ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exercise sheet 4

Recall the 'classification theorem' of complex tori.

The set of equivalence classes of complex tori is in one-to-one correspondence with the upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$, quotient by the action of $\mathrm{SL}(2,\mathbb{Z})$:

$$\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \xrightarrow{1:1} \{\, \mathbb{C}/\Lambda \,\}\,/\text{biholomorphism}\,, \qquad [\tau] \longmapsto [\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})]\,. \tag{1}$$

The group $\mathrm{SL}(2,\mathbb{Z})$ is the group of 2×2 matrices with integer coefficients and determinant 1. Its action on the upper-half plane is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .\tau := \frac{a\tau + b}{c\tau + d}\,. \tag{2}$$

Exercise 1. *Prove the theorem above by following the steps below.*

- *Let $\Lambda, \Lambda' \subset \mathbb{C}$ be two lattices. Suppose $\exists \alpha \in \mathbb{C}^*$ such that $\alpha\Lambda \subseteq \Lambda'$. Show that the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \alpha z$ induces a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$, which is biholomorphic if and only if $\alpha\Lambda = \Lambda'$.*
- *Show that every torus $\mathbb{C}/\Lambda$ is isomorphic to a torus of the form $T(\tau) := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{H}$.*
- *Suppose $\gamma \in \mathrm{SL}(2,\mathbb{C})$ and $\tau \in \mathbb{H}$. Let $\tau' := \gamma.\tau$, according to the action defined in equation (2). Prove that $T(\tau)$ and $T(\tau')$ are isomorphic.*

Exercise 2. *Let $F \in \mathbb{C}[z_0, z_1, z_2]$. Prove that the following are equivalent.*

- *F is homogeneous of degree d.*
- *Every monomial in F has degree d.*
- *F satisfies the so-called Euler identity: $z_0 \frac{\partial F}{\partial z_0} + z_1 \frac{\partial F}{\partial z_1} + z_2 \frac{\partial F}{\partial z_2} = d \cdot F$.*

Exercise 3 (The Puzzle of the Doctor of Physic 💀)**.** *Find a positive rational solution (different from the trivial solution $\{1, 2\}$) to the equation*

$$x^3 + y^3 = 9\,. \tag{3}$$

This is a version of Fermat's Last Theorem, with the right-hand side equal to 9 (rather than 1).

The question is a reformulation of the 20th puzzle from The Canterbury Puzzles and Other Curious Problems *(1907) by Henry Dudeney.*

> *This Doctor, learned though he was, for "In all this world to him there was none like To speak of physic and of surgery," and "He knew the cause of every malady," yet was he not indifferent to the more material side of life. "Gold in physic is a cordial; Therefore he lovéd gold in special." The problem that the Doctor propounded to the assembled pilgrims was this. He produced two spherical phials, as shown in our illustration, and pointed out that one phial was exactly a foot in circumference, and the other two feet in circumference.*
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> *"I do wish," said the Doctor, addressing the company, "to have the exact measures of two other phials, of a like shape but different in size, that may together contain just as much*

liquid as is contained by these two." To find exact dimensions in the smallest possible numbers is one of the toughest nuts I have attempted. Of course the thickness of the glass, and the neck and base, are to be ignored.

In mathematical terms, the puzzle can be rephrased as follows. The Doctor has two spherical phials of circumference 1 and 2 feet respectively, that is radii $(2\pi)^{-1}$ *and* $(\pi)^{-1}$ *respectively. Hence, the total volume contained in the two phials is* $\frac{4}{3}\pi((2\pi)^{-3} + (\pi)^{-3}) = \frac{3}{2\pi^2}$*. The Doctor is asking for two more spherical phials with the same total volume, but different rational circumferences. That is, he is looking for a pair of positive rational numbers* (x, y)*, different from* $(1, 2)$ *and* $(2, 1)$ *such that*

$$\frac{4}{3}\pi\left(\left(\frac{x}{2\pi}\right)^3 + \left(\frac{y}{2\pi}\right)^3\right) = \frac{3}{2\pi^2}\,. \tag{4}$$

Simplifying, we get to find a non-trivial positive rational solution to equation (3).

Hint. *Notice that* $E = Z(z_0^3 + z_1^3 - 9z_2^3)$ *is a smooth projective plane curve of degree* 3*, hence an elliptic curve with its group structure. Can you define the group law (even if E is not in Weierstraß form)? And can you use the group law to find new rational solutions, starting from the trivial ones?*