
Exercise 1. *For a fixed identification polygon decomposition, prove that there exists a choice of basis of holomorphic differentials* $(\omega_1, \dots, \omega_g)$ *such that*

$$\int_{a_i} \omega_j = \delta_{i,j} \,.$$

In this case, it is customary to denote the matrix of B-periods as $\tau = (\tau_{i,j})_{1 \le i,j \le g}$, *where* $\tau_{i,j} = \oint_{b_i} \omega_j$. *Prove that the matrix* τ *is symmetric and its imaginary part is positive-definite.*

The space of $g \times g$ symmetric matrices over the complex numbers whose imaginary part is positive definite is called the Siegel upper-half space of degree g, and it generalises the upper-half space $\mathbb{H}$ (corresponding to $g = 1$) to higher dimensions. Lie-theoretically, it is the symmetric space associated to the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$.

Exercise 2. *Prove the Weierstraß gap theorem on the torus* $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ *using Abel's theorem.*

Exercise 3. *Prove that*

$$H_{0 \xrightarrow{d} 0}\big((d), (d)\big) = \frac{1}{d} \,.$$

Hint. *Choose* $y_1 = \mathbf{0}$ *and* $y_2 = \boldsymbol{\infty}$ *on the target* $\mathbb{P}^1$. *Write down all possible maps* $f \colon \mathbb{P}^1 \to \mathbb{P}^1$ *branched over* $\mathbf{0}$ *and* $\boldsymbol{\infty}$, *and show that they are all isomorphic. Can you write down the automorphism group of* f?

Exercise 4. *Compute* $x + y$ *and* $x \cdot y$ *in the group algebra* $\mathbb{C}[S_3]$, *where* $x = 3(1\,2) + 5(1\,2\,3)$ *and* $y = 4(1\,3) - 6(1\,2\,3)$. *Compute* C_μ *for* $\mu = (2, 1)$.

Exercise 5. *Compute* $H^{\bullet}_{0 \xrightarrow{3} 0}\big((3), (3)\big)$ *by counting permutation. Can you generalise the computation* $H^{\bullet}_{0 \xrightarrow{d} 0}\big((d), (d)\big)$? *And can you motivate why* $H^{\bullet}_{0 \xrightarrow{d} 0}\big((d), (d)\big) = H_{0 \xrightarrow{d} 0}\big((d), (d)\big)$ *in this case?*

Exercise 6. *Show that* $H^{\bullet}_{0 \xrightarrow{4} 0}\big((3,1), (2,2), (2,2)\big) = 0$ *by counting permutation.*

The previous exercise demonstrates that, even if the discrete data satisfies the Riemann–Hurwitz formula, it is still possible that no Hurwitz covers exist for that data. Remarkably, determining the necessary and sufficient conditions for a Hurwitz number to be non-zero remains an open problem, known as the Hurwitz existence problem.