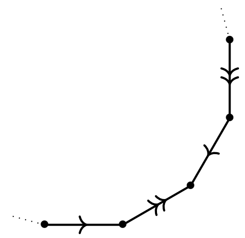
ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exam sheet

The first problem on your exam will be one of the following 15 problems, chosen randomly.

Problem 1. Compute the genus of the surface X obtained by gluing the sides of a regular 20-gon according to the pattern in the picture below.



Problem 2. Let $f\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Show that if $g_X = g_Y \geq 2$, then X and Y are biholomorphic.

Problem 3. Compute the genus of a double cover of $\mathbb{P}^1$ branched at 12 points.

Problem 4. Let f be a meromorphic function on a compact Riemann surface X with one simple pole and no other poles. Prove that $f\colon X \to \mathbb{P}^1$ is a biholomorphism.

Problem 5. Consider the meromorphic function $f(z) = \frac{z}{(z-1)^2}$ on $\mathbb{C}$. Write it as a holomorphic function on $\mathbb{P}^1$ and compute the ramification points and their multiplicities.

Problem 6. Consider the meromorphic function $f(z) = \frac{z^2+4}{z^3-z}$ on $\mathbb{C}$. Write it as a holomorphic function on $\mathbb{P}^1$ and compute $\operatorname{div}(f)$.

Problem 7. Show that $\operatorname{Pic}(\mathbb{P}^1) = \mathbb{Z}$.

Problem 8. Show that, up to biholomorphism, $\mathbb{P}^1$ is the only compact Riemann surface of genus 0.

Problem 9. Let $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be a torus and denote by $O \in X$ the unit. Prove that the functions $1, \wp, \wp'$ form a basis of $\mathcal{L}(3[O])$.

Problem 10. Let D be a divisor on $\mathbb{P}^1$. Show that $\ell(D) = \max\{\, 0, 1 + \deg D \,\}$.

Problem 11. Let D be a divisor on a compact Riemann surface X of genus g, such that $\deg D = 2g - 2$ and $\ell(D) = g$. Show that D is a canonical divisor.

Problem 12. Show that every genus 2 Riemann surface is hyperelliptic, that is, it admits a degree two map to $\mathbb{P}^1$.

Problem 13. Let X be a genus 0 Riemann surface with canonical divisor K. Compute $\ell(mK)$ for $m \in \mathbb{Z}$.

Problem 14. Let X be a torus. Show that for every divisor D of degree 1 there is a unique point $x \in X$ such that D is linearly equivalent to the divisor $[x]$.

Problem 15. Compute the Hurwitz number $H^{\bullet}_{0 \xrightarrow{3} 0}\big((3),(3)\big)$ by counting permutations.