ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exercise sheet 1

Exercise 1 (Cauchy–Riemann equations)**.** *Let f be a holomorphic function on an open $U \subseteq \mathbb{C}$. Write $f(z) = u(x,y) + \mathrm{i}v(x,y)$ for $z = x + \mathrm{i}y$, considered as a real differentiable function on $\mathbb{R}^2$. Prove that*

$$u_x = v_y\,, \qquad v_x = -u_y\,. \tag{1}$$

Exercise 2. *Let γ_r be the counter-clock wise oriented circle of radius r centred at the origin $0 \in \mathbb{C}$. Prove that, for $n \in \mathbb{Z}$,*

$$\frac{1}{2\pi\mathrm{i}} \oint_{\gamma_r} z^{n-1}\, dz = \delta_{n,0}\,. \tag{2}$$

Exercise 3. *Show that if f has a pole of order 1 at z_0 (also called a simple pole), then the residue can be computed as*

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z)\,. \tag{3}$$

Exercise 4 (Residue theorem)**.** *Let γ be a counter-clock wise oriented loop in an open set $U \subseteq \mathbb{C}$ that does not cross itself and contains the points $z_1, \dots, z_N$. Let f be a holomorphic function in $U \setminus \{z_1, \dots, z_N\}$ with poles at $z_1, \dots, z_N$. Then*

$$\frac{1}{2\pi\mathrm{i}} \oint_{\gamma} f(z)\, dz = \sum_{i=1}^{N} \operatorname*{Res}_{z=z_i} f(z)\,. \tag{4}$$

💡 Hint. *Consider (small enough) counter-clock wise oriented loops around the poles, $\gamma_1, \dots, \gamma_N$, and the deformation of the contour $\gamma \cup \gamma_1 \cup \dots \cup \gamma_N$ depicted in figure 1.*

Exercise 5 (Basel problem 💀)**.** *Prove the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{2}$. The left-hand side is the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ evaluated at $s = 2$. Can you use the same strategy to prove the following equation?*

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k} (2\pi)^{2k}}{2(2k)!}\,, \tag{5}$$

where B_m is the m-th Bernoulli number. What is wrong with the odd values of the zeta function?

💡 Hint. *Consider the function $f(z) = \frac{\pi}{z^2} \cot(\pi z)$ and the residue theorem.*

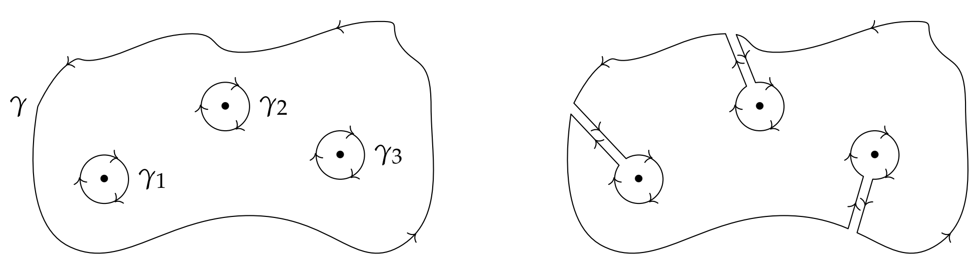

Figure 1. Contour deformation in the proof of the residue theorem.