $\Lambda^{k-2}_{\mathbb{C}}$, which is in bijection with $\Lambda^{n-k+2}_{\mathbb{C}}$ via L^{n-k+2}. Thus, $\Lambda\alpha = 0$, *i.e.* α is primitive. □

Remark 2.2.8. The above isomorphisms can be schematically pictured as follows.



Further, as the operators L, Λ and H are pure operators of type (1,1), $(-1,-1)$ and (0,0) respectively, we have that the Lefschetz decomposition is compatible with the type decomposition. More precisely, we have

$$P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}, \tag{2.2.22}$$

where $P^{p,q} = P^k_{\mathbb{C}} \cap \Lambda^{p,q}$ and $\overline{P^{p,q}} = P^{q,p}$ and the Lefschetz decomposition theorem can be written as

$$\Lambda^{p,q} = \bigoplus_{r \geqslant 0} L^r P^{p-r,q-r}. \tag{2.2.23}$$

Proposition 2.2.9 (Weyl's formula)**.** *For every* $\alpha \in P^k_{\mathbb{C}}$*,*

$$* L^r\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(n-k-r)!} L^{n-k-r} J\alpha. \tag{2.2.24}$$

Proof. We perform induction on n. For $n = 1$, we can split $\Lambda^{\bullet}_{\mathbb{C}} = \Lambda^0_{\mathbb{C}} \oplus \Lambda^1_{\mathbb{C}} \oplus \Lambda^2_{\mathbb{C}}$, where

$$\Lambda^0_{\mathbb{C}} = \langle 1 \rangle, \qquad \Lambda^1_{\mathbb{C}} = \langle dx, dy \rangle, \qquad \Lambda^2_{\mathbb{C}} = \langle \omega \rangle.$$

Further, $P^0_{\mathbb{C}} = \Lambda^0_{\mathbb{C}}$ and $P^1_{\mathbb{C}} = \Lambda^1_{\mathbb{C}}$. We just to need to check Weyl's formula for $r = 0,1$. Take $r = 0$ and $k = 0$. We have to prove that $*1 = LJ(1)$. Indeed, $LJ(1) = \omega$, while

$$*1 = dx \wedge dy = \omega.$$

For $k = 1$, we have $*dx = dy$ and $*dy = -dx$, while $J(dx) = -dy$ and $J(dy) = dx$, so that

$$*dx = -J(dx), \qquad *dy = -J(dy).$$

The case $r = 1$ is similar. We proceed now by induction: for $n > 1$, choose a non-trivial g_p-orthogonal decomposition of $T_{\mathbb{C},p}M = W_1 \oplus W_2$ with $J(W_i) \subset W_i$. Then the Lefschetz operators split as

$$L = L_1 \otimes \mathrm{id} + \mathrm{id} \otimes L_2, \qquad \Lambda = \Lambda_1 \otimes \mathrm{id} + \mathrm{id} \otimes \Lambda_2.$$

Moreover, for a split form $\alpha_1 \otimes \alpha_2$, with $\alpha_i \in \Lambda^{k_i} W_i^*$,

$$*(\alpha_1 \otimes \alpha_2) = (-1)^{k_1 k_2} (*\alpha_1) \otimes (*\alpha_2).$$

In order to understand the explicit form of primitive elements, we can suppose $\dim_{\mathbb{C}} W_2 = 1$, so that $W_2^* = \langle dx, dy \rangle$. Such decomposition always exists: take W_2 as a one-dimensional J-eigenspace and W_1 its orthogonal complement. Then $\alpha \in P^k_{\mathbb{C}}$ can be written as

$$\alpha = \beta + \gamma' \otimes dx + \gamma'' \otimes dy + \delta \otimes \omega,$$

where $\omega = dx \wedge dy$ (we have omitted $\otimes 1$ for β). Since $\Lambda_2 1 = \Lambda_2 dx = \Lambda_2 dy = 0$, $\Lambda_2 \omega = 1$, we find

$$\Lambda\alpha = \Lambda_1\beta + \Lambda_1\gamma' \otimes dx + \Lambda_1\gamma'' \otimes dy + \Lambda_1\delta \otimes \omega + \delta.$$

By degree arguments, from $\Lambda\alpha = 0$ we find

$$\begin{cases} \Lambda_1\beta + \delta = 0, \\ \Lambda_1\gamma' = \Lambda_1\gamma'' = \Lambda_1\delta = 0. \end{cases}$$

So γ', γ'' and δ are primitive, while in general β is not. However, note that $\Lambda_1^2\beta = 0$. By Lefschetz decomposition theorem, we can write

$$\beta = \sum_{r \geqslant 0} L^r \eta_{k-2r}, \qquad \eta_{k-2r} \text{ primitive elements in } \Lambda^{k-2r} W_1^*.$$

The condition $\Lambda_1^2\beta = 0$ implies that $\beta = \beta' + L_1\delta'$, with the relation $\delta = (k-n-1)\delta'$. Collecting all together, we write

$$\alpha = \beta + L_1\delta + \gamma' \otimes dx + \gamma'' \otimes dy + (k-n-1)\delta \otimes \omega,$$

with $\beta, \gamma', \gamma'', \delta$ primitive (we removed the prime in β and δ). In this way, we have "parametrized" primitive elements in $\Lambda^\bullet_{\mathbb{C}}$ with those in $\Lambda^\bullet W_1^*$. Now, as $\dim_{\mathbb{C}} W_2 = 1$, we find

$$L^r = L_1^r \otimes \mathrm{id} + r\, L_1^{r-1} \otimes L_2,$$

so that (writing simply ℓ instead of L_1)

$$\begin{aligned} L^r\alpha &= \ell^r\beta + \ell^{r+1}\delta + r\left(\ell^{r-1}\beta\right)\otimes\omega + r\left(\ell^r\delta\right)\otimes\omega + \left(\ell^r\gamma'\right)\otimes dx + \\ &\qquad\qquad + \left(\ell^r\gamma''\right)\otimes dy + (k-n+1)\left(\ell^r\delta\right)\otimes\omega \\ &= \ell^r\beta + \ell^{r+1}\delta + \left(\ell^r\gamma'\right)\otimes dx + \left(\ell^r\gamma''\right)\otimes dy + \\ &\qquad + \Big(r\left(\ell^{r-1}\beta\right) + (k-n+r+1)\left(\ell^r\delta\right)\Big)\otimes\omega. \end{aligned}$$

Computing the Hodge dual and using Weyl's formula on primitive elements in dimension $< n$, we find an expression that can be compared with the right hand side of Weyl's formula applied to α in expansion of primitive forms of W_1, obtaining the thesis. □

2.2.2 Harmonic forms on Kähler manifolds

Let us move to differential operators on differential forms. We firstly find the closed expression for the formal adjoint of the exterior derivative and the del and del-bar operators (see [Hori et al., 2003] for a physical interpretation of these differential operators in terms of a supersymmetric quantum mechanical theory).

Lemma 2.2.10. *Let* M *be a compact Hermitian manifold. The following equalities hold:*

$$d^* = -*\,d*, \qquad \partial^* = -*\,\partial*, \qquad \bar\partial^* = -*\,\bar\partial\,*\,. \tag{2.2.25}$$

Proof. Fix α a k-form. For the exterior derivative,

$$\begin{aligned} \langle d\alpha, \beta\rangle &= \int_M d\alpha\wedge *\bar\beta = (-1)^{k+1}\int_M \alpha\wedge d*\bar\beta \\ &= -\int_M \alpha\wedge *\big(*d*\bar\beta\big) = \langle \alpha, -*\,d*\beta\rangle\,. \end{aligned}$$

Analogously for the other operators. □

We are now ready to show the key result in local theory of compact Kähler manifold, *i.e.* the commutation relations between the Lefschetz operators and the differential operators. In the proof, the Kähler condition $d\omega = 0$ will be crucial. Before stating the result, we introduce the Laplacians on Kähler manifold.

Definition 2.2.11. Let (M,ω) be a compact Kähler manifold. We define the Laplacians

$$\Delta = dd^* + d^*d, \qquad \Delta_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial. \tag{2.2.26}$$

They are all sef-adjoint, positive definite, elliptic operators acting on $\mathcal{A}^k_{\mathbb{C}}(M)$.

Theorem 2.2.12 (Kähler identities)**.** *Let* (M,ω) *be a compact Kähler manifold. The following identities hold.*

1) $[\partial, L] = [\bar\partial, L] = 0, \qquad [\partial^*, \Lambda] = [\bar\partial^*, \Lambda] = 0.$

2) $[\partial^*, L] = -\sqrt{-1}\bar{\partial}$, $[\bar{\partial}^*, L] = \sqrt{-1}\partial$ *and* $[\partial, \Lambda] = -\sqrt{-1}\bar{\partial}^*$, $[\bar{\partial}, \Lambda] = \sqrt{-1}\partial^*$.

3) $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$ *and they commute with* $*, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L$ *and* Λ.

Proof. Before proceeding with the proof, note that $0 = d\omega = \partial\omega + \bar{\partial}\omega$ implies, by type argument, that both $\partial\omega$ and $\bar{\partial}\omega$ vanishes.
1) We have that

$$\begin{aligned}[\partial, L]\alpha &= \partial(\omega \wedge \alpha) - \omega \wedge \partial\alpha \\ &= \partial\omega \wedge \alpha + \omega \wedge \partial\alpha - \omega \wedge \partial\alpha = 0.\end{aligned}$$

Similarly for $[\bar{\partial}, L] = 0$. Recall now that $\Lambda = *^{-1}L*$ and $\partial^* = - * \partial *$, so that

$$\begin{aligned}[\partial^*, \Lambda]\alpha &= -(*\partial*)(*^{-1}L*)\alpha + (*^{-1}L*)(*\partial*)\alpha \\ &= -(*\partial L*)\alpha + (*L\partial*)\alpha \\ &= - * [\partial, L] * \alpha = 0.\end{aligned}$$

Similarly for $[\bar{\partial}^*, \Lambda] = 0$.
2) A simpler proof can be given in terms of the operator

$$d^c = -\sqrt{-1}\big(\partial - \bar{\partial}\big), \qquad (d^c)^* = \sqrt{-1}\big(\partial^* - \bar{\partial}^*\big).$$

By type argument, the commutation relations (2) are equivalent to

$$[d^*, L] = -d^c, \qquad [d, \Lambda] = d^{c*}$$

respectively. Note that $d^c = J^{-1}dJ$. Indeed, on $\mathcal{A}^{p,q}(M)$ we have

$$Jd^c\alpha = (\sqrt{-1})^{p-q}\partial\alpha + (\sqrt{-1})^{p-q}\bar{\partial}\alpha = dJ\alpha.$$

Further, $d^{c*} = - * d^c *$. Let us prove now the commutation relations for d and Λ. By Lefschetz decomposition theorem, it suffices to prove the relation for elements of the form $L^r\alpha$, with α primitive.

$$[d, \Lambda](L^r\alpha) = \underbrace{d\Lambda L^r\alpha}_{\text{(I)}} - \underbrace{\Lambda d L^r\alpha}_{\text{(II)}}.$$

To compute the two terms, decompose the differential of α by Lefschetz decomposition theorem:

$$d\alpha = \beta_0 + L\beta_1 + L^2\beta_2 + \cdots, \qquad \beta_s \text{ a primitive } (k+1-2s)\text{-form}.$$

As α is primitive and $[d, L] = 0$, $L^{n-k+1}d\alpha = 0$. Thus, as the decomposition is a direct sum, $L^{n-k+s+1}\beta_s = 0$ for every $s \geqslant 0$. Note that L^t on primitive m-forms is into for $t \leqslant n - m$, so that $\beta_s = 0$ for $s \geqslant 2$. Hence, $d\alpha = \beta_0 + L\beta_1$. Now,

$$\begin{aligned}\text{(I)} = -d[L^r, \Lambda]\alpha &= -d\Big(r(k-n+r-1)L^{r-1}\alpha\Big) \\ &= -r(k-n+r-1)L^{r-1}(\beta_0 + L\beta_1) \\ &= -r(k-n+r-1)L^{r-1}\beta_0 - r(k-n+r-1)L^r\beta_1.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{II}) &= \Lambda L^r(\beta_0 + L\beta_1) = -[L^r,\Lambda]\beta_0 - [L^{r+1},\Lambda]\beta_1 \\ &= -r(k-n+r)L^{r-1}\beta_0 - (r+1)(k-n+r-1)L^r\beta_1.\end{aligned}$$

Thus,

$$[d,\Lambda](L^r\alpha) = -rL^{r-1}\beta_0 + (k-n+r-1)L^r\beta_1.$$

On the other hand, using Weyl's identity,

$$d^{c*}L^r\alpha = -(*J^{-1}dJ*)(L^r\alpha) = -(-1)^{\frac{k(k+1)}{2}}\frac{r!}{(n-k-r)!}(*J^{-1}dJ)(L^{n-k-r}J\alpha)$$

Now, $[L,d]=0$ and it can be simply shown that $[L,J] = [*,J^{-1}] = 0$, so that

$$\begin{aligned}d^{c*}L^r\alpha &= -(-1)^{\frac{(k-1)k}{2}}\frac{r!}{(n-k-r)!}(J^{-1}*L^{n-k-r})(d\alpha) \\ &= -(-1)^{\frac{(k-1)k}{2}}\frac{r!}{(n-k-r)!}J^{-1}\Big(*L^{n-k-r}\beta_0 + *L^{n-k-r+1}\beta_1\Big).\end{aligned}$$

By Weyl's formula again,

$$\begin{aligned}*L^{n-k-r}\beta_0 &= (-1)^{\frac{(k-1)k}{2}}\frac{(n-k-r)!}{(r-1)!}L^{r-1}J\beta_0, \\ *L^{n-k-r+1}\beta_1 &= (-1)^{\frac{(k-1)k}{2}}\frac{(n-k-r+1)!}{r!}L^rJ\beta_1,\end{aligned}$$

so finally, using $[L,J]=0$,

$$\begin{aligned}d^{c*}L^r\alpha &= -r\,J^{-1}L^{r-1}J\beta_0 - (n-k-r+1)\,J^{-1}L^rJ\beta_1 \\ &= -rL^{r-1}\alpha + (k-n+r-1)L^r\beta_1.\end{aligned}$$

This proves that $[d,\Lambda] = d^{c*}$. The relation for d^* and L follows:

$$\begin{aligned}[d^*,L] &= *(-*d^**)(*^{-1}L*)* - *(*^{-1}L*)(-*d^**)* \\ &= *[d,\Lambda]* = *d^{c*}* = -d^c.\end{aligned}$$

3) Let us prove that $\Delta_\partial = \Delta_{\bar\partial}$. We have from (2) that

$$\begin{aligned}\Delta_\partial &= \partial\partial^* + \partial^*\partial = \sqrt{-1}[\Lambda,\bar\partial]\partial + \sqrt{-1}\partial[\Lambda,\bar\partial] \\ &= \sqrt{-1}\big(\Lambda\bar\partial\partial - \bar\partial\Lambda\partial + \partial\Lambda\bar\partial - \partial\bar\partial\Lambda\big) \\ &= \sqrt{-1}\big(\Lambda\bar\partial\partial - (\bar\partial[\Lambda,\partial] + \bar\partial\partial\Lambda) + ([\partial,\Lambda]\bar\partial + \Lambda\partial\bar\partial) - \partial\bar\partial\Lambda\big) \\ &= \Delta_{\bar\partial} + \Lambda\{\partial,\bar\partial\} - \{\partial,\bar\partial\}\Lambda = \Delta_{\bar\partial}.\end{aligned}$$

Here we used the fact that $\{\partial,\bar\partial\}=0$. Let us show now that $\Delta_d = 2\Delta_\partial$.

$$\begin{aligned}\Delta_d &= dd^* + d^*d = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial) \\ &= \Delta_\partial + \Delta_{\bar\partial} + (\partial\bar\partial^* + \bar\partial^*\partial) + (\partial^*\bar\partial + \bar\partial\partial^*) \\ &= 2\Delta_\partial + 2\,\mathrm{Re}(\partial\bar\partial^* + \bar\partial^*\partial).\end{aligned}$$

On the other hand,

$$\partial\bar\partial^* + \bar\partial^*\partial = \sqrt{-1}\big(\partial[\partial,\Lambda] + [\partial,\Lambda]\partial\big) = \sqrt{-1}[\partial^2,\Lambda] = 0.$$

Thus, $\Delta_d = 2\Delta_\partial$. Finally, the commutation relations for the Laplacians follows from the previous ones. For example,

$$\begin{aligned}\Delta_\partial L &= \partial\partial^* L + \partial^*\partial L\\ &= \partial L\partial^* - \sqrt{-1}\partial\bar\partial + \partial^* L\partial\\ &= L\partial\partial^* - \sqrt{-1}\partial\bar\partial + L\partial^*\partial - \sqrt{-1}\bar\partial\partial\\ &= L\Delta_\partial - \sqrt{-1}\{\partial,\bar\partial\} = L\Delta_\partial.\end{aligned}$$

The remaining ones are analogous. □

Remark 2.2.13. The Lefschetz operator and the Kähler identities allow us to express the scalar curvature in different useful ways. Recall that $S = g^{i\bar j}R_{i\bar j}$, where $\mathrm{Ric} = \sqrt{-1}R_{i\bar j}dz^i \wedge d\bar z^j$ is the Ricci form. Then we can write

$$S = \Lambda\,\mathrm{Ric}\,. \tag{2.2.27}$$

Indeed, for a general form of type (1,1), say $\eta = \sqrt{-1}\eta_{i\bar j}dz^i \wedge \bar z^j$, and a function $f \in C^\infty(M)$, we have in local holomorphic coordinates

$$\begin{aligned}\langle Lf,\eta\rangle &= \int_M g(f\omega,\eta)\ \mathrm{Vol}_g = \int_M f\sqrt{-1}g_{i\bar j}g^{i\bar k}g^{\bar j l}\overline{\sqrt{-1}\eta_{k\bar l}}\ \mathrm{Vol}_g\\ &= \int_M f\, g^{\bar k l}\overline{\eta_{k\bar l}}\ \mathrm{Vol}_g = \int_M f\,\overline{g^{k\bar l}\eta_{k\bar l}}\ \mathrm{Vol}_g\\ &= \langle f, g^{k\bar l}\eta_{k\bar l}\rangle\,.\end{aligned}$$

Thus, $g^{k\bar l}\eta_{k\bar l} = \Lambda\eta$. On the other hand, the Lefschetz dual allows us to express the $\bar\partial$-Laplacian on functions as $\Delta_{\bar\partial} f = -\sqrt{-1}\Lambda\partial\bar\partial f$. Indeed,

$$-\sqrt{-1}\Lambda\partial\bar\partial f = -\sqrt{-1}\partial\Lambda\bar\partial f + \bar\partial^*\bar\partial f = \Delta_{\bar\partial} f.$$

In coordinates, $\Delta_{\bar\partial} f = -g^{i\bar j}\partial_i\partial_{\bar j} f$. Applying this to the local expression for the Ricci form, $\mathrm{Ric} = -\sqrt{-1}\partial\bar\partial\log(\det g)$, we find

$$S = \Delta_{\bar\partial}\log(\det g). \tag{2.2.28}$$

These identities will be useful in the next chapter.

We can proceed now discussing harmonic forms.

Definition 2.2.14. For M a compact Hermitian manifold, we define the space of *harmonic* k*-forms*

$$\mathcal{H}^k(M,\mathbb{C}) = \left\{\, \alpha \in \mathcal{A}^k_{\mathbb{C}}(M) \;\middle|\; \Delta_d\alpha = 0 \,\right\}, \tag{2.2.29}$$

and the space of *harmonic forms of type* (p,q)

$$\mathcal{H}^{p,q}(M) = \left\{\, \alpha \in \mathcal{A}^{p,q}(M) \;\middle|\; \Delta_d\alpha = 0 \,\right\}. \tag{2.2.30}$$

We have

$$\mathcal{H}^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M). \tag{2.2.31}$$

Analogously, one could define the spaces of ∂ and $\bar{\partial}$-harmonic forms: $\mathcal{H}^k_{\partial}(M)$, $\mathcal{H}^{p,q}_{\partial}(M)$ and $\mathcal{H}^k_{\bar{\partial}}(M)$, $\mathcal{H}^{p,q}_{\bar{\partial}}(M)$. Note that the harmonicity actually depends on the Hermitian metric.

If M is Kähler, by the Kähler identities we have

$$\mathcal{H}^k(M,\mathbb{C}) = \mathcal{H}^k_{\partial}(M) = \mathcal{H}^k_{\bar{\partial}}(M) \tag{2.2.32}$$

and the same holds for harmonic forms of type (p, q). This is peculiar of Kähler manifolds: in general, for M just Hermitian, there is no such relation between the different Laplacians.

Remark 2.2.15. The Laplace equation $\Delta_d \alpha = 0$, which is a second order differential equation, is equivalent to the system of first order equations $d\alpha = 0$ and $d^*\alpha = 0$ (it is sometimes said that α is *closed and coclosed*). Indeed, if α is closed and coclosed, than it is obviously harmonic. On the other hand, if α is harmonic, then

$$0 = \langle \Delta\alpha, \alpha\rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2,$$

so that $d\alpha = 0$ and $d^*\alpha = 0$. The same holds for ∂ and $\bar{\partial}$.

We can state now the fundamental result in Hodge theory.

Theorem 2.2.16 (Hodge decomposition theorem)**.** *Let M be a compact Hermitian manifold. There are natural orthogonal decompositions*

$$\begin{aligned} \mathcal{A}^{p,q}(M) &= \partial\mathcal{A}^{p-1,q}(M) \oplus \mathcal{H}^{p,q}_{\partial}(M) \oplus \partial^*\mathcal{A}^{p+1,q}(M), \\ \mathcal{A}^{p,q}(M) &= \bar{\partial}\mathcal{A}^{p,q-1}(M) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}(M) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(M), \end{aligned} \tag{2.2.33}$$

where the splits depend on the Hermitian metric. Further, the spaces $\mathcal{H}^{p,q}_{\partial}(M)$ and $\mathcal{H}^{p,q}_{\bar{\partial}}(M)$ are finite dimensional $\mathbb{C}$-vector spaces.

The proof uses the same techniques of the Riemannian case. See [Griffiths and Harris, 1994] for a complete reference. From the decomposition theorem, we immediately obtain some interesting corollaries.

Corollary 2.2.17. *Let (M, ω) be a compact Hermitian manifold. There is a natural isomorphism between the $\bar{\partial}$-harmonic forms of type (p, q) and the (p, q)th Dolbeault cohomology group:*

$$\begin{aligned} \mathcal{H}^{p,q}_{\bar{\partial}}(M) &\longrightarrow H^{p,q}_{\bar{\partial}}(M) \\ \alpha &\longmapsto [\alpha]. \end{aligned} \tag{2.2.34}$$

The same holds for the del operator.

Proof. Firstly, note that the map is well-defined, since $\Delta_{\bar\partial}\alpha = 0$ implies $\bar\partial\alpha = 0$. Let us prove the injectivity. Suppose $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(M)$ with $[\alpha] = 0$. Then $\alpha = \bar\partial\beta$ and by Hodge decomposition theorem,

$$\alpha = \bar\partial\left(\bar\partial\beta_1 + \beta_2 + \bar\partial^*\beta_3\right) = \bar\partial\bar\partial^*\beta_3$$

since β_2 is $\bar\partial$-harmonic, hence $\bar\partial$-closed. Then $0 = \Delta_{\bar\partial}\alpha = \bar\partial\bar\partial^*\bar\partial\bar\partial^*\beta_3$. Thus,

$$0 = \langle \bar\partial\bar\partial^*\bar\partial\bar\partial^*\beta_3, \beta_3\rangle = \langle \bar\partial\bar\partial^*\beta_3, \bar\partial\bar\partial^*\beta_3\rangle = \|\alpha\|^2,$$

and $\alpha = 0$. For the surjectivity, take $[\alpha] \in H^{p,q}_{\bar\partial}(M)$. By Hodge decomposition theorem, $\alpha = \bar\partial\beta_1 + \beta_2 + \bar\partial^*\beta_3$, with β_2 $\bar\partial$-harmonic. But $\bar\partial\alpha = 0$, so that $\bar\partial\bar\partial^*\beta_3 = 0$. As a consequence,

$$0 = \langle \bar\partial\bar\partial^*\beta_3, \beta_3\rangle = \left\|\bar\partial^*\beta_3\right\|^2$$

and we obtain $\bar\partial^*\beta_3 = 0$. Thus, $[\alpha] = [\beta_2]$, *i.e.* the map is surjective. □

Another useful corollary is $\partial\bar\partial$-lemma, peculiar of the Kähler case and crucial in many applications.

Corollary 2.2.18 ($\partial\bar\partial$-lemma)**.** *Let (M, ω) be a compact Kähler manifold. For a d-closed form α of type (p, q), the following conditions are equivalent.*

1) *α is d-exact: exists $\beta \in \mathcal{A}^{p+q-1}_{\mathbb{C}}(M)$ such that $d\beta = \alpha$.*

2) *α is ∂-exact: exists $\beta \in \mathcal{A}^{p-1,q}(M)$ such that $\partial\beta = \alpha$.*

3) *α is $\bar\partial$-exact: exists $\beta \in \mathcal{A}^{p,q-1}(M)$ such that $\bar\partial\beta = \alpha$.*

4) *α is $\partial\bar\partial$-exact: exists $\gamma \in \mathcal{A}^{p-1,q-1}(M)$ such that $\partial\bar\partial\gamma = \alpha$.*

5) *α is orthogonal to $\mathcal{H}^{p+q}(M, \mathbb{C})$ or $\mathcal{H}^{p,q}_{\partial}(M)$ or $\mathcal{H}^{p,q}_{\bar\partial}(M)$.*

Proof. By Hodge decomposition theorem, (5) is implied by any of the other conditions. Further, (4) implies (1)-(3), so it suffices to prove that (5) implies (4). Take α a d-closed form. By ∂-Hodge decomposition, $\alpha = \partial\beta + \beta' + \partial^*\beta''$. As before, $\partial^*\beta'' = 0$. Further, since α is orthogonal to $\mathcal{H}^{p,q}_{\partial}(M)$, $\beta' = 0$. Thus, $\alpha = \partial\beta$. Applying $\bar\partial$-Hodge decomposition to β, we find $\beta = \bar\partial\gamma + \gamma' + \bar\partial^*\gamma''$. As γ' is harmonic, $\partial\gamma' = 0$. On the other hand, $\partial\bar\partial^* = -\bar\partial^*\partial$, so that $\alpha = \partial\bar\partial\gamma - \bar\partial^*\partial\gamma''$. Finally, $0 = \bar\partial\alpha = -\bar\partial\bar\partial^*\partial\gamma''$, so that

$$0 = \langle \bar\partial\bar\partial^*\partial\gamma'', \partial\gamma''\rangle = \left\|\bar\partial^*\partial\gamma''\right\|^2.$$

Thus, $\alpha = \partial\bar\partial\gamma$. □

A consequence of the $\partial\bar\partial$-lemma is that on compact Kähler manifolds the Kähler classes are parametrized by real function.

Corollary 2.2.19 ($\partial\bar\partial$-lemma for Kähler classes)**.** *Let (M, ω) be a compact Kähler manifold, η a real (1,1) form in the same cohomology class of ω. Then there exists $\phi \colon M \to \mathbb{R}$ a smooth real function such that*

$$\eta = \omega + \sqrt{-1}\partial\bar\partial\phi. \tag{2.2.35}$$

Remark 2.2.20. Since every closed form is locally exact, every Kähler form can be locally represented as $\omega|_U = \sqrt{-1}\partial\bar{\partial}\phi$, with $\phi\colon U \to \mathbb{R}$ smooth. Such a function is called a *local Kähler potential* for ω. For the compact case, there cannot exists a global Kähler potential due to positivity of the volume. Note there is no comparable way of describing a general Riemannian metric in terms of a single function.

A consequence of the $\partial\bar{\partial}$-lemma is the existence of a *Hodge structure of weight* k on the kth cohomology group of a compact complex manifold. See [Griffiths, 1984] for further readings on Hodge structures.

Proposition 2.2.21. *Let M be a compact Kähler manifold. The following decomposition holds:*

$$H^k_{dR}(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M). \tag{2.2.36}$$

Further, the decomposition does not depend on the Kähler metric.

Proof. The decomposition directly follows from the Riemannian and $\bar{\partial}$-isomorphism between harmonic forms and cohomology groups:

$$H^k_{dR}(M,\mathbb{C}) \cong \mathcal{H}^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar{\partial}}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M).$$

Although the first and last term do not depend on the metric, the equality in the middle do. To prove the independence on the metric, we need to show that if $\alpha \in \mathcal{H}^{p,q}(M,\omega)$ and $\alpha' \in \mathcal{H}^{p,q}(M,\omega')$ with $[\alpha] = [\alpha'] \in H^{p,q}_{\bar{\partial}}(M)$, then $[\alpha] = [\alpha'] \in H^k_{dR}(M,\mathbb{C})$ for arbitrary Kähler forms ω and ω' (here we emphasized the dependence of the harmonic forms on the metric). Indeed, since α and α' are $\bar{\partial}$-cohomologous,

$$\alpha - \alpha' = \bar{\partial}\beta.$$

On the other hand, by (d,ω)-Hodge decomposition theorem, $\bar{\partial}\beta = d\gamma + \gamma' + d^*\gamma''$, where $\gamma' \in \mathcal{H}^k(M,\omega,\mathbb{C})$. The claim is that $\gamma' = d^*\gamma'' = 0$. Indeed, since $\bar{\partial}\beta$ is d-closed (as α and α' are so), we have $dd^*\gamma'' = 0$. By the usual argument, $\gamma'' = 0$. On the other hand, the (d,ω)-Hodge decomposition is ω-orthogonal and

$$\langle\bar{\partial}\beta, \gamma'\rangle_\omega = \langle\beta, \bar{\partial}^*\gamma'\rangle_\omega = 0$$

since, by harmonicity, γ' is $\bar{\partial}$-coclosed. As a consequence, $\gamma' = 0$ and $\alpha - \alpha' = d\gamma$. Thus, $[\alpha] = [\alpha'] \in H^k_{dR}(M,\mathbb{C})$. □

Let us analyse the symmetries on the spaces of harmonic forms and on the Dolbeault cohomology groups of a Kähler manifold we have encountered so far.

Complex conjugation. It leads to a isomorphism

$$\mathcal{H}^{p,q}_{\bar{\partial}}(M) \xrightarrow{\cong} \mathcal{H}^{q,p}_{\bar{\partial}}(M). \tag{2.2.37}$$

Note that *a priori* the conjugation has image in the space of ∂-harmonic forms, but it coincides with the $\bar{\partial}$-harmonic forms thanks to the Kähler identities. As before, the induced isomorphism in cohomology does not depend on the Kähler metric.

Serre duality. The intersection pairing, associating to $\alpha \in \mathcal{A}^{p,q}(M)$ and $\beta \in \mathcal{A}^{n-p,n-q}(M)$ the complex number

$$(\alpha, \beta) = \int_M \alpha \wedge \beta, \tag{2.2.38}$$

descends to a non-degenerate pairing between $\mathcal{H}^{p,q}_{\bar\partial}(M)$ and $\mathcal{H}^{n-p,n-q}_{\bar\partial}(M)$. Indeed, if $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(M)$ with $\alpha \neq 0$, then $*\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(M)$ and

$$(\alpha, *\bar\alpha) = \int_M \alpha \wedge *\bar\alpha = \|\alpha\|^2 \neq 0. \tag{2.2.39}$$

Thus, we have the so-called Serre duality

$$\mathcal{H}^{p,q}_{\bar\partial}(M) \xrightarrow{\cong} \left(\mathcal{H}^{n-p,n-q}_{\bar\partial}(M)\right)^*. \tag{2.2.40}$$

Again, the induced isomorphism in cohomology does not depend on the choice of a Kähler metric.

Hodge reflection. The Hodge dual $*$ induces the isomorphism

$$\mathcal{H}^{p,q}_{\bar\partial}(M) \xrightarrow{\cong} \mathcal{H}^{n-q,n-p}_{\bar\partial}(M). \tag{2.2.41}$$

The map is well-defined since $[*, \Delta_{\bar\partial}] = 0$. *A priori*, the induced map in cohomology depends on the Kähler metric ω, but it actually depends just on the Kähler class $[\omega]$. Indeed, take $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(M, \omega)$. By Lefschetz decomposition, write α as

$$\alpha = \sum_{r \geqslant 0} L^r \beta_r. \tag{2.2.42}$$

The decomposition depends just on $[\omega]$, since the Lefschetz operator in cohomology is the product $[\alpha] \mapsto [\omega] \wedge [\alpha]$. Then (emphasizing the dependence of the Hodge dual and the Lefschetz operator on the metric)

$$*_\omega \alpha = \sum_{r \geqslant 0} *_\omega L_\omega \beta_r. \tag{2.2.43}$$

Using Weyl's formula, and passing to cohomology,

$$\left[*_\omega \alpha\right] = \sum_{r \geqslant 0} (-1)^{\frac{k(k+1)}{2}+r} \frac{r!}{(n-k+r)!} L^{n-k+r}_{[\omega]} J\left[\beta_r\right]. \tag{2.2.44}$$

The right hand side depends just on $[\omega]$, so that the left hand side depends on the Kähler class too. This proves the claim.

These symmetries can be summarised as follow.

Definition 2.2.22. Let M be a compact complex manifold. Define the (p, q)*th Hodge numbers* as

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M). \tag{2.2.45}$$

These are finite, by Hodge decomposition theorem (every complex manifold can be equipped with a Hermitian metric) and they depend just on the complex structure of the manifold.

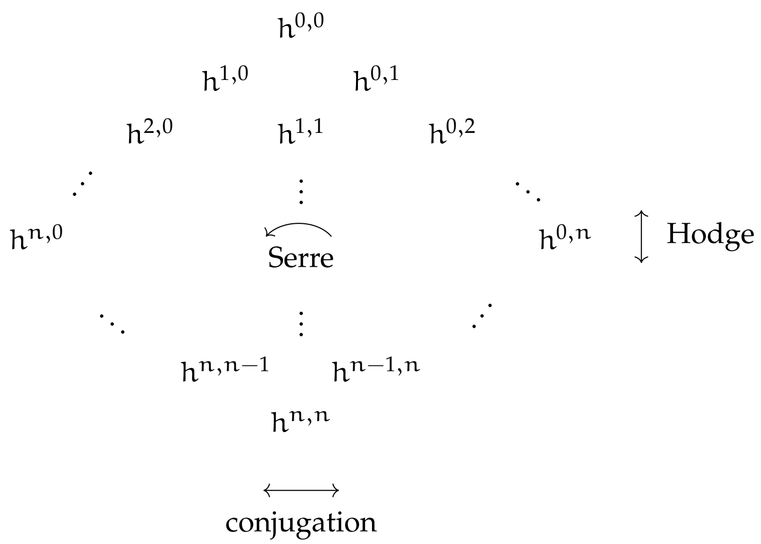


Figure 2.1: Hodge diamond and symmetries.

If M is Kähler, we have $h^{p,q} = h^{q,p}$ by conjugation, $h^{p,q} = h^{n-p,n-q}$ by Serre duality and $h^{p,q} = h^{n-q,n-p}$ by Hodge reflection. Representing the Hodge numbers within the *Hodge diamond* (see Figure 2.1), the symmetries can be pictured as follows: the conjugation is the reflection through the vertical axis, Serre duality is a rotation of π and Hodge reflection is the reflection through the horizontal axis.

An interesting topological constraint on Kähler manifolds that follows from the conjugation symmetry is the following

Corollary 2.2.23. *Let M be compact a Kähler manifold. Then the odd Betti numbers β_{2k+1} are even.*

Proof. We simply have, from $h^{p,q} = h^{q,p}$,

$$\beta_{2k+1} = \sum_{p+q=2k+1} h^{p,q} = 2 \sum_{p \leqslant k} h^{p,2k+1-p}.$$

□

The property of having odd Betti numbers which are even turns out to be sufficient in the case of compact complex surfaces (this is known as the Kodaira conjecture, definitely proved in [Siu, 1983]). Thus for dimension 2, as in the case of complex curves, being compact Kähler is a topological property. This is no longer true for higher dimension, as shown by an example due to Hironaka of Kähler manifolds of dimension 3 that can be smoothly deformed to non-Kähler ones.

Let us discuss now how to extend the Lefschetz decomposition theorem to cohomology on compact Kähler manifolds.

Definition 2.2.24. Let (M, ω) be a compact Kähler manifold. Define the *primitive cohomology groups* as

$$H^k_{\mathrm{prim}}(M, \mathbb{C}) = \ker\big(L^{n-k+1} \colon H^k_{\mathrm{dR}}(M, \mathbb{C}) \to H^{2n-k+2}_{\mathrm{dR}}(M, \mathbb{C})\big). \tag{2.2.46}$$

Analogously, we define

$$H^{p,q}_{\mathrm{prim}}(M) = \ker\big(L^{n-k+1} \colon H^{p,q}_{\bar\partial}(M, \mathbb{C}) \to H^{n-p+1,n-q+1}_{\bar\partial}(M)\big). \tag{2.2.47}$$

The definition are well-posed thanks to Kähler identities: $[L, \partial] = [L, \bar\partial] = 0$. Further, the primitive cohomology groups depend just on the cohomology class of the metric, which has been omitted in the notation.

As for the de Rham cohomology, the primitive one admits a type decomposition, *i.e.* the primitive cohomology defines a Hodge structure. However, while for the de Rham case the decomposition does not depend on the Kähler metric, in the primitive case we have dependence on the Kähler class. Further, it is equipped with the so-called Hodge-Riemann bilinear pairing. This is related to the concept of *polarized Hodge structure* (see [Griffiths, 1984] for further readings). All the following results come from the fact that L commutes with the Laplace operators, so that we can take harmonic representatives. Further, since L is of type (1,1), it respects the type decomposition.

Lemma 2.2.25. *For a compact Kähler manifold (M, ω), the following decomposition holds:*

$$H^k_{\mathrm{prim}}(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathrm{prim}}(M). \tag{2.2.48}$$

Proof. It suffices to prove that if $[\alpha] \in H^k_{\mathrm{prim}}(M, \mathbb{C})$ is primitive, then the harmonic representative is primitive too (as differential form). Indeed, take $\alpha' \in [\alpha]$ the harmonic representative. Then $L^{n-k+1}\alpha'$ is d-exact. By the $\partial\bar\partial$-lemma, it is orthogonal to $\mathcal{H}^k(M, \mathbb{C})$. On the other hand, from the Kähler identities, we have that it is harmonic:

$$\Delta_d L^{n-k+1}\alpha' = L^{n-k+1}\Delta_d \alpha' = 0.$$

Thus, $L^{n-k+1}\alpha' = 0$, *i.e.* α' is a primitive form. The decomposition then follows from the one for harmonic forms. □

Similarly, from the bundle Lefschetz decomposition theorem we have the analogous results in cohomology.

Theorem 2.2.26 (Hard Lefschetz theorem)**.** *Let (M, ω) be a compact Kähler manifold. Then for every $k \leqslant n$, the map*

$$L^{n-k} \colon H^k_{\mathrm{dR}}(M, \mathbb{C}) \to H^{2n-k}_{\mathrm{dR}}(M, \mathbb{C}) \tag{2.2.49}$$

is an isomorphism. The map is also compatible with the type decomposition.

Theorem 2.2.27 (Lefschetz decomposition theorem)**.** *Let* (M, ω) *be a compact Kähler manifold. The following orthogonal decomposition holds:*

$$H^k_{dR}(M, \mathbb{C}) = \bigoplus_{r \geqslant 0} L^r H^{k-2r}_{prim}(M, \mathbb{C}). \tag{2.2.50}$$

The decomposition is also compatible with the type decomposition.

Finally, let us introduce the Hodge-Riemann bilinear pairing.

Definition 2.2.28. For a compact Kähler manifold (M, ω), define the *Hodge-Riemann bilinear pairing* Q on k-forms as

$$Q(\alpha, \beta)\,\mathrm{Vol}_\omega = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}. \tag{2.2.51}$$

The sign is just a normalization convention. From the definition it is clear that Q is graded anticommutative and $Q(\alpha, \beta) = 0$, unless α is of type (p, q) and β of type (q, p).

Lemma 2.2.29. *For* α *a primitive form of type* (p, q)*, with* $p + q = k$*, the following relation holds:*

$$(\sqrt{-1})^{p-q} Q(\alpha, \bar{\alpha}) = (n-k)!\,\langle \alpha, \alpha \rangle\,. \tag{2.2.52}$$

Proof. We have that

$$\begin{aligned} Q(\alpha, \bar{\alpha})\,\mathrm{Vol}_\omega &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k}\bar{\alpha} \\ &= (-1)^{\frac{k(k-1)}{2}} (-1)^k \alpha \wedge *^2 L^{n-k}\bar{\alpha} = (-1)^{\frac{k(k+1)}{2}} \alpha \wedge *^2 L^{n-k}\bar{\alpha} \\ &= (-1)^{\frac{k(k+1)}{2}} \langle \alpha, *L^{n-k}\alpha \rangle\,\mathrm{Vol}_\omega\,. \end{aligned}$$

Using Weyl's formula, we have the thesis:

$$\begin{aligned} Q(\alpha, \bar{\alpha})\,\mathrm{Vol}_\omega &= (n-k)!\,\langle \alpha, J\alpha \rangle\,\mathrm{Vol}_\omega = (n-k)!\,\langle \alpha, (\sqrt{-1})^{p-q}\alpha \rangle\,\mathrm{Vol}_\omega \\ &= (\sqrt{-1})^{q-p}(n-k)!\,\langle \alpha, \alpha \rangle\,\mathrm{Vol}_\omega\,. \end{aligned}$$

□

Corollary 2.2.30 (Hodge-Riemann bilinear relations)**.** *Let* (M, ω) *be a compact Kähler manifold. There is a well-defined bilinear pairing on the* k*th primitive cohomology group*

$$Q([\alpha], [\beta]) = \int_M Q(\alpha, \beta) \tag{2.2.53}$$

satisfying the two Hodge-Riemann bilinear relations*:*

1) $Q([\alpha], [\beta]) = 0$*, unless* α *is of type* (p, q) *and* β *of type* (q, p)*,*

2) $(\sqrt{-1})^{p-q} Q([\alpha], [\bar{\alpha}]) > 0$ *for any non-zero element in* $H^{p,q}_{prim}(M)$*.*

Proof. The first property comes from the one at the level of forms. For the second property, consider a harmonic representative $\alpha' \in [\alpha]$, which is a primitive form of type (p, q). Since $[\alpha] \neq 0$, there exists a point $p \in M$ such that $\langle \alpha'_p, \alpha'_p \rangle > 0$. Thus,

$$(\sqrt{-1})^{p-q} Q([\alpha], [\bar{\alpha}]) = (n-k)! \int_M \langle \alpha', \alpha' \rangle \operatorname{Vol}_\omega > 0.$$

□

To conclude this section, we show how Hodge theory allows us to find again topological constraints for Kähler manifolds. Indeed, note that on compact Kähler of even dimension $\dim_{\mathbb{C}} M = 2m$, the Hodge-Riemann bilinear form on the middle cohomology group coincides with the intersection pairing: on $H^{2m}_{dR}(M, \mathbb{R})$,

$$\int_M \alpha \wedge \beta = (-1)^m Q([\alpha], [\beta]). \tag{2.2.54}$$

The Hodge index theorem expresses the signature of this topological pairing in terms of the Hodge numbers.

Theorem 2.2.31 (Hodge index theorem)**.** *For a compact Kähler manifold (M, ω) of even dimension $\dim_{\mathbb{C}} M = 2m$, the intersection pairing on the middle real cohomology group $H^{2m}_{dR}(M, \mathbb{R})$,*

$$\big([\alpha], [\beta]\big) = \int_M \alpha \wedge \beta, \tag{2.2.55}$$

has signature

$$\operatorname{sgn} = \sum_{p,q} (-1)^p h^{p,q}. \tag{2.2.56}$$

Proof. For p, q with $p + q = k$, set

$$H^{p,q}_{prim}(M, \mathbb{R}) = H^{p,q}_{prim}(M) \cap H^k_{dR}(M, \mathbb{R}).$$

The above results imply that for a real, primitive form of type (p, q) with $p + q = k$,

$$\int_M \alpha \wedge \bar{\alpha} \wedge \omega^{2m-k} = (\sqrt{-1})^{q-p} (-1)^{\frac{k(k-1)}{2}} (2m-k)! \langle \alpha, \alpha \rangle .$$

Note that, thanks to Lefschetz decomposition theorem,

$$H^{2m}_{dR}(M, \mathbb{R}) = \Big(\bigoplus_{r \geqslant 0} L^r H^{m-r,m-r}_{prim}(M, \mathbb{R}) \Big) \oplus \\ \oplus \Big(\bigoplus_{r \geqslant 0} \bigoplus_{\substack{p+q=2m \\ p<q}} L^r \Big(H^{p-r,q-r}_{prim}(M, \mathbb{R}) \oplus H^{q-r,p-r}_{prim}(M, \mathbb{R}) \Big) \Big).$$

Further, the decomposition is orthogonal with respect to the intersection pairing, so that the signature is the sum of the signatures in every space. For $[\alpha] \in$

$H^{m-r,m-r}_{\text{prim}}(M,\mathbb{R})$, taking the harmonic representative which is primitive as a differential form, we have that

$$\big(L^r[\alpha], L^r[\alpha]\big) = \int_M \alpha \wedge \bar{\alpha} \wedge \omega^{2r} = (-1)^{m-r}(2r)!\,\langle \alpha, \alpha \rangle\,,$$

so that on $L^r H^{m-r,m-r}_{\text{prim}}(M)$ the intersection pairing has signature $(-1)^{m-r}$. For the second term, noting that

$$\overline{H^{q-r,p-r}_{\text{prim}}(M,\mathbb{R})} = H^{p-r,q-r}_{\text{prim}}(M,\mathbb{R}),$$

we have for $[\alpha] \in H^{p-r,q-r}_{\text{prim}}(M,\mathbb{R})$, $p+q=2m$, with harmonic representative

$$\begin{aligned}\big(L^r[\alpha+\bar{\alpha}], L^r[\alpha+\bar{\alpha}]\big) &= \int_M (\alpha+\bar{\alpha}) \wedge (\alpha+\bar{\alpha}) \wedge \omega^{2r} \\ &= 2\int_M \alpha \wedge \bar{\alpha} \wedge \omega^{2r} \\ &= 2(\sqrt{-1})^{q-p}(-1)^{m-r}(2r)!\,\langle \alpha, \alpha \rangle \\ &= 2(-1)^{p+r}(2r)!\,\langle \alpha, \alpha \rangle\,.\end{aligned}$$

Here we used the fact that $Q(\alpha,\alpha) = Q(\bar{\alpha},\bar{\alpha}) = 0$, as $p<q$. Let us set now

$$h^{p,q}_{\text{prim}} = \dim_{\mathbb{R}} H^{p,q}_{\text{prim}}(M,\mathbb{R}).$$

From the above results, we obtain

$$\begin{aligned}\text{sgn} &= \sum_{r\geqslant 0}(-1)^{m-r}h^{m-r,m-r}_{\text{prim}} + \sum_{r\geqslant 0}\sum_{\substack{p+q=2m\\ p<q}}(-1)^{p+r}\big(h^{p-r,q-r}_{\text{prim}} + h^{q-r,p-r}_{\text{prim}}\big) \\ &= \sum_{r\geqslant 0}\sum_{p+q=2m}(-1)^{p+r}h^{p-r,q-r}_{\text{prim}} = \sum_{p+q=2m}(-1)^p\sum_{r\geqslant 0}(-1)^r h^{p-r,q-r}_{\text{prim}}.\end{aligned}$$

Here we used the fact that, thanks to conjugation, $h^{p,q}_{\text{prim}} = h^{q,p}_{\text{prim}}$. We can link now the Hodge numbers to the values $h^{p,q}_{\text{prim}}$ with the relation

$$h^{p,q} = \sum_{r\geqslant 0} h^{p-r,q-r}_{\text{prim}},$$

which follows from Lefschetz decomposition and the usual argument of taking harmonic representatives. Performing the trick

$$\begin{aligned}\sum_{r\geqslant 0}(-1)^r h^{p-r,q-r}_{\text{prim}} &= \sum_{r\geqslant 0} h^{p-r,q-r}_{\text{prim}} + 2\sum_{s>0}(-1)^s\sum_{r\geqslant 0} h^{p-s-r,q-s-r}_{\text{prim}} \\ &= h^{p,q} + 2\sum_{s>0}(-1)^s h^{p-s,q-s},\end{aligned}$$

where the first relation can be proved by induction, we find

$$\text{sgn} = \sum_{p+q=2m}(-1)^p\left(h^{p,q} + 2\sum_{s>0}(-1)^s h^{p-s,q-s}\right).$$

By Hodge reflection and $p + q = 2m$, we have $h^{p-s,q-s} = h^{p+s,q+s}$, so that $h^{p,q} + 2\sum_{s>0} h^{p-s,q-s} = \sum_{s\in\mathbb{Z}}(-1)^s h^{p+s,q+s}$. Thus, with the notation $a \equiv_2 b$ for $a - b \in 2\mathbb{Z}$, we can rewrite the signature as

$$\mathrm{sgn} = \sum_{p+q=2m} (-1)^p \sum_{s\in\mathbb{Z}} (-1)^s h^{p+s,q+s} = \sum_{p+q\equiv_2 0} (-1)^p h^{p,q}.$$

Finally, since $(-1)^p h^{p,q} + (-1)^q h^{q,p} = 0$ for $p + q \equiv_2 1$, we find the thesis:

$$\mathrm{sgn} = \sum_{p,q} (-1)^p h^{p,q}.$$

□

Chapter 3

The J-equation

3.1 The J-flow and the J-equation

In this section we introduce the J-flow and the associated J-equation, which will be the main topic in the following analysis. The first approach is due to Donaldson in [Donaldson, 1999], where a moment map point-of-view naturally leads to the study of the equation. At around the same time, Chen in [Chen, 2000] independently discovered the same flow as the gradient flow of his J-functional, which appeared in the expression of the Mabuchi functional. Let us briefly summarise the two arguments leading to the equation.

Suppose that (M, α) is a compact Kähler manifold and let ω_0 be another Kähler form, possibly in a different Kähler class. For simplicity, we will assume $H^1_{dR}(M, \mathbb{R}) = 0$. Consider the infinite-dimensional manifold $\mathcal{X}$ of orientation-preserving diffeomorphisms $f \colon M \to M$ in a fixed homotopy class. The tangent space at a point $f \in \mathcal{X}$ is the space of section of the pull-back bundle f^*TM. Then $\mathcal{X}$ carries a natural symplectic form Ω defined by

$$\Omega_f(v, w) = \int_M \alpha(v, w) \frac{\omega_0^n}{n!} \tag{3.1.1}$$

for any $v, w \in \Gamma(f^*TM)$. Consider now the infinite-dimensional Lie group $\mathcal{G}$ of exact ω_0-symplectomorphisms, which acts on $\mathcal{X}$ by composition on the right by the inverse: $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ is given by $g.f = f \circ g^{-1}$. The action preserves Ω. As $H^1_{dR}(M, \mathbb{R}) = 0$, its Lie algebra $\mathfrak{g}$ can be identified with the Hamiltonian vector fields and, due compactness, with the space of smooth functions modulo constants. In other terms, we have the identification

$$\mathfrak{g} \cong \left\{ H \in C^\infty(M, \mathbb{R}) \;\middle|\; \textstyle\int_M H\, \omega_0^n = 0 \right\}. \tag{3.1.2}$$

A moment map $\mu \colon \mathcal{X} \to \mathfrak{g}^*$ for the group action is given by

$$\mu(f) = \frac{f^*\alpha \wedge \omega_0^{n-1}}{\omega_0^n} - c, \tag{3.1.3}$$

where the constant, which depends just on the Kähler classes of α and ω_0, is such that $\mu(f)$ has zero average and is given by

$$c = \frac{f^*[\alpha] \smile [\omega_0]^{n-1}}{[\omega_0]^n}. \tag{3.1.4}$$

Here we used the L^2 inner product in $C^\infty(M, \mathbb{R})$ to identify $\mathfrak{g}$ with its dual. A natural construction is the symplectic quotient, given by solutions of $\mu(f) = 0$ modulo $\mathcal{G}$. Under particular conditions, one would hope that a certain gradient flow would converge to a solution of $\mu(f) = 0$. This flow can be expressed via the complex structure $\mathcal{I}$ on $\mathcal{X}$, induced by the natural one on M, as

$$\frac{\partial f_t}{\partial t} = \mathcal{I} X_{\mu(f_t)}(f_t), \tag{3.1.5}$$

where $X_{\mu(f)}$ is the vector field on $\mathcal{X}$ associated to the element $\mu(f) \in \mathfrak{g}$. The flow can be rewritten as a flow of Kähler forms $(f_t^*)^{-1}\omega_0$. In terms of Kähler potentials

$$\mathcal{H} = \left\{ \phi \in C^\infty(M, \mathbb{R}) \;\middle|\; \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0 \right\}, \tag{3.1.6}$$

we find

$$\frac{\partial \phi_t}{\partial t} = c - \frac{\alpha \wedge \omega_{\phi_t}^{n-1}}{\omega_{\phi_t}^n}. \tag{3.1.7}$$

This is known as J-flow. Similarly, the equation $\mu(f) = 0$ can be rewritten in terms of ω_ϕ as $\alpha \wedge \omega_\phi^{n-1} = c\omega_\phi^n$. To fix the notation, let us give the following

Definition 3.1.1. Let (M, α) be a compact Kähler manifold, ω_0 another Kähler metric on M. We define the J-*equation* on the space of Kähler potentials associated to ω_0, *i.e.* $\mathcal{H} = \{ \phi \in C^\infty(M, \mathbb{R}) \mid \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0 \}$, as

$$\alpha \wedge \omega_\phi^{n-1} = c\omega_\phi^n, \tag{3.1.8}$$

where c is the constant $c = \frac{[\alpha] \smile [\omega_0]^{n-1}}{[\omega_0]^n}$ depending just on the Kähler classes $[\alpha]$ and $[\omega_0]$. We will call it the J-constant.

With a different approach, Chen found a new expression for the Mabuchi energy functional, introduced in [Mabuchi, 1986], which plays a key role in the study of constant scalar curvature Kähler metrics. Consider again a compact Kähler manifold (M, ω_0). Let $\mathcal{H}$ be the space of Kähler potentials associated to ω_0. The Mabuchi functional (or K-energy functional) is defined as follows: consider a path ϕ_t in $\mathcal{H}$ such that $\phi_0 = 0$ and $\phi_1 = \phi$ and set

$$\mathcal{M}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} (\hat{S} - S_t) \frac{\omega_t^n}{n!}\, dt, \tag{3.1.9}$$

where for brevity we set $\omega_t = \omega_{\phi_t}$, S_t is the scalar curvature associated to the metric ω_t and $\hat{S}$ is the average of the scalar curvature, which depends just on the Kähler class $[\omega_0]$ and the first Chern class of M:

$$\hat{S} = \frac{\int_M S_0\, \omega_0^n}{\int_M \omega_0^n} = \frac{n \int_M \mathrm{Ric}(\omega_0) \wedge \omega_0^{n-1}}{\int_M \omega_0^n} = \frac{2\pi n\, c_1(M) \smile [\omega_0]^{n-1}}{[\omega_0]^n}. \tag{3.1.10}$$

For the second equality, we used the fact that $S_0 = \Lambda_{\omega_0} \mathrm{Ric}(\omega_0)$ (see Remark 2.2.13) and Weyl's formula. It can be shown that the integral defining the Mabuchi functional is independent of the path connecting 0 and ϕ, so that we have a well-defined functional $\mathcal{M}\colon \mathcal{H} \to \mathbb{R}$. The alternative expression found by Chen is

$$\mathcal{M}(\phi) = \int_M \log \frac{\det g_\phi}{\det g_0} \frac{\omega_\phi^n}{n!} + J(\phi) + \hat{S} I(\phi), \tag{3.1.11}$$

where g_ϕ and g_0 are the metrics associated to ω_ϕ and ω_0 respectively, while the J and I functionals are defined as

$$J(\phi) = -\int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \mathrm{Ric}(\omega_0) \wedge \frac{\omega_{\phi_t}^{n-1}}{(n-1)!} dt, \tag{3.1.12}$$

$$I(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \frac{\omega_{\phi_t}^n}{n!} dt. \tag{3.1.13}$$

In his article, Chen pointed out that if $c_1(M) < 0$, thanks to Aubin-Yau theorem, we can choose ω_0 such that $\mathrm{Ric}(\omega_0) = -\omega_0$. In this case, the J-functional becomes

$$J(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \omega_0 \wedge \frac{\omega_{\phi_t}^{n-1}}{(n-1)!} dt. \tag{3.1.14}$$

Subsequently Chen in a second work (see [Chen, 2004]) studied the more general functional defined on then normalized space $\mathcal{H}_0 = \{ \phi \in \mathcal{H} \mid I(\phi) = 0 \}$

$$J(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \alpha \wedge \frac{\omega_{\phi_t}^{n-1}}{(n-1)!} dt \tag{3.1.15}$$

for a fixed Kähler form α. Alternatively, we can consider the normalized J-functional defined on $\mathcal{H}$, whose gradient flow is nothing but the flow discovered by Donaldson.

Definition 3.1.2. Let (M, α) be a compact Kähler manifold, ω_0 another Kähler metric on M. We define the *normalized* J*-functional* on the space $\mathcal{H}$ of Kähler potentials associated to ω_0 as

$$\hat{J}(\phi) = J(\phi) - ncI(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \left(\alpha \wedge \omega_{\phi_t}^{n-1} - c\omega_{\phi_t}^n \right) \frac{dt}{(n-1)!}, \tag{3.1.16}$$

where c is the J-constant defined before: $c = \frac{[\alpha] \smile [\omega_0]^{n-1}}{[\omega_0]^n}$.

3.1.1 Uniqueness and dependence on Kähler classes

Let us study now the J-equation in more details. We firstly show two equivalent formulations of the equation: the contraction of α with the metric ω is constant or α is an ω-harmonic form. Further, we will prove a uniqueness statement thanks to the comparison principle for nonlinear elliptic PDEs and we will briefly summarise an interesting result by Collins and Székelyhidi, showing that the solvability of the J-equation depends just on the Kähler class $[\alpha]$.

Proposition 3.1.3. *The following statements are equivalent:*

1) $\alpha \wedge \omega^{n-1} = c\omega^n$,

2) $\Lambda_\omega \alpha = nc$,

3) α *is* ω*-harmonic.*

Proof. Let us prove that (1) is equivalent to (2). Noting that, thanks to Weyl's formula, $\alpha \wedge \omega^{n-1} = (n-1)!\alpha \wedge *\omega$, we have

$$\alpha \wedge \omega^{n-1} = \langle \alpha, \omega \rangle \frac{\omega^n}{n} = \langle \Lambda_\omega \alpha, 1 \rangle \frac{\omega^n}{n} = \Lambda_\omega \alpha \frac{\omega^n}{n}$$

and the statement is proved. For the equivalence between (2) and (3), note that by compactness $\Lambda_\omega \alpha = nc$ is equivalent to $d\Lambda_\omega \alpha = 0$. As α is closed, this is equivalent to $[d, \Lambda_\omega]\alpha = 0$ and using the Kähler identities, we obtain $d^{c*}\alpha = 0$, where the adjoint is taken with respect to the metric ω. Again by closedness of α, this is equivalent to α being ω-harmonic. □

In local holomorphic coordinates (z^i) on M, writing $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j$, the second formulation of the J-equation reads

$$g^{i\bar{j}} \alpha_{i\bar{j}} = nc. \tag{3.1.17}$$

In terms of Kähler potential, it is a second-order nonlinear elliptic PDE. The ellipticity condition has to be intended as follows: consider the operator $F(\phi) = nc - \Lambda_{\omega_\phi} \alpha$ defined on the space of Kähler potentials. Then the variation of F at ϕ is the linear operator $D_\phi F \colon C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ defined as

$$D_\phi F(\xi) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\phi + \epsilon\xi). \tag{3.1.18}$$

Setting $g_{\epsilon, i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}}(\phi + \epsilon\xi)$, we have

$$\begin{aligned} D_\phi F(\xi) &= -\left.\frac{dg_\epsilon^{k\bar{l}}}{d\epsilon}\right|_{\epsilon=0} \alpha_{k\bar{l}} = g_0^{k\bar{j}} \left.\frac{dg_{\epsilon, i\bar{j}}}{d\epsilon}\right|_{\epsilon=0} g_0^{i\bar{l}} \alpha_{k\bar{l}} \\ &= \left(g_0^{i\bar{l}} g_0^{k\bar{j}} \alpha_{k\bar{l}}\right) \partial_i \partial_{\bar{j}} \xi. \end{aligned}$$

Setting $F^{i\bar{j}}(\phi) = g_0^{i\bar{l}} g_0^{k\bar{j}} \alpha_{k\bar{l}}$, we have that the linear operator

$$D_\phi F \colon \xi \longmapsto F^{i\bar{j}}(\phi) \partial_i \partial_{\bar{j}} \xi \tag{3.1.19}$$

is elliptic: for any point $p \in M$ and any coordinate system, the matrix $F^{i\bar{j}}(\phi)|_p$ is positive definite, as it can be seen in normal holomorphic coordinates at p for the metric ω_ϕ, where $F^{i\bar{j}}(\phi)|_p = \alpha_{i\bar{j}}|_p$. As a consequence, we can prove a comparison principle for the operator F and we will derive a uniqueness statement for the J-equation.

Proposition 3.1.4 (Comparison principle)**.** *In the above assumptions, let* $\phi, \psi \in \mathcal{H}$ *be Kähler potentials such that* $F(\phi) \geqslant F(\psi)$*. Then* $\phi \geqslant \psi$ *on* M*.*

Proof. Fix a holomorphic atlas $\{(U, z^i)\}$. We need to show that $\phi(p) \geqslant \psi(p)$ for every point $p \in M$. Let us set $\chi_t = t\phi + (1-t)\psi$, which is still a Kähler potential. By the mean value theorem, on U we have

$$0 \leqslant F(\phi) - F(\psi) = \int_0^1 D_{\chi_t} F(\phi - \psi) dt = a^{i\bar{j}} \partial_i \partial_{\bar{j}} (\phi - \psi).$$

Here we have defined the smooth functions on U

$$a^{i\bar{j}} = \int_0^1 F^{i\bar{j}}(\chi_t) dt.$$

Note that the ellipticity condition for F implies that $a^{i\bar{j}}$ is positive definite. Setting $Lf = a^{i\bar{j}} \partial_i \partial_{\bar{j}} f$ for a general function $f \in C^\infty(M, \mathbb{R})$, we have that L is a second order, linear, elliptic operator on M with non-positive first order coefficients (it is actually uniformly elliptic by compactness of M and the first order coefficients are identically zero). By the maximum principle for linear elliptic operators (see for instance [Aubin, 1998]), from $L(\phi - \psi) \geqslant 0$ it follows that $\phi - \psi \geqslant 0$, that is $\phi \geqslant \psi$. □

Corollary 3.1.5. *In the above assumption, if a solution on $\mathcal{H}$ for the J-equation $F(\phi) = 0$ exists, then it is unique.*

Let us discuss now the dependence of the solvability on the Kähler class $[\alpha]$.

Theorem 3.1.6 (Collins and Székelyhidi, 2014). *Let $\omega \in [\omega_0]$ be a solution of $\Lambda_\omega \alpha = nc$. Then if $\beta \in [\alpha]$ is another Kähler form, there exists $\eta \in [\omega_0]$ such that $\Lambda_\eta \beta = nc$.*

Sketch of the proof. The main ingredient of the proof is the equivalence between the solvability of the equation and the properness of the normalized J-functional. The functional $\hat{J}$ is said to be *proper* if there exist constants $C, \delta > 0$ such that

$$\hat{J}(\phi) \geqslant -C + \delta \int_M \phi \left(\omega_0^n - \omega_\phi^n\right).$$

For clarity, let us denote the dependence of the functional on α writing $\hat{J}_\alpha$. With this result, it is easy to show that if $\hat{J}_\alpha$ is proper, then so is $\hat{J}_\beta$ for any Kähler form $\beta \in [\alpha]$. Indeed, setting $\beta = \alpha + \sqrt{-1}\partial\bar{\partial}\psi$ and choosing $\phi_t = t\phi$, we have

$$\begin{aligned} \hat{J}_\beta(\phi) - \hat{J}_\alpha(\phi) &= \int_0^1 \int_M \phi(\beta - \alpha) \wedge \frac{\omega_{t\phi}^{n-1}}{(n-1)!} dt = \int_0^1 \int_M \psi(\sqrt{-1}\partial\bar{\partial}\phi) \wedge \frac{\omega_{t\phi}^{n-1}}{(n-1)!} dt \\ &= \int_0^1 \int_M \psi \frac{d}{dt} \left(\frac{\omega_{t\phi}^n}{n!} \right) dt = \int_M \psi \, \frac{(\omega_\phi^n - \omega_0^n)}{n!}, \end{aligned}$$

so that $|\hat{J}_\beta - \hat{J}_\alpha| \leqslant 2 \sup_M |\psi| \operatorname{Vol}_{\omega_0}(M)$. Thus, if $\hat{J}_\alpha$ is proper then so is $\hat{J}_\beta$. □

3.1.2 A numerical criterion

The rest of the thesis will be devoted to the study of a numerical criterion for the solvability of the J-equation. In his original work [Donaldson, 1999], Donaldson found that a necessary condition for the solvability of $\Lambda_\omega \alpha = nc$ is the positivity of the form $nc\omega - \alpha$. Indeed, choosing a normal holomorphic system for α where ω is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n$, the J-equation reads

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} = nc. \tag{3.1.20}$$

Thus, a necessary condition, as $\lambda_i > 0$, is that $nc\lambda_i > 1$ for all $i = 1, \dots, n$. The condition translates into

$$[nc\omega - \alpha] > 0. \tag{3.1.21}$$

Donaldson conjectured that this is also sufficient for the existence of a solution of the J-equation. The positivity condition is actually sufficient for complex surfaces, where the J-equation can be reduced to a complex Monge-Ampère equation solved by Yau. However, in [Lejmi and Székelyhidi, 2015] the authors showed that the conjecture is false, finding a more general necessary condition and an example where $[nc\omega - \alpha]$ is positive, but the new condition fails. The conjecture, proposed in the same article, is the following.

Conjecture 3.1.7 (Lejmi and Székelyhidi, 2015)**.** There exists a solution of $\Lambda_\omega \alpha = nc$ in $[\omega_0]$ if and only if, for all irreducible subvarieties $V \subset M$ of dimension $k < n$, the *numerical criterion*

$$\int_V \left(nc\omega_0^k - k\alpha \wedge \omega_0^{k-1}\right) > 0 \tag{3.1.22}$$

holds.

One direction of the conjecture can be easily proved: if a solution exists, then the numerical criterion (3.1.22) holds. Indeed if ω is a solution, denoting by α_V and ω_V the restrictions of α and ω along V respectively, we have (as in the proof of Proposition 3.1.3)

$$k\alpha_V \wedge \omega_V^{k-1} = \left(\Lambda_{\omega_V} \alpha_V\right) \omega_V^k \tag{3.1.23}$$

along V. Further, looking at the eigenvalues in normal coordinates, we have for $k < n$ that $\Lambda_{\omega_V} \alpha_V < \Lambda_\omega \alpha = nc$. Thus, we have $nc\omega_V^k - k\alpha_V \wedge \omega_V^{k-1} > 0$ and integrating we obtain the "only if" part. A first step in this direction was done by Collins and Székelyhidi, where the conjecture was proved for toric manifolds.

Definition 3.1.8. A *compact Kähler toric manifold* is a compact Kähler manifold (M, ω) equipped with an action by isometries of the real n-torus $\mathbb{T}^n$, such that the extension of the action to the complex torus $(\mathbb{C}^*)^n$ is holomorphic and with a free, open, dense orbit $M_0 \subset M$.

Theorem 3.1.9 (Collins and Székelyhidi, 2014)**.** *Let M be a compact toric manifold with Kähler forms α and ω_0. Suppose that for all toric subvarieties $V \subset M$ of dimension $k < n$ we have*

$$\int_V \left(nc\omega_0^k - k\alpha \wedge \omega_0^{k-1}\right) > 0. \tag{3.1.24}$$

Then there exists $\omega \in [\omega_0]$ *such that* $\Lambda_\omega \alpha = nc$.

We will not go through the proof of the above theorem. However, it is interesting to show that the conjecture firstly proposed by Donaldson is actually true for complex surfaces.

Theorem 3.1.10 (Chen, 2004)**.** *Let* (M, α) *be a compact Kähler surface,* ω_0 *another Kähler form. If* $[2c\omega_0 - \alpha] > 0$*, then there exists a solution to* $\Lambda_\omega \alpha = 2c$ *in the Kähler class defined by* ω_0*.*

Proof. The idea is to reduce the J-equation to a complex Monge-Ampére equation, solved by Yau (see Theorem 2.1.18). Multiplying $[\omega_0]$ for a constant if necessary, we can suppose $c = \frac{1}{2}$. Define the form

$$\chi_0 = \omega_0 - \alpha,$$

which is positive by hypothesis. Consider now a Kähler potential for χ_0: set $\chi_\phi = \chi_0 + \sqrt{-1}\partial\bar{\partial}\phi$. The J-equation for $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ in terms of χ_ϕ becomes

$$\alpha \wedge (\chi_\phi + \alpha) = \frac{1}{2}(\chi_\phi + \alpha)^2 = \frac{1}{2}(\chi_\phi^2 + 2\chi_\phi \wedge \alpha + \alpha^2),$$

which is equivalent to the Monge-Ampére equation $\chi_\phi^2 = \alpha^2$. By Yau's theorem, the equation has a solution $\chi_\phi > 0$. Thus, $\omega_\phi = \chi_\phi + \alpha$ is still a Kähler form, solution of the J-equation. With the proper constant, we can obtain a solution of the J-equation in the original fixed Kähler class. □

As said before, the conjecture is false in dimension $n > 2$, as shown in [Lejmi and Székelyhidi, 2015], where the authors built two metrics α and ω_0 on a threefold where $nc\omega_0 - \alpha$ is Kähler, but the numerical criterion does not hold.

3.2 Solvability on blowups

3.2.1 Blowups of Kähler manifolds

A common construction in Complex Geometry is the blowup of a complex manifold along a submanifold. Heuristically speaking, the process consists in substituting the submanifold with all the directions pointing out from it. For instance, if we consider a single point, the blowup construction substitutes the point with a copy of the complex projective space, leaving the complement unchanged. For a submanifold, the same procedure is done with the projectivisation of the normal bundle. The result is that of "zooming in" the submanifold.

Let us start by blowing up a linear subspace $\mathbb{C}^k$ in $\mathbb{C}^n$ of dimension k, following [Huybrechts, 2005], [Griffiths, 1984] and [Voisin, 2002]. We can suppose that $\mathbb{C}^k$ is given by the equations $z^{k+1} = \cdots = z^n = 0$. Denote by $[Z^{k+1} : \cdots : Z^n]$ the homogeneous coordinates in $\mathbb{P}^{n-k-1}$ and define the blowup of $\mathbb{C}^n$ along $\mathbb{C}^k$ as

$$\mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) = \left\{ ([Z], z) \in \mathbb{P}^{n-k-1} \times \mathbb{C}^n \;\middle|\; z^i Z^j = z^j Z^i \quad \forall i, j = k+1, \dots, n \right\}. \tag{3.2.1}$$

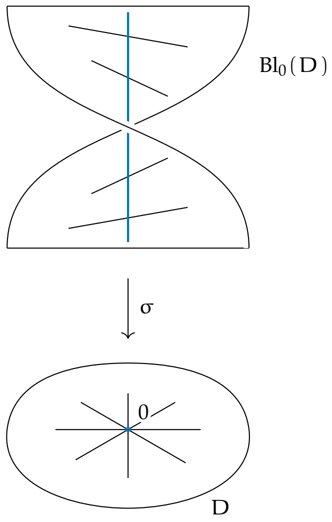


Figure 3.1: The real points of the blowup of the unitary disk $D = \{ z \in \mathbb{C}^2 \mid |z|^2 < 1 \}$.

Define the projection map onto the projective space as $\pi\colon \mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) \to \mathbb{P}^{n-k-1}$ and the blowdown map $\sigma\colon \mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) \to \mathbb{C}^n$. Alternatively, we can realise the blowup as the set

$$\mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) = \left\{ (\ell, z) \in \mathbb{P}^{n-k-1} \times \mathbb{C}^n \;\middle|\; z \in \mathbb{C}^k \oplus \ell \right\}, \tag{3.2.2}$$

where ℓ is interpreted as a line in the orthogonal complement of $\mathbb{C}^k$. Thus, it is clear that the fibre over a line $\ell \in \mathbb{P}^{n-k-1}$ is just the linear space $\pi^{-1}(\ell) \cong \mathbb{C}^k \oplus \ell$, so that $\pi\colon \mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) \to \mathbb{P}^{n-k-1}$ realises $\mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n)$ as the total space of a holomorphic vector bundle of rank $k+1$ over the projective space. Hence, it is a complex manifold of dimension n. Further, the blowdown map $\sigma\colon \mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) \to \mathbb{C}^n$ is an isomorphism over $\mathbb{C}^n \setminus \mathbb{C}^k$ and $\sigma^{-1}(\mathbb{C}^k) \cong \mathbb{P}(N_{\mathbb{C}^k/\mathbb{C}^n})$, the projectivisation of the normal bundle to $\mathbb{C}^k$, with $\sigma|_{\sigma^{-1}(\mathbb{C}^k)}$ the bundle map.

Note that if $k = 0$, *i.e.* the linear subspace is trivial, then the blowup of $\mathbb{C}^n$ in the origin is the tautological bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and we can think of $\mathrm{Bl}_0(\mathbb{C}^n)$ as obtained from $\mathbb{C}^n$ by replacing the origin by the space of all lines through it. The real points of the blowup of the unitary disk in $\mathbb{C}^2$ can be imagined as in Figure 3.1.

The construction can be globalised to arbitrary complex manifolds.

Proposition 3.2.1. *Let* M *be a complex manifold,* S *a complex submanifold of dimension* k*. Then there exists a complex manifold* $\mathrm{Bl}_S(M)$ *of the same dimension of* M*, called the* blowup of M along S, *together with a holomorphic map* $\sigma\colon \mathrm{Bl}_S(M) \to M$ *such that* σ *is a biholomorphism outside* S *and* $\sigma\colon \sigma^{-1}(S) \to S$ *is the bundle map* $\mathbb{P}(N_{S/M}) \to S$.

Proof. Consider an atlas $\{ (U_\lambda, \varphi_\lambda) \}$ of M such that $\varphi_\lambda(U_\lambda \cap S) \cong \varphi_\lambda(U_\lambda) \cap \mathbb{C}^k$, with $\mathbb{C}^k$ a subspace of $\mathbb{C}^n$. Let $\tau\colon \mathrm{Bl}_{\mathbb{C}^k}(\mathbb{C}^n) \to \mathbb{C}^n$ be the blowup of $\mathbb{C}^n$ along $\mathbb{C}^k$ and set $X_\lambda = \tau^{-1}(\phi_\lambda(U_\lambda))$ with the restrictions of the blowdown map $\tau_\lambda\colon X_\lambda \to \phi_\lambda(U_\lambda)$. We just need to check that the blowups glue on the intersections, so that we can define

$$\sigma\colon \mathrm{Bl}_S(M) = {\textstyle\bigsqcup_\lambda X_\lambda} / {\sim} \to M,$$

with $\sigma|_{X_\lambda} = \tau_\lambda$. In order to prove it, consider $U, V \subset \mathbb{C}^n$ with a biholomorphism $\phi\colon U \to V$ such that $\phi(U \cap \mathbb{C}^k) = V \cap \mathbb{C}^k$, where $\mathbb{C}^k$ is defined by the equations $z^{k+1} = \cdots = z^n = 0$. We can write in components $\phi = (\phi^i)$. Then the condition $\phi^i(z^1, \dots, z^k, 0, \dots, 0) = 0$ for all $i > k$, implies that for $i > k$ we can write

$$\phi^i(z) = \sum_{j=k+1}^{n} a_{ij}(z) z^j$$

for suitable holomorphic functions a_{ij}. They can be described as a matrix of functions $A = (a_{ij})_{i,j=k+1,\dots,n}$. Define the biholomorphism $\hat{\phi}\colon \tau^{-1}(U) \to \tau^{-1}(V)$ as

$$\hat{\phi}\big([Z], z\big) = \big([A(z)Z], \phi(z)\big).$$

From the definition of $A(z)$, it follows that $\hat{\phi}([Z], z)$ actually belongs to $\tau^{-1}(V)$ for every $([Z], z) \in \tau^{-1}(U)$. Now, as τ is a biholomorphism on $\mathbb{C}^n \setminus \mathbb{C}^k$, it is clear that the gluings are compatible over $M \setminus S$. On the other hand, over S the matrices $a_{ij}|_{U \cap \mathbb{C}^k}$ are nothing but the transition functions for the normal bundle $N_{S/M}$. Thus, the gluings are compatible also over S and, moreover, this proves that $\sigma^{-1}(S) \cong \mathbb{P}(N_{S/M})$. □

Definition 3.2.2. Let $\sigma\colon \mathrm{Bl}_S(M) \to M$ be the blowup of M along S. The hypersurface $\sigma^{-1}(S) \cong \mathbb{P}(N_{S/M})$ is called the *exceptional divisor* of the blowup. We will usually denote it by E.

In the following, we will be interested in the blowup of a manifold M at a point p, denoted by $\mathrm{Bl}_p(M)$. Let us study how the Kähler condition can be transferred from the manifold to its blowup at a point, following [Voisin, 2002]. The naive idea would be to consider the pull-back on $\mathrm{Bl}_p(M)$ of metric ω on M. The problem is that the pull-back is only semi-positive on the exceptional divisor. Thus, we have to "correct" it on E with a suitable form, without breaking the positiveness outside E.

Lemma 3.2.3. *The line bundle $\mathcal{O}_{\mathrm{Bl}_p(M)}(-E)$ is trivial outside E and its restriction to $E \cong \mathbb{P}^{n-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$.*

Proof. Let us set $\widetilde{M} = \mathrm{Bl}_p(M)$. Consider a coordinate chart U centred at p and set $\widetilde{U} = \sigma^{-1}(U)$. We can suppose $U \cong \{ z \in \mathbb{C}^n \mid |z| < 1 \} = D$, so that we can identify

$$\widetilde{U} \cong \left\{ ([Z], z) \in \mathbb{P}^{n-1} \times D \;\middle|\; z^i Z^j = z^j Z^i \right\}.$$

Under this identification, set $\widetilde{U}_i \cong \{ ([Z], z) \in \widetilde{U} \mid Z^i \neq 0 \}$, where we have local coordinates

$$w(i)^j = \frac{Z^j}{Z^i} = \frac{z^j}{z^i}, \qquad j = 1, \dots, \widehat{i}, \dots, n$$

and

$$w(i)^i = z^i.$$

Then the blowdown map $\sigma\colon \widetilde{M} \to M$ is given on $\widetilde{U}_i$ by

$$(w(i)^1, \dots, w(i)^n) \mapsto (w(1)^i \cdot w(i)^1, \dots, w(i)^i, \dots, w(1)^i \cdot w(i)^n)$$

and the exceptional divisor by $E \cap \widetilde{U}_i = \{ w(i)^i = 0 \}$. Thus, on $\widetilde{U}_i \cap \widetilde{U}_j$, we find

$$w(i)^i = z^i = \frac{z^i}{z^j} \cdot z^j = \frac{z^i}{z^j} \cdot w(j)^j.$$

and the transition maps for $\mathcal{O}_{\mathrm{Bl}_p(M)}(E)|_E$ are precisely those of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$:

$$g_{ij} = \frac{z^i}{z^j}.$$

Taking the dual, we obtain the thesis. □

Proposition 3.2.4. *If* (M, ω) *is Kähler and* $p \in M$*, the blowup manifold* $\mathrm{Bl}_p(M)$ *is Kähler, and it is compact if* M *is compact.*

Proof. Let us set $\widetilde{M} = \mathrm{Bl}_p(M)$. It follows from the above construction that the blowdown map is proper. Thus, $\widetilde{M}$ is compact if M is compact. Let us construct a metric form. As we said above, the pull-back form $\sigma^*\omega$ is a real, closed (1,1)-form with the following properties:

$$\sigma^*\omega = \begin{cases} \geqslant 0 & \text{everywhere,} \\ > 0 & \text{on } \widetilde{M} \setminus E, \\ > 0 & \text{on } T^{1,0}_q\widetilde{M} \big/ T^{1,0}_q E \text{ for all } q \in E. \end{cases}$$

So we need to find a correction to $\sigma^*\omega$ on $T^{1,0}E$. Consider a coordinate chart U centred at p as before and the closed set C containing p with $C \subset U$ identified with $C \cong \{ z \in \mathbb{C}^n \mid |z| \leqslant \frac{1}{2} \}$. Set $\widetilde{U} = \sigma^{-1}(U)$, $\widetilde{C} = \sigma^{-1}(C)$. Note that in the proof of the above lemma we have shown that the fibres of $\mathcal{O}(E)|_{\widetilde{U}}$ over $([Z], z) \in \widetilde{U}$ can be identified with

$$\{ \lambda Z \mid \lambda \in \mathbb{C} \}.$$

In particular, we can consider the metric h_1 on $\mathcal{O}(-E)|_{\widetilde{U}}$ given by

$$h_{1,([Z],z)}(\lambda Z) = \frac{1}{|\lambda|^2 \, \|Z\|^2}.$$

Note that on E, the curvature form is 2π times the Fubini-Study metric. On the other hand, the restriction of $\mathcal{O}(-E)$ to $\widetilde{M} \setminus E$ is trivial, so we can take the flat metric h_2. Consider now a partition of unity $\{ \rho_1, \rho_2 \}$ subordinate to $\{ \widetilde{U}, \widetilde{M} \setminus \widetilde{C} \}$ and define the global metric

$$h = \rho_1 h_1 + \rho_2 h_2$$

on $\mathcal{O}(-E)$. Let λ be the differential form given by $\frac{1}{2\pi}$ times the curvature form of h, which is a real closed (1,1)-form whose cohomology class is $c_1(\mathcal{O}(-E))$. Note that

- $\lambda > 0$ on $T^{1,0}_p E$ for all $p \in E$, as λ on $E \cong \mathbb{P}^{n-1}$ is the Fubini-Study metric;
- it is positive definite on $\widetilde{C}$, as $\rho_1 \equiv 1$ on it;

- $\lambda = 0$ on $\widetilde{M} \setminus \widetilde{U}$, as $\rho_2 \equiv 1$ on it.

In particular, due to compactness, we can choose ϵ sufficiently small such that

$$\tilde{\omega} = \sigma^* \omega + \epsilon \lambda$$

is positive definite. Indeed, in the closure of the annulus $\widetilde{U} \setminus \widetilde{C}$ the form λ is bounded, so that for small ϵ the positivity of $\sigma^* \omega$ is not broken. On the other hand, $\lambda > 0$ on the tangent spaces of the exceptional divisor, so that $\tilde{\omega}$ is a Kähler form. □

It will be useful in the following section to understand the cohomology class of the new metric on the blowup. The above construction turns out to give us a metric on the pull-back class of the metric, minus a small multiple of the Poicaré dual of the exceptional divisor. This is a consequence of Proposition 1.3.15:

$$c_1(\mathcal{O}(-E)) = -\,\mathrm{PD}[E]\,,$$

i.e. the Poicaré dual of a divisor is the first Chern class of the associated line bundle.

Corollary 3.2.5. *Let* $\sigma\colon \mathrm{Bl}_p(M) \to M$ *be the blowup of a Kähler manifold* (M, ω) *at a point* $p \in M$*. Then we can construct a Kähler metric on* $\mathrm{Bl}_p(M)$ *in the cohomology class*

$$\sigma^*[\omega] - \epsilon\,\mathrm{PD}[E] \in H^{1,1}(\mathrm{Bl}_p(M), \mathbb{R}) \tag{3.2.3}$$

for a sufficiently small $\epsilon > 0$.

The greatest $\epsilon > 0$ which makes the above cohomology class positive is called the Seshadri constant. Intuitively, it expresses the positivity of the Kähler form ω near the point p. See [Hartshorne, 1977] for the blowup construction in Algebraic Geometry and [Lazarsfeld, 2004] for further readings on the Seshadri constant.

Definition 3.2.6. Let (M, ω) be a Kähler manifold, $p \in M$. The *Seshadri constant* of ω at p is defined to be the positive real number

$$\Sigma(M, \omega, p) = \sup\{\, \epsilon > 0 \mid \sigma^*[\omega] - \epsilon\,\mathrm{PD}[E] > 0 \,\}. \tag{3.2.4}$$

A particular feature of this construction is that it carries infinitesimal information around p to global phenomena on the blowup. See for example [Griffiths, 1984]. On the other hand, from the topological point of view, $\mathrm{Bl}_p(M)$ is nothing but the connected sum between the manifold and a negatively oriented projective space.

Proposition 3.2.7. *The blowup of a manifold* M *at a point* p *is diffeomorphic as an oriented manifold to the connected sum* $M \# \overline{\mathbb{P}^n}$*, where* $\overline{\mathbb{P}^n}$ *is the complex projective space with orientation opposite to the one induced by the complex structure.*

Proof. Recall that the connected sum of two oriented manifolds X and Y of real dimension m is defined as follows. Consider the unit disk $D = \{\, x \in \mathbb{R}^m \mid |x| < 1 \,\}$

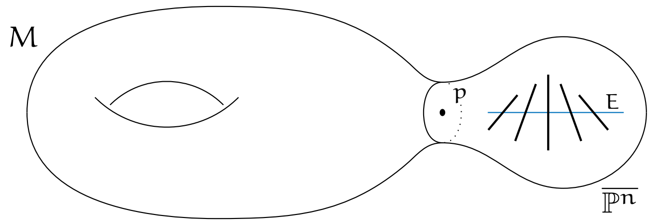


Figure 3.2: A representation of the blowup of a Riemann surface at a point. The figure takes inspiration from [Huybrechts, 2005].

and two open sets $U \subset X$, $V \subset Y$ with diffeomorphisms $f\colon D \to U$ and $g\colon D \to V$, f orientation-preserving and g orientation-reversing. Consider the gluing map

$$\begin{aligned} \gamma\colon D \setminus \left(\tfrac{1}{2}\bar{D}\right) &\longrightarrow D \setminus \left(\tfrac{1}{2}\bar{D}\right) \\ x &\longmapsto \frac{x}{2|x|^2} \end{aligned}$$

on the annulus. Then the connected sum is defined as the union of $X \setminus f\left(\frac{1}{2}\bar{D}\right)$ and $Y \setminus g\left(\frac{1}{2}\bar{D}\right)$, glued via $g \circ \gamma \circ f^{-1}$:

$$X \# Y = {X \setminus f\left(\tfrac{1}{2}\bar{D}\right) \cup Y \setminus g\left(\tfrac{1}{2}\bar{D}\right)}\Big/{\sim}.$$

To prove the proposition, as the statement is local, we can suppose M to be the unitary disk in $\mathbb{C}^n$, that is $D = \{ z \in \mathbb{C}^n \mid |z| < 1 \}$, and $p = 0$ to be the origin. Then

$$\mathrm{Bl}_0(D) = \left\{ ([Z], z) \in \mathbb{P}^{n-1} \times D \;\middle|\; Z^i z^j = Z^j z^i \quad \forall i, j = 1, \ldots, n \right\}.$$

We can take $U = D$ with f the identity and $g\colon D \to \overline{\mathbb{P}^n}$ defined by $g(z) = [1\colon z]$. Here the coordinates on $\overline{\mathbb{P}^n}$ are $[\bar{Z}^0 : Z]$, so that g is orientation-reversing. Note that, as $g\left(\frac{1}{2}\bar{D}\right) = \{ [\bar{Z}^0 : Z] \mid |Z| \leqslant \frac{1}{2}|\bar{Z}^0| \}$, we have

$$\overline{\mathbb{P}^n} \setminus g\left(\tfrac{1}{2}\bar{D}\right) = \left\{ [\bar{Z}^0 : Z] \;\middle|\; |Z| > \tfrac{1}{2}|\bar{Z}^0| \right\}.$$

In order to prove the statement, we just need to find two orientation-preserving smooth maps

$$\begin{aligned} \overline{\mathbb{P}^n} \setminus g\left(\tfrac{1}{2}\bar{D}\right) &\xrightarrow{a} \mathrm{Bl}_0(D) \\ D \setminus \left(\tfrac{1}{2}\bar{D}\right) &\xrightarrow{b} \mathrm{Bl}_0(D) \end{aligned}$$

such that $a^{-1} \circ b$ coincides with $g \circ \gamma$. If we set $a(\bar{Z}^0 : Z) = \left(\frac{Z^0}{2|Z|^2} Z, [Z]\right)$ and $b(z) = (z, [z])$, then

$$(a^{-1} \circ b)(z) = \left[2|z|^2 : z\right],$$

while

$$(g \circ \gamma)(z) = \left[1 : \frac{z}{2|z|^2}\right] = \left[2|z|^2 : z\right].$$

□

3.2.2 Blowing up the numerical criterion

In the same direction of a result by Arezzo and Pacard for the constant scalar curvature Kähler metric equation (see [Arezzo and Pacard, 2006, 2009]), we want to establish whether a solution to the J-equation exists on the blowup $\mathrm{Bl}_p(M)$, provided that a solution exists on M. The cohomology classes we are interested in are

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - \eta\,\mathrm{PD}[E]\,.$$

If we look for sufficiently small values of ϵ and η, it turns out that, to obtain a solution, η must be of the same order of ϵ. This can be directly seen by the positiveness condition of Donaldson (3.1.21). Rather than directly work with the equation, we will analyse the behaviour of the numerical criterion (3.1.22). We firstly state the result for surfaces, where Chen's result 3.1.10 can be applied. Before doing that, let us show the behaviour of the J-constant on the blowup.

Lemma 3.2.8. *Consider a compact Kähler manifold (M, α) and another Kähler form ω. Let $\sigma\colon \mathrm{Bl}_p(M) \to M$ be the blowup at a point p and take $\epsilon, \eta > 0$ sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - \eta\,\mathrm{PD}[E]\,, \tag{3.2.5}$$

are positive. Suppose that η is of the same order of ϵ, i.e. *$\eta = a\epsilon$ for a positive a. Then the J-constant on $\mathrm{Bl}_p(M)$, that is $\tilde{c} = \frac{[\tilde{\alpha}]\smile[\tilde{\omega}]^{n-1}}{[\tilde{\omega}]^n}$, satisfies*

$$\tilde{c} = \frac{c - a\nu\epsilon^n}{1 - \nu\epsilon^n} = c + \nu(c-a)\epsilon^n + O(\epsilon^{2n}), \tag{3.2.6}$$

where $\nu = \frac{1}{\int_M \omega^n}$.

Proof. Let us set $\widetilde{M} = \mathrm{Bl}_p(M)$ and take the representative form λ for $-\mathrm{PD}[E]$ as in the proof of Proposition 3.2.4. We first compute the volume of $(\widetilde{M}, \tilde{\omega})$:

$$\begin{aligned}
\mathrm{Vol}_{\tilde{\omega}}(\widetilde{M}) &= \int_{\widetilde{M}} \frac{\tilde{\omega}^n}{n!} \\
&= \int_{\widetilde{M}} \frac{\sigma^*\omega \wedge \tilde{\omega}^{n-1}}{n!} + \frac{\epsilon}{n}\int_{\widetilde{M}} \lambda \wedge \frac{\tilde{\omega}^{n-1}}{(n-1)!} \\
&= \int_{\widetilde{M}} \frac{\sigma^*\omega \wedge \tilde{\omega}^{n-1}}{n!} - \frac{\epsilon}{n}\int_E \frac{\tilde{\omega}|_E^{n-1}}{(n-1)!}.
\end{aligned}$$

Note that in the identification $E \cong \mathbb{P}^{n-1}$, we have $[\tilde{\omega}]|_E = \epsilon[\omega_{FS}]$. Thus,

$$\int_E \frac{\tilde{\omega}|_E^{n-1}}{(n-1)!} = \epsilon^{n-1}\,\mathrm{Vol}_{FS}(\mathbb{P}^{n-1}) = \frac{\epsilon^{n-1}}{(n-1)!}.$$

On the other hand,

$$\begin{aligned}
\int_{\widetilde{M}} \frac{\sigma^*\omega \wedge \tilde{\omega}^{n-1}}{n!} &= \int_{\widetilde{M}} \frac{\sigma^*\omega^n}{n!} + \frac{1}{n!}\sum_{k=1}^{n-1}\binom{n-1}{k}\int_{\widetilde{M}} \sigma^*\omega^{n-k} \wedge (\epsilon\lambda)^k \\
&= \int_{\widetilde{M}\setminus E} \frac{\sigma^*\omega^n}{n!} - \frac{\epsilon}{n!}\sum_{k=1}^{n-1}\binom{n-1}{k}\int_E \sigma^*\omega|_E^{n-k} \wedge (\epsilon\lambda|_E)^{k-1}.
\end{aligned}$$

The first term is simply the volume of M, as σ is a biholomorphism on $\widetilde{M} \setminus E$, while the second term vanishes as $\sigma^*[\omega]|_E = 0$. Thus, we find

$$\mathrm{Vol}_{\tilde{\omega}}(\widetilde{M}) = \mathrm{Vol}_{\omega}(M) - \frac{\epsilon^n}{n!}.$$

A similar computation shows that

$$\int_{\widetilde{M}} \tilde{\alpha} \wedge \tilde{\omega}^{n-1} = \int_M \alpha \wedge \omega^{n-1} - a\epsilon^n.$$

Thus, we finally find

$$\begin{aligned}\tilde{c} &= \frac{\int_M \alpha \wedge \omega^{n-1} - a\epsilon^n}{\int_M \omega^n - \epsilon^n} = \left(c - \frac{a\epsilon^n}{\int_M \omega^n}\right) \frac{1}{1 - \frac{\epsilon^n}{\int_M \omega^n}} \\ &= \frac{c - a\nu\epsilon^n}{1 - \nu\epsilon^n} = c + \nu(c - a)\epsilon^n + O(\epsilon^{2n}).\end{aligned}$$

□

Remark 3.2.9. It is interesting to note that the blowup procedure actually decreases the volume of the manifold. This is due to the fact that blowing up is a collapsing process, where the interior of a disk, neighbourhood of the blowup centre, is cut out and its boundary collapses to the exceptional divisor. The procedure has been formalised by Lerman in the symplectic category, see [Lerman, 1995].

Theorem 3.2.10. *Let* (M, α) *be a compact Kähler surface admitting a Kähler form* ω *such that* $2c\omega - \alpha > 0$*. Consider the blowup* $\sigma\colon \mathrm{Bl}_p(M) \to M$ *at a point* p*. Then there exists* $\epsilon > 0$ *sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\, \mathrm{PD}[E], \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\, \mathrm{PD}[E] \tag{3.2.7}$$

are positive and $2\tilde{c}\tilde{\omega} - \tilde{\alpha}$ *is positive too, provided that* $a < 2c$*.*

Proof. Let us set $\widetilde{M} = \mathrm{Bl}_p(M)$ and take the representative form λ for $-\mathrm{PD}[E]$ as in the proof of Proposition 3.2.4. We know that, as $\chi = 2c\omega - \alpha$ is a Kähler form, there exists a sufficiently small $\theta > 0$ such that

$$\sigma^*(2c\omega - \alpha) + \theta\lambda > 0.$$

We can introduce the parameter $\eta > 0$ on the expression, rewriting it as

$$2c\left(\sigma^*\omega + \frac{\theta + \eta}{2c}\lambda\right) - (\sigma^*\alpha + \eta\lambda) > 0.$$

Let us set $\epsilon = \frac{\theta+\eta}{2c}$, that is $\theta = 2c\epsilon - \eta$. Then we must have

$$\begin{cases} \eta < \Sigma(M, \alpha, p) \\ \epsilon < \Sigma(M, \omega, p) \\ 0 < 2c\epsilon - \eta < \Sigma(M, \chi, p)\,. \end{cases}$$

The last equation implies that η must go to zero with ϵ. Hence, it is more convenient to reparametrize $\eta = a\epsilon$ for a positive a. Then the conditions can be rewritten as follows:

$$\begin{cases} a < 2c \\ \epsilon < \min\left\{ \Sigma(M,\omega,p)\,, \frac{\Sigma(M,\alpha,p)}{a}, \frac{\Sigma(M,\chi,p)}{2c-a} \right\}. \end{cases}$$

Thus, the forms

$$\tilde{\omega}_\epsilon = \sigma^*\omega + \epsilon\lambda \qquad \tilde{\alpha}_\epsilon = \sigma^*\alpha + a\epsilon\lambda$$

$$\hat{\chi}_\epsilon = 2c\tilde{\omega}_\epsilon - \tilde{\alpha}_\epsilon$$

are positive definite and the cohomology classes of the first two metrics are precisely those of the statement. Now we have to prove the positivity of the form with the "right" constant, that is $\tilde{c}$. The above lemma shows us that, thinking of a as fixed, the value of $\tilde{c}$ will be close to that of c for small ϵ. Let us set

$$\tilde{\chi}_\epsilon = 2\tilde{c}\tilde{\omega}_\epsilon - \tilde{\alpha}_\epsilon.$$

To prove positiveness for sufficiently small ϵ, we can express $\tilde{c}$ as a function of c and ϵ using Lemma 3.2.8:

$$\begin{aligned} \tilde{\chi}_\epsilon &= 2\tilde{c}\sigma^*\omega - \sigma^*\alpha + (2\tilde{c} - a)\epsilon\lambda \\ &= 2c\sigma^*\omega - \sigma^*\alpha + \Big(2\nu(c-a)\epsilon^2 + O(\epsilon^4)\Big)\sigma^*\omega + \\ &\qquad\qquad + \Big((2c-a) + 2\nu(c-a)\epsilon^2 + O(\epsilon^4)\Big)\epsilon\lambda \\ &= 2c\sigma^*\omega + (2c-a)\epsilon\lambda + \Big(C\epsilon^2 + O(\epsilon^4)\Big)\sigma^*\omega + \Big(C\epsilon^3 + O(\epsilon^5)\Big)\lambda. \end{aligned}$$

Note that the first term is $\hat{\chi}_\epsilon$, which is positive definite and has a first-order dependence on ϵ, while the corrections goes like $O(\epsilon^2)$. Thus, there exists a positive $\bar{\epsilon}$ such that for every $\epsilon \in (0, \bar{\epsilon})$ we have

$$\tilde{\chi}_\epsilon > \frac{1}{2}\hat{\chi}_\epsilon > 0.$$

This concludes the proof. □

Applying Theorem 3.1.10, we obtain that for compact Kähler surfaces the J-equation has a solution on suitable Kähaler classes of $\mathrm{Bl}_p(M)$, provided that a critical metric exists on M.

Corollary 3.2.11. *Let (M, α) be a compact Kähler surface admitting a solution to the J-equation in the Kähler class $[\omega_0]$. Then there exists $\epsilon > 0$ sufficiently small such that*

$$[\tilde{\omega}_0] = \sigma^*[\omega_0] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\,\mathrm{PD}[E] \tag{3.2.8}$$

are positive and there exists a solution to the J-equation $\Lambda_{\tilde{\omega}}\tilde{\alpha} = 2\tilde{c}$ on $\mathrm{Bl}_p(M)$ in the Kähler class $[\tilde{\omega}_0]$, provided that $a < 2c$.

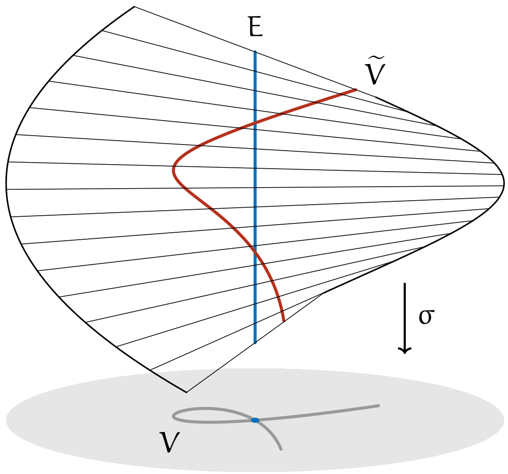


Figure 3.3: The real points of the nodal curve $V = \{ y^2 - x^2(x+1) = 0 \}$ and its proper transform.

Let us move now to the general case. To prove that the numerical criterion still holds on blowups, we have to understand the subvarieties of $\mathrm{Bl}_p(M)$. To that end, let us give the following

Definition 3.2.12. Let $V \subset M$ be a closed irreducible nonsingular subvariety of M, $p \in V$ a point and $\sigma\colon \mathrm{Bl}_p(M) \to M$ the blowup. Then the subvariety $\sigma^{-1}(V)$ consists of two irreducible components:

$$\sigma^{-1}(V) = E \cup \widetilde{V}, \tag{3.2.9}$$

where the subvariety $\widetilde{V} = \overline{\sigma^{-1}(V \setminus \{ p \})}$ is called the *proper transform* of V.

Theorem 3.2.13. *Let* (M, α) *be a compact Kähler manifold admitting a Kähler class* $[\omega]$ *such that the numerical criterion* (3.1.22) *holds. Consider the blowup* $\sigma\colon \mathrm{Bl}_p(M) \to M$ *at a point* p*. Then there exists* $\epsilon > 0$ *sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\, \mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\, \mathrm{PD}[E] \tag{3.2.10}$$

are positive and the numerical criterion holds on $\mathrm{Bl}_p(M)$ *in the above classes, provided that* $a < \frac{n}{n-1}c$.

Proof. Consider ϵ such that $\epsilon < \min\{ \Sigma(M, \omega, p), \frac{1}{a}\Sigma(M, \alpha, p) \}$, so that $[\tilde{\omega}]$, $[\tilde{\alpha}]$ are Kähler classes. We have to prove that, for a sufficiently small ϵ,

$$\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha} \wedge \tilde{\omega}^{k-1} \right) > 0$$

for every irreducible k-subvariety $\widetilde{V}$ of $\mathrm{Bl}_p(M)$, with $k = 1, \dots, n-1$. We have to check the positivity in three different cases: the subvarieties of the exceptional divisor, the preimage of subvarieties not passing through p and the proper transform of those passing through p.

Case 1: subvarieties of the exceptional divisor. Consider an irreducible k-subvariety $\widetilde{V}$ of the exceptional divisor, $k = 1, \dots, n-1$. As $\tilde{\omega}|_E = \epsilon\lambda|_E$, $\tilde{\alpha}|_E = a\epsilon\lambda|_E$, we find

$$\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha} \wedge \tilde{\omega}^{k-1}\right) = \epsilon^k\left(n\tilde{c} - ka\right) \int_{\widetilde{V}} \lambda^k.$$

Noting that the integral on the right hand side is bounded from below by a positive constant C_1, as $\lambda|_E$ is the Fubini-Study metric on the projective space, we obtain

$$\begin{aligned}\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha} \wedge \tilde{\omega}^{k-1}\right) &= C_1\left(n\tilde{c} - ka\right)\epsilon^k \\ &= C_1\left(nc - ka\right)\epsilon^k + O(\epsilon^{n+k}).\end{aligned}$$

Thus, if $nc > (n-1)a$, there exists a positive ϵ_1 such that for every $\epsilon \in (0, \epsilon_1)$ the above integral is positive.

Case 2: consider an irreducible k-subvariety $V \subset M$ with $p \notin V$. As σ is a biholomorphism outside E, $\sigma^{-1}(V) \cong V$ and we find

$$\begin{aligned}\int_{\sigma^{-1}(V)} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha} \wedge \tilde{\omega}^{k-1}\right) &= \int_V \left(n\tilde{c}\omega^k - k\alpha \wedge \omega^{k-1}\right) \\ &= \frac{1}{1-\nu\epsilon^n} \int_V \left(nc\omega^k - k\alpha \wedge \omega^{k-1}\right) - \\ &\quad - \frac{\nu\epsilon^n}{1-\nu\epsilon^n} \int_V \left(na\omega^k - k\alpha \wedge \omega^{k-1}\right).\end{aligned}$$

Let us look at the terms multiplying the functions in ϵ: we can think of them as linear functionals on the real 2k-homology group $A, B \colon H_{2k}(M, \mathbb{R}) \to \mathbb{R}$

$$\begin{aligned}A(\gamma) &= \int_\gamma \left(nc\omega^k - k\alpha \wedge \omega^{k-1}\right), \\ B(\gamma) &= \int_\gamma \left(na\omega^k - k\alpha \wedge \omega^{k-1}\right),\end{aligned}$$

applied to the class associated to V. We can choose a norm $\|\cdot\|$ on the finite-dimensional vector space $H_{2k}(M, \mathbb{R})$, such that there exists an irreducible k-subvariety V_0 of M not passing through p with $\|[V_0]\| = 1$. Now, by linearity we have that B is bounded: $|B([V])| \leqslant \delta\,\|V\|$. On the other hand,

$$\inf_V \frac{A([V])}{\|[V]\|} = \inf_{\|[V]\|=1} A([V]).$$

Here the infimum is taken over all irreducible k-subvarieties of M not passing through p. By hypothesis, $A([V])$ is bounded from below by a positive constant C_2:

$$A([V]) = \int_V (nc\omega^k - k\alpha \wedge \omega^{k-1}) > C_2 \qquad \text{for every } V.$$

Thus, the infumum is bounded from below by the same C_2. As a consequence, $A([V]) \geqslant C_2\,\|[V]\|$ for every irreducible k-subvariety V of M not passing through p and we can estimate the initial expression as

$$\int_{\sigma^{-1}(V)} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) \geqslant \frac{1}{1-\nu\epsilon^n} C_2 \left\|[V]\right\| - \frac{\nu\epsilon^n}{1-\nu\epsilon^n}\delta\left\|[V]\right\| = \frac{C_2 - \delta\nu\epsilon^n}{1-\nu\epsilon^n}\left\|[V]\right\|.$$

The term in front of the norm tends to C_2 for $\epsilon \to 0$, so there exists a positive ϵ_2 such that for every $\epsilon \in (0, \epsilon_2)$ the above expression is greater than $\frac{C_2}{2}\left\|[V]\right\|$. Thus, we have

$$\int_{\sigma^{-1}(V)} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) > \frac{C_2}{2}\left\|[V]\right\| > 0.$$

Note that the above choice of ϵ_2 depends just on the dimension k of the subvarieties and on the Kähler classes $[\alpha]$, $[\omega]$ in M. As we have just a finite number of possible dimensions, that are $k = 1, \dots, n-1$, we can take ϵ_2 independent of k.

Case 3: consider an irreducible k-subvariety $V \subset M$ with $p \in V$ and consider its proper transform $\widetilde{V}$. Note that the blowdown map for $\widetilde{V}$ coincides with the map $\sigma\colon \mathrm{Bl}_p(M) \to M$. Splitting the integral as we did in case 2, we find

$$\begin{aligned}\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) &= \int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^{k-1} - k\tilde{\alpha}\wedge\tilde{\omega}^{k-2}\right)\wedge\sigma^*\omega - \\ &\quad + \epsilon\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^{k-1} - k\tilde{\alpha}\wedge\tilde{\omega}^{k-2}\right)\wedge\lambda \\ &= \int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^{k-1} - k\tilde{\alpha}\wedge\tilde{\omega}^{k-2}\right)\wedge\sigma^*\omega - \\ &\quad - \epsilon\int_{\widetilde{V}\cap E} \left(n\tilde{c}\tilde{\omega}^{k-1} - k\tilde{\alpha}\wedge\tilde{\omega}^{k-2}\right) \\ &= \int_{V} \left(n\tilde{c}\omega^k - k\alpha\wedge\omega^{k-1}\right) - \\ &\quad - \epsilon^k(n\tilde{c} - ka)\int_{\widetilde{V}\cap E}\lambda^{k-1}.\end{aligned}$$

Let us set $F = \widetilde{V}\cap E$, which defines a class in $H_{2\bullet}(E,\mathbb{R})$. The first term is again the difference between the linear functionals A and B in $\left(H_{2k}(M,\mathbb{R})\right)^*$. They can be extended to $H_{2k}(M,\mathbb{R}) \oplus H_{2\bullet}(E,\mathbb{R})$ by setting $A(\gamma) = B(\gamma) = 0$ for $\gamma \in H_{2\bullet}(E,\mathbb{R})$. Thus, choosing a norm on $H_{2k}(M,\mathbb{R}) \oplus H_{2\bullet}(E,\mathbb{R})$ such that there exist an irreducible k-subvariety V_0 of M passing through p and a cycle $\gamma_0 \in H_{2\bullet}(E,\mathbb{R})$ with $\left\|[V_0] + \gamma_0\right\| = 1$, we find as before that

$$\int_{V} \left(n\tilde{c}\omega^k - k\alpha\wedge\omega^{k-1}\right) \geqslant \frac{C_2 - \delta\nu\epsilon^n}{1-\nu\epsilon^n}\left\|[V] + [F]\right\|.$$

Note that the choice of the norm does not depend on V, but just on the dimension k. Similarly, we have the functional

$$C([F]) = \int_F \lambda^{k-1},$$

extended to $H_{2k}(M,\mathbb{R}) \oplus H_{2\bullet}(E,\mathbb{R})$ by setting $C(\gamma) = 0$ for all $\gamma \in H_{2k}(M,\mathbb{R})$. Then $C([F]) = C([V]+[F])$ is bounded by $\delta' \,\|[V]+[F]\|$, so that

$$\begin{aligned}\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) &\geqslant \frac{C_2 - \delta\nu\epsilon^n}{1-\nu\epsilon^n}\,\|[V]+[F]\| - \\ &\qquad - \epsilon^k(n\tilde{c} - ka)\delta'\,\|[V]+[F]\| \\ &= \frac{C_2 - C_3\epsilon^k - \delta\nu\epsilon^n + \gamma\nu\epsilon^{n+k}}{1-\nu\epsilon^n}\,\|[V]+[F]\|\,.\end{aligned}$$

The constants C_3 and γ are determined by expressing $\tilde{c}$ in terms of c and ϵ. The factor in front of the norm tends again to C_2 for $\epsilon \to 0$, so there exists a positive ϵ_3 such that for every $\epsilon \in (0,\epsilon_3)$ it is greater that $\frac{C_2}{2}$. Thus, we have

$$\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) > \frac{C_2}{2}\,\|[V]+[F]\| > 0.$$

Again, we can choose ϵ_3 not depending on the dimension k.

To conclude, choosing $\epsilon < \min\{\epsilon_1,\epsilon_2,\epsilon_3\}$, we find that

$$\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - k\tilde{\alpha}\wedge\tilde{\omega}^{k-1}\right) > 0$$

for every irreducible k-subvariety $\widetilde{V}$ of $\mathrm{Bl}_p(M)$, for $k = 1,\dots,n-1$. □

Combining this result with Theorem 3.1.9 of Collins and Székelyhidi, we obtain the existence of non-trivial solutions on toric blowups.

Corollary 3.2.14. *Let (M,ω_0) be a compact, toric, Kähler manifold, $p \in M$ a point invariant under the torus action. Then the blowup $\mathrm{Bl}_p(M)$ admits non-trivial solutions to the J-equation $\Lambda_\omega\alpha = nc$ in the classes*

$$[\omega] = \sigma^*[\omega_0] - \epsilon\,\mathrm{PD}[E]\,, \qquad [\alpha] = \sigma^*[\omega_0] - a\epsilon\,\mathrm{PD}[E]\,, \tag{3.2.11}$$

for ϵ sufficiently small, provided that $a < \frac{n}{n-1}$.

3.3 Inverse σ_m equations

The same arguments of the above section extend to the study of the more general inverse σ_m equations, introduced in [Fang and Lai, 2012; Fang, Lai, and Ma, 2011] with a geometric flow approach, which include the J-equation and the complex Monge-Ampère equation as special cases.

The equation is defined as follows: let (M,α) be a compact Kähler manifold, ω_0 another Kähler metric. We define the *inverse σ_m equation* on the space of Kähler potentials $\mathcal{H} = \{\, \phi \in C^\infty(M,\mathbb{R}) \mid \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0 \,\}$ as

$$\binom{n}{m}\alpha^m \wedge \omega_\phi^{n-m} = nc\,\omega_\phi^n, \tag{3.3.1}$$

where c is the constant

$$c = \binom{n}{m} \frac{[\alpha]^m \smile [\omega_0]^{n-m}}{n[\omega_0]^n} \tag{3.3.2}$$

depending just on the Kähler classes $[\alpha]$ and $[\omega_0]$. We will call it the σ_m *constant*. Note that the inverse σ_1 equation is the J-equation, while the σ_n equation is a complex Monge-Ampère equation. Applying the arguments of the above section, we obtain without any difficulties the following results.

Proposition 3.3.1. *The following statements are equivalent:*

1) $\binom{n}{m}\alpha^m \wedge \omega^{n-m} = nc\omega^n$,

2) $\frac{1}{m!}\Lambda_\omega^m \alpha^m = nm!c$.

The proof easily follows from Weyl's formula. We can also state a maximum principle for the general inverse σ_m equation: in local holomorphic coordinates, it is a second-order nonlinear elliptic PDE in the Kähler potential, written in terms of the mth elementary symmetric function evaluated on the eigenvalues of the matrix $g^{i\bar{k}}\alpha_{j\bar{k}}$. Indeed, the symmetric functions σ_m are defined via the relation

$$\det\left(\delta_i^j + tg^{i\bar{k}}\alpha_{j\bar{k}}\right) = \sum_{m=0}^{n} \sigma_m\left(g^{i\bar{k}}\alpha_{j\bar{k}}\right)t^m. \tag{3.3.3}$$

On the other hand, the determinant on the right hand side can be expressed as

$$\begin{aligned}\det\left(\delta_i^j + tg^{i\bar{k}}\alpha_{j\bar{k}}\right) &= \frac{\det\left(g_{i\bar{j}} + t\alpha_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)} = \frac{(\omega + t\alpha)^n}{\omega^n} \\ &= \sum_{m=0}^{n} \binom{n}{m} \frac{\alpha^m \wedge \omega^{n-m}}{\omega^n} t^m,\end{aligned}$$

so that the inverse σ_m equation can be written as

$$\sigma_m\left(g^{i\bar{k}}\alpha_{j\bar{k}}\right) = nc. \tag{3.3.4}$$

We consider the operator

$$F_m(\phi) = nc - \binom{n}{m} \frac{\alpha^m \wedge \omega_\phi^{n-m}}{\omega_\phi^n} \tag{3.3.5}$$

defined on the space of Kähler potentials. Recall that the variation of F_m at ϕ is the linear operator $D_\phi F_m \colon C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ defined as

$$D_\phi F_m(\xi) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} F_m(\phi + \epsilon\xi). \tag{3.3.6}$$

Set $g_{\epsilon,i\bar{j}} = g_{i\bar{j}} + \partial_i\partial_{\bar{j}}(\phi + \epsilon\xi)$ and define for simplicity the matrix $A_\epsilon = (g_\epsilon^{i\bar{k}}\alpha_{j\bar{k}})$. We have $F_m(\phi + \epsilon\xi) = nc - \sigma_m(A_\epsilon)$, so that

$$\begin{aligned} D_\phi F_m(\xi) &= -\frac{\partial\sigma_m(A_0)}{\partial a^i_j}\frac{dg_\epsilon^{i\bar{k}}}{d\epsilon}\bigg|_{\epsilon=0}\alpha_{j\bar{k}} \\ &= \underbrace{\left(\frac{\partial\sigma_m(A_0)}{\partial a^i_j} g_0^{i\bar{s}}\, g_0^{r\bar{k}}\, \alpha_{j\bar{k}}\right)}_{=F_m^{r\bar{s}}(\phi)} \partial_r\partial_{\bar{s}}\xi. \end{aligned}$$

Then we have that the linear operator

$$D_\phi F_m \colon \xi \longmapsto F_m^{r\bar{s}}(\phi)\partial_r\partial_{\bar{s}}\xi \tag{3.3.7}$$

is elliptic. Indeed, for any point $p \in M$, choose a normal holomorphic coordinate system for the metric ω_ϕ at p, such that α is diagonal: $(g_{0,i\bar{j}}) = \mathrm{Id}$ and $(\alpha_{i\bar{j}}) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Here everything is computed in p. Thus

$$A_0 = \left(g_0^{i\bar{k}}\alpha_{j\bar{k}}\right) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

and, as a consequence,

$$\sigma_m(A_0) = \sum_{1\leqslant i_1<\cdots<i_m\leqslant n} \lambda_{i_1}\cdots\lambda_{i_m}.$$

On the other hand, setting $A_{0,i} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n)$, a simple computation shows that the derivative of σ_m at a diagonal matrix is

$$\frac{\partial\sigma_m(A_0)}{\partial a^i_j} = \begin{cases} \sigma_{m-1}(A_{0,i}) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we find that

$$F_m^{r\bar{s}}(\phi) = \begin{cases} \sigma_{m-1}(A_{0,r})\lambda_r & \text{if } r = s, \\ 0 & \text{otherwise,} \end{cases}$$

so that $F_m^{r\bar{s}}(\phi)$ is positive definite, as all λ_i's are positive:

$$F_m^{r\bar{r}}(\phi) = \sum_{\substack{i_1<\cdots<i_{m-1} \\ i_j\neq r}} \lambda_{i_1}\cdots\lambda_{i_{m-1}}\cdot\lambda_r.$$

As for the J-equation, we can prove a comparison principle for the operators F_m and a uniqueness statement for the inverse σ_m equations.

Proposition 3.3.2. *In the above assumptions, let $\phi, \psi \in \mathcal{H}$ be Kähler potentials such that $F_m(\phi) \geqslant F_m(\psi)$. Then $\phi \geqslant \psi$ on M. Further, if a solution on $\mathcal{H}$ for the inverse σ_m equation $F_m(\phi) = 0$ exists, then it is unique.*

The next step in the analysis of the equations is the identification of a criterion for the existence of a solution. Lejmi and Székelyhidi conjectured a numerical test for the solvability for the inverse σ_m equations, generalising the criterion studied in the previous section.

Conjecture 3.3.3 (Lejmi and Székelyhidi, 2015)**.** There exists a solution of the inverse σ_m equation $\binom{n}{m}\alpha^m \wedge \omega^{n-m} = nc\omega^n$ in $[\omega_0]$ if and only if, for all irreducible subvarieties $V \subset M$ of dimension k, with $k = m, m+1, \dots, n-1$, the *numerical criterion*

$$\int_V \left(nc\omega_0^k - \binom{k}{m}\alpha^m \wedge \omega_0^{k-m} \right) > 0 \tag{3.3.8}$$

holds.

As for the J-equation, the "only if" part can be proved looking at the eigenvalues of the metrics. Further, it must be said that the only case where the above conjecture is known to be true is the Monge-Ampère equation, that is $m = n$: for the inverse σ_n equation the above condition is trivial, reflecting the fact that the complex Monge-Ampère is always solvable.

The numerical criterion can be blown up at a point in the Kähler classes

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\, \mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\, \mathrm{PD}[E]\,, \tag{3.3.9}$$

provided that a is small enough compared to the σ_m constant on M and ϵ is taken sufficiently small. The theorem follows from the same analysis of the σ_m constant on the blowup and arguments similar to those of Theorem 3.2.13.

Lemma 3.3.4. *Consider a compact Kähler manifold (M, α) and another Kähler form ω. Let $\sigma\colon \mathrm{Bl}_p(M) \to M$ be the blowup at a point p and take $\epsilon, \eta > 0$ sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\, \mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - \eta\, \mathrm{PD}[E]\,, \tag{3.3.10}$$

are positive. Suppose that η is of the same order of ϵ, i.e. $\eta = a\epsilon$ for a positive a. Then the σ_m constant on $\mathrm{Bl}_p(M)$, that is $\tilde{c} = \binom{n}{m}\frac{[\tilde{\alpha}]^m \smile [\tilde{\omega}]^{n-m}}{n[\tilde{\omega}]^n}$, satisfies

$$\tilde{c} = \frac{c - \frac{a^m}{n}\binom{n}{m}\nu\epsilon^n}{1 - \nu\epsilon^n} = c + \nu\left(c - \frac{a^m}{n}\binom{n}{m}\right)\epsilon^n + O(\epsilon^{2n}), \tag{3.3.11}$$

where $\nu = \frac{1}{\int_M \omega^n}$.

Theorem 3.3.5. *Let (M, α) be a compact Kähler manifold admitting a Kähler class $[\omega]$ such that the numerical criterion (3.3.8) holds. Consider the blowup $\sigma\colon \mathrm{Bl}_p(M) \to M$ at a point p. Then there exists $\epsilon > 0$ sufficiently small such that*

$$[\tilde{\omega}] = \sigma^*[\omega] - \epsilon\, \mathrm{PD}[E]\,, \qquad [\tilde{\alpha}] = \sigma^*[\alpha] - a\epsilon\, \mathrm{PD}[E] \tag{3.3.12}$$

are positive and the numerical criterion holds on $\mathrm{Bl}_p(M)$ in the above classes, provided that

$$\binom{n-1}{m} a^m < nc. \tag{3.3.13}$$

Sketch of the proof. The interesting part, where the blowing up condition on the parameter a arises, is the case of subvarieties of the exceptional divisor. Consider an irreducible k-subvariety $\widetilde{V}$ of the exceptional divisor, $k = 1, \dots, n-1$. As $\tilde{\omega}|_E = \epsilon\lambda|_E$, $\tilde{\alpha}|_E = a\epsilon\lambda|_E$, we find

$$\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - \binom{k}{m} \tilde{\alpha}^m \wedge \tilde{\omega}^{k-m} \right) = \epsilon^k \left(n\tilde{c} - \binom{k}{m} a^m \right) \int_{\widetilde{V}} \lambda^k.$$

The integral on the right hand side is bounded from below by a positive constant C_1, as $\lambda|_E$ is the Fubini-Study metric on the projective space. Thus, we obtain

$$\begin{aligned}\int_{\widetilde{V}} \left(n\tilde{c}\tilde{\omega}^k - \binom{k}{m} \tilde{\alpha} \wedge \tilde{\omega}^{k-1} \right) &= C_1 \left(n\tilde{c} - \binom{k}{m} a^m \right) \epsilon^k \\ &= C_1 \left(nc - \binom{k}{m} a^m \right) \epsilon^k + O(\epsilon^{n+k}).\end{aligned}$$

Hence, if $\binom{n-1}{m} a^m < nc$, there exists a positive ϵ_1 such that for every $\epsilon \in (0, \epsilon_1)$ the above integral is positive. The other two cases are similar to those of Theorem 3.2.13. □

Again, we have the existence of non-trivial solutions on toric blowups, as Conjecture 3.3.3 was proved by Collins and Székelyhidi in the toric case.

Theorem 3.3.6 (Collins and Székelyhidi, 2014)**.** *Let M be a compact toric manifold with Kähler forms α and ω_0. Suppose that for all toric subvarieties $V \subset M$ of dimension k, with $k = m, \dots, n-1$, we have*

$$\int_V \left(nc\omega_0^k - \binom{k}{m} \alpha^m \wedge \omega_0^{k-m} \right) > 0. \tag{3.3.14}$$

Then there exists $\omega \in [\omega_0]$ such that $\binom{n}{m} \alpha^m \wedge \omega^{n-m} = nc\omega^n$.

Corollary 3.3.7. *Let (M, ω_0) be a compact, toric, Kähler manifold, $p \in M$ a point invariant under the torus action. Then $\mathrm{Bl}_p(M)$ admits a solution to the inverse σ_m equation $\binom{n}{m} \alpha^m \wedge \omega^{n-m} = nc\omega^n$ in the classes*

$$[\omega] = \sigma^*[\omega_0] - \epsilon\, \mathrm{PD}[E]\,, \qquad [\alpha] = \sigma^*[\omega_0] - a\epsilon\, \mathrm{PD}[E]\,, \tag{3.3.15}$$

for ϵ sufficiently small, provided that $\binom{n-1}{m} a^m < n$.

Let us study now the blowup of the projective space: $\mathrm{Bl}_p(\mathbb{P}^n)$ for a generic point $p \in \mathbb{P}^n$. This case was analysed in details by Fang and Lai via a geometric flow approach. More precisely, the following theorem holds.

Theorem 3.3.8 (Fang and Lai, 2013)**.** *Let $M = \mathrm{Bl}_p(\mathbb{P}^n)$ be the blowup of the projective space and denote with $\sigma\colon M \to \mathbb{P}^n$ the blowdown map. Consider the metrics*

$$\tilde{\beta} \in r\,\sigma^*[\omega_{FS}] - \mathrm{PD}[E]\,, \qquad \tilde{\eta} \in s\,\sigma^*[\omega_{FS}] - \mathrm{PD}[E]\,, \qquad r, s > 1, \tag{3.3.16}$$

satisfying the Calabi ansatz. Let ϕ_t *be a solution of the inverse* σ_m*-flow on the space of Kähler potentials* $\mathcal{H} = \{ \phi \in C^\infty(M, \mathbb{R}) \mid \tilde{\eta} + \sqrt{-1}\partial\bar{\partial}\phi > 0 \}$*:*

$$\frac{\partial \phi_t}{\partial t} = \tilde{c}^{\frac{1}{m}} - \left(\binom{n}{m} \frac{\tilde{\beta}^m \wedge \tilde{\eta}_t^{n-m}}{\tilde{\eta}^n} \right)^{\frac{1}{m}}, \tag{3.3.17}$$

where $\tilde{\eta}_t = \tilde{\eta} + \sqrt{-1}\partial\bar{\partial}\phi_t$ *and* $\tilde{c}$ *is the* σ_m *constant on* M*. Then, if the numerical condition*

$$\frac{r^m s^{n-m} - 1}{s^n - 1} > \frac{n-m}{n} \tag{3.3.18}$$

holds, the potential ϕ_t *converges as* $t \to +\infty$ *to a smooth Kähler potential for* $\tilde{\eta}$ *(in the topology of* $C^\infty(M, \mathbb{R})$*), and the limit metric* $\tilde{\eta}_\infty$ *satisfies the critical equation*

$$\binom{n}{m} \tilde{\beta}^m \wedge \tilde{\eta}_\infty^{n-m} = n\tilde{c}\tilde{\eta}_\infty^n. \tag{3.3.19}$$

With the above notation, set $a\epsilon = \frac{1}{r}$ and $\epsilon = \frac{1}{s}$. Then the numerical condition (3.3.18) becomes

$$\frac{1 - a^m \epsilon^n}{1 - \epsilon^n} > a^m \frac{n-m}{n}. \tag{3.3.20}$$

The left hand side can be expanded in ϵ as $1 + O(\epsilon^n)$, so that for small ϵ we have the condition $a^m < \frac{n}{n-m}$. On the other hand, as we are blowing up the trivial equation in the Kähler classes $[\alpha] = [\omega] = [\omega_{FS}]$ on $\mathbb{P}^n$, we have

$$c = \binom{n}{m} \frac{[\omega_{FS}]^m \smile [\omega_{FS}]^{n-m}}{n[\omega_{FS}]^n} = \frac{1}{n}\binom{n}{m}. \tag{3.3.21}$$

Thus, the blowup condition (3.3.13) becomes exactly

$$\binom{n-1}{m} a^m < n \frac{1}{n} \binom{n}{m} \qquad \Longleftrightarrow \qquad a^m < \frac{n}{n-1}, \tag{3.3.22}$$

in accordance with the result by Fang and Lai.

We can also directly study the numerical criterion (3.3.14), as far as we blowup a T-invariant point p of $\mathbb{P}^n$. In this case, $\mathrm{Bl}_p(\mathbb{P}^n)$ is still toric and we can check the positivity condition on all toric subvarieties. For simplicity, we will consider the J-equation on the blowup of $\mathbb{P}^3$ at $p = [1:0:0:0]$. To that end, let us consider the fan Σ associated to $\mathbb{P}^3$ (see [Fulton, 1993] and [Hori et al., 2003] as references for Toric Geometry): it is the fan generated by the four edges $\Sigma(1) = \{ u, e_1, e_2, e_3 \}$, where $\{ e_1, e_2, e_3 \}$ is the standard basis in $\mathbb{R}^3$ and

$$u = -(e_1 + e_2 + e_3) = (-1, -1, -1). \tag{3.3.23}$$

The cones of dimension 2 are

$$\Sigma(2) = \{ \langle u, e_1 \rangle, \langle u, e_2 \rangle, \langle u, e_3 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_3 \rangle, \langle e_2, e_3 \rangle \}, \tag{3.3.24}$$

while those of dimension 3 are

$$\Sigma(3) = \{ \langle u, e_1, e_2 \rangle, \langle u, e_1, e_3 \rangle, \langle u, e_2, e_3 \rangle, \langle e_1, e_2, e_3 \rangle \}. \tag{3.3.25}$$

Here $\langle v_1, \ldots, v_k\rangle$ denotes the cone generated by the edges $v_1, \ldots, v_k$.

Consider the coordinates (z_0, z_1, z_2, z_3) on $\mathbb{C}^4$ associated to the ordered set of edges (u, e_1, e_2, e_3), which after the proper quotient by $\mathbb{C}^*$ define the homogeneous coordinates $[Z^0 : Z^1 : Z^2 : Z^3]$ on $\mathbb{P}^3$. In the quotient, the point $p = [1 : 0 : 0 : 0]$ corresponds to the cone $c = \langle e_1, e_2, e_3\rangle$, so that to blowup $\mathbb{P}^3$ at p, we have to introduce the edge $v = e_1 + e_2 + e_3$ and, with the proper subdivision of c, we obtain the fan Σ' associated to $\mathrm{Bl}_p(\mathbb{P}^3)$. The edges are $\Sigma'(1) = \{u, e_1, e_2, e_3, v\}$, the cones of dimension 2 are given by

$$\Sigma'(2) = \Sigma(2) \cup \{ \langle e_1, v\rangle , \langle e_2, v\rangle , \langle e_3, v\rangle \}, \tag{3.3.26}$$

and those of dimension 3 are

$$\Sigma'(3) = \big(\Sigma(3) \setminus \{ c \}\big) \cup \{ \langle e_1, e_2, v\rangle , \langle e_1, e_2, v\rangle , \langle e_2, e_3, v\rangle \}. \tag{3.3.27}$$

Consider the blowdown map $\sigma\colon \mathrm{Bl}_p(\mathbb{P}^3) \to \mathbb{P}^3$. The correspondence

$$\big\{ \text{cones of } \Sigma' \big\} \longleftrightarrow \big\{ \text{non-empty T-invariant subvarieties of } \mathrm{Bl}_p(\mathbb{P}^3) \big\} \tag{3.3.28}$$

allows us to immediately find the subvarieties we need to check for the numerical criterion. Let us list the toric subvarieties, excluding $\mathrm{Bl}_p(\mathbb{P}^3)$ and the T-invariant points, which correspond to $\{0\}$ and the 3-cones respectively.

cone	subvariety
u	$\sigma^{-1}(Z_0 = 0) \cong \{ Z_0 = 0 \}$
e_i	$\sigma^{-1}(Z_i = 0)$
v	$E = \sigma^{-1}(p)$
$\langle u, e_i\rangle$	$\sigma^{-1}(Z_0 = Z_i = 0) \cong \{ Z_0 = Z_i = 0 \}$
$\langle e_i, e_j\rangle$	$\sigma^{-1}(Z_i = Z_j = 0)$
$\langle v, e_i\rangle$	$\sigma^{-1}(Z_i = 0) \cap E$

Consider as before the Kähler metrics on $\mathrm{Bl}_p(\mathbb{P}^3)$

$$\widetilde{\omega} \in \sigma^*[\omega_{FS}] - \epsilon\,\mathrm{PD}[E] , \qquad \widetilde{\alpha} \in \sigma^*[\omega_{FS}] - a\epsilon\,\mathrm{PD}[E] . \tag{3.3.29}$$

We have the associated J-constant

$$\tilde{c} = \frac{1 - a\epsilon^3}{1 - \epsilon^3} = 1 - a\epsilon^3 + O(\epsilon^6). \tag{3.3.30}$$

Let us just compute the integral along the toric subvarieties associated to the five edges of Σ'.

1) The hypersurface V_u associated to u does not meet the exceptional divisor, so that we can reduce it to the integral along a hypersurface in $\mathbb{P}^3$:

$$\begin{aligned}\int_{V_u} \big(3\tilde{c}\tilde{\omega}^2 - 2\tilde{\alpha} \wedge \tilde{\omega}\big) &= (3\tilde{c} - 2) \int_{Z^0=0} \omega_{FS}^2 \\ &= (3\tilde{c} - 2) = 1 - 3a\epsilon^3 + O(\epsilon^6).\end{aligned} \tag{3.3.31}$$

Thus, for small ϵ the integral on the left hand side is positive.

2) Consider now the hypersurface V_i defined as the proper transform of $\{Z^i = 0\}$, which are identified by the edges e_i. With the same calculations as in the proof of Theorem 3.2.13, we have

$$\begin{aligned}\int_{V_i} (3\tilde{c}\tilde{\omega}^2 - 2\tilde{\alpha}\wedge\tilde{\omega}) &= (3\tilde{c}-2)\int_{Z^i=0}\omega_{FS}^2 - (3\tilde{c}-2a)\epsilon^2\int_{V_i\cap E}\lambda \\ &= (3\tilde{c}-2) - (3\tilde{c}-2a)\epsilon^2 \\ &= 1-(3-2a)\epsilon^2 + O(\epsilon^3).\end{aligned} \tag{3.3.32}$$

Here we have used the fact that

$$\int_{V_i\cap E}\lambda = \int_{W^i=0}\omega_{FS,\mathbb{P}^2} = 1, \tag{3.3.33}$$

where $[W^1 : W^2 : W^3]$ are the homogeneous coordinates in $E \cong \mathbb{P}^2$.

3) Finally, the integral along the exceptional divisor is

$$\begin{aligned}\int_E (3\tilde{c}\tilde{\omega}^2 - 2\tilde{\alpha}\wedge\tilde{\omega}) &= (3\tilde{c}-2a)\epsilon^2\int_E\lambda^2 \\ &= (3\tilde{c}-2a)\epsilon^2 = (3-2a)\epsilon^2 + O(\epsilon^5).\end{aligned} \tag{3.3.34}$$

Here we have used the fact that

$$\int_E\lambda = \int_{\mathbb{P}^2}\omega_{FS,\mathbb{P}^2} = 1, \tag{3.3.35}$$

where again $[W^1 : W^2 : W^3]$ are the homogeneous coordinates in $E \cong \mathbb{P}^2$. For small ϵ, the integral along E is positive, provided that $a < \frac{3}{2}$.

The same calculations can be done for the subvarieties associated to 2-cones. The result is that the positivity condition holds for ϵ sufficiently small, if the blowup parameter satisfies $a < \frac{3}{2}$. This is again the blowup condition found by Fang and Lai and the same of Theorem 3.2.13.

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