ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exercise sheet 5

Exercise 1. *Let $f\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces.*

- *Show that $g_X \geq g_Y$.*
- *Suppose that both X and Y have genus 1. Conclude that f, must be unramified.*
- *Suppose that $Y = P^1(\mathbb{C})$ and f is unramified. Conclude that X is biholomorphic to $\mathbb{P}^1(\mathbb{C})$.*
- *Show that if $g_X = g_Y \geq 2$, then X and Y are biholomorphic.*

Exercise 2. *Consider the meromorphic function on $\mathbb{C}$ given by $f(z) = \frac{z^3}{1-z^2}$. Can you define a map $F\colon P^1(\mathbb{C}) \to P^1(\mathbb{C})$ that equals f in the affine chart $\{ [z:w] \mid w \neq 0 \}$? Show that F has degree 3, find its ramification points, and verify the Riemann–Hurwitz formula in this case.*

Exercise 3 (Fermat's Last Theorem for polynomials)**.** *Let $F, G, H \in \mathbb{C}[z,w]$ be non-constant, coprime, homogeneous polynomials such that $F^n + G^n = H^n$. Show that $n \leq 2$.*

Exercise 4. *For an open set $U \subseteq \mathbb{C}$, prove that $\mathrm{ord}_{z_0}\colon \mathcal{M}(U) \to \mathbb{Z} \cup \{\infty\}$ is a discrete valuation:*

- $\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$,
- $\mathrm{ord}_{z_0}(f + g) \geq \min\{ \mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g) \}$,
- $\mathrm{ord}_{z_0}(f) = \infty$ *if and only if* $f = 0$.