ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exercise sheet 3

Exercise 1. *Let $A = \{ a, b \}$, so that $A \cup \bar{A} = \{ a, b, \bar{a}, \bar{b} \}$. Consider the identification polygons*

$$w_1 = a\bar{a}b\bar{b}\,, \qquad w_2 = aab\bar{b}\,, \qquad w_3 = ab\bar{a}\bar{b}\,, \qquad w_4 = aba\bar{b}\,.$$

Which surfaces do they represent?

Exercise 2. *Convince yourself that $P^2(\mathbb{R})\#P^2(\mathbb{R})$ is the Klein bottle, and prove using polygon identification that $P^2(\mathbb{R})\#T \cong P^2(\mathbb{R})^{\#3}$.*

Exercise 3. *Come up with a list of identification polygons that produce the list in the classification theorem of connected, compact surfaces. Prove that any identification polygon gives a surface homeomorphic to one in the list.*

Hint. *For the second part, you can follow the steps below.*

- *Prove that if an identification polygon contains the subword $a\bar{a}$ or $\bar{a}a$, then you can remove the subword and obtain an homeomorphic surface.*
- *Prove that you can always obtain an equivalent identification polygon where all vertices are identified with each other.*
- *Prove that you can always reduce yourself to the case $S \cong S'\#P^2(\mathbb{R})^{\#m}$ and S' only contains pairs of conjugate letters.*
- *Prove that you can always reduce yourself to the case $S \cong T^{\#g}\#P^2(\mathbb{R})^{\#m}$.*
- *Conclude that if there is one $P^2(\mathbb{R})$ factor, then the surfaces can be re-arranged to have only $P^2(\mathbb{R})$ factors.*

Exercise 4. *Prove that the Euler characteristic of $T^{\#g}$ is $2 - 2g$ and the Euler characteristic of $P^2(\mathbb{R})^{\#m}$ is $2 - m$. Conclude that the genus and the demigenus uniquely characterise the surface within the orientable and the non-orientable classes.*