Exercise 1 (Stereographic projections). *On the circle*

$$S^1 := \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}, \tag{1}$$

let $\mathrm{N} := (0,1)$ *and* $\mathrm{S} := (0,-1)$ *be the north and south poles respectively. Define* $U'_{\mathrm{N}} := S^1 \setminus \mathrm{N}$ *and the map* $\varphi'_{\mathrm{N}} \colon U'_{\mathrm{N}} \to \mathbb{R}$ *by declaring that* $\varphi'_{\mathrm{N}}(x,y)$ *is the unique intersection between the x-axis and the line passing through* (x,y) *and* N *(see figure 1). Similarly, define* $U'_{\mathrm{S}} := S^1 \setminus \mathrm{S}$ *and the map* $\varphi'_{\mathrm{S}} \colon U'_{\mathrm{S}} \to \mathbb{R}$ *declaring that* $\varphi'_{\mathrm{S}}(x,y)$ *is the unique intersection between the x-axis and the line passing through* (x,y) *and* S.

- *Find a formula for* φ'_{N} *and* φ'_{S}.
- *Prove that* φ'_{N} *and* φ'_{S} *are homeomorphisms.*
- *Prove that transition function* φ'_{NS} *is smooth.*

Can you generalise this construction to S^n*?*

FIGURE 1. The atlas for the circle provided by the stereographic projections.

Exercise 2 (Graphs of smooth functions). *Consider an open set* $U \subseteq \mathbb{R}^n$ *and a smooth function* $f \colon U \to \mathbb{R}^m$. *Show that the graph of* f, *that is*

$$\Gamma_f = \{ (x,y) \in U \times \mathbb{R}^m \mid y = f(x) \} \tag{2}$$

is a smooth manifold of dimension n.

Exercise 3 (The sphere model of the complex projective line). *Prove that any point* $[z_0 : z_1] \in P^1(\mathbb{C})$ *can be realised as* $[x_1 : x_2 + \mathrm{i}x_3]$, *with* x_i *real,* $x_1^2 + x_2^2 + x_3^2 = 1$, *and* $x_1 \geq 0$. *Deduce that* $P^1(\mathbb{C})$, *as a 2-dimensional real smooth manifold, is the sphere* S^2.

Exercise 4 (The cross-cap). *Repeat the same argument for* $P^2(\mathbb{R})$, *and deduce that the real projective plane can be identified with a hemisphere with boundary glued along the antipodal map.*

Exercise 5 (Torus). *Let* ω_1 *and* ω_2 *be two complex numbers which are linearly independent over* $\mathbb{R}$ *(that is, they do not lie on the same real line through* 0 *in* $\mathbb{C}$*). The set of all integral linear combinations of* ω_1 *and* ω_2, *that is*

$$\Lambda := \{ n_1\omega_1 + n_2\omega_2 \in \mathbb{C} \mid n_1,\ n_2 \in \mathbb{Z} \}, \tag{3}$$

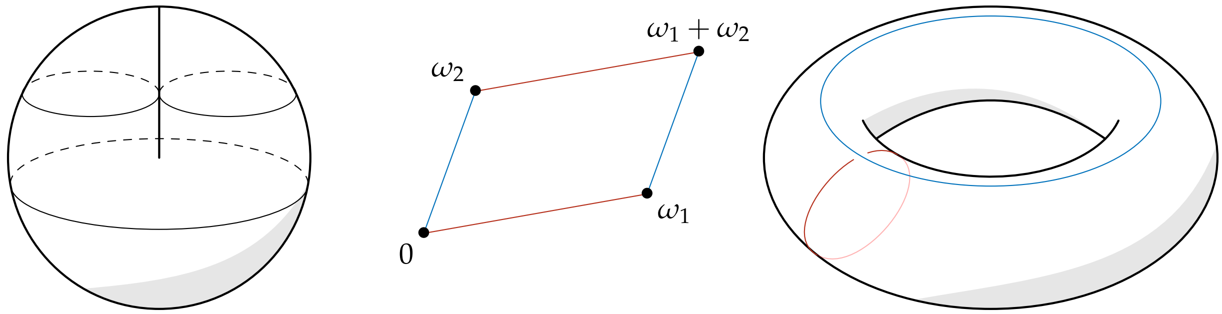

FIGURE 2. On the left, the cross-cap. On the right, the the gluing of a polygon and its identification with a torus.

is called a lattice. Define $T := \mathbb{C}/\Lambda$*, equipped with the quotient topology induced by the projection map* $\pi\colon \mathbb{C} \to T$*.*

- *Consider the closed polygon* $P \subset \mathbb{C}$ *with vertices* $0, \omega_1, \omega_2, \omega_1 + \omega_2$*. Show that for any* $z \in \mathbb{C}$*, there exists* $z_0 \in P$ *such that* $z - z_0 \in \Lambda$*. Thus,* $\pi|_P\colon P \to T$ *is surjective. What happens at the sides of* P*?*
- *Deduce that every point in* T *has a neighbourhood homeomorphic to a disc in* $\mathbb{C}$*, and that the transitions maps are translations.*

Since translations are holomorphic, we deduce that T *is a Riemann surface.*