Exercise 1 (Meromorphic functions on $\mathbb{P}^1$)**.** *Prove that all meromorphic on $\mathbb{P}^1$ are of the form*

$$[z:w] \longmapsto [F(z,w):G(z,w)] \tag{1}$$

where $F, G \in \mathbb{C}[z,w]$ are homogeneous polynomials of the same degree with no common components.

Exercise 2 (Weierstraß gap theorem for the torus)**.** *Fix a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and assume $f\colon \mathbb{C} \to \mathbb{P}^1$ is a Λ-periodic function with no poles on the boundary of the closed polygon $P \subset \mathbb{C}$ with vertices 0, ω_1, ω_2, $\omega_1 + \omega_2$ (the fundamental domain). Show that the sum of the residues of all poles of f inside P is zero. Conclude that no such f can have a unique simple pole in P.*

Exercise 3 (Weierstraß $\wp$-function)**.** *Fix a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and consider*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \tag{2}$$

the Weierstraß $\wp$-function associated to Λ.

(i) Show that the series defining $\wp$ converges absolutely for every $z \in \mathbb{C} \setminus \Lambda$ and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$.

Hint. *Use Weierstraß's M-test.*

(ii) Show that the derivative $\wp'$ is Λ-periodic and odd and that $\wp$ is even. Use this to show that $\wp$ is Λ-periodic. Conclude that $\wp$ is a meromorphic function on $\mathbb{C}/\Lambda$ with a single double pole at $[0]$.

(iii) Show that $\wp'$ has simple zeroes exactly at the points $[\frac{\omega_1}{2}], [\frac{\omega_1}{2}], [\frac{\omega_1+\omega_2}{2}] \in \mathbb{C}/\Lambda$.

Hint. *Use that $\wp'$ satisfies $\sum_{[z]} \operatorname{ord}_{[z]}(\wp') = 0$.*

(iv) Prove that every Λ-periodic function f with double poles exactly at the elements of Λ is of the form $f(z) = a\wp(z) + b$ for $a \in \mathbb{C}^*, b \in \mathbb{C}$. Hence, up to translation, scaling and adding constant functions, $\wp$ is the unique meromorphic function on $\mathbb{C}/\Lambda$ with a single double pole.

Hint. *Use Weierstraß's gap theorem.*

(v) Let the Laurent expansion of (the even function) $\wp$ around $z = 0$ be given by

$$\wp(z) = \frac{1}{z^2} + c + \frac{1}{20} g_2 \, z^2 + \frac{1}{28} g_3 \, z^4 + \mathrm{O}(z^6) \,. \tag{3}$$

Show that $c = 0$ and that $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\,\wp^3 - g_2\,\wp - g_3 \,. \tag{4}$$

Exercise 4 (Complex tori and elliptic curves 💀)**.** *Let $E_\Lambda = Z(z_1^2 z_2 - z_0^3 + \frac{g_2}{4} z_0 z_2^2 + \frac{g_3}{4} z_0^3) \subset \mathbb{P}^2$. Show that E_Λ is a smooth projective plane curve if and only if $\Delta = g_2^3 - 27 g_3^2 \neq 0$. Prove that the function*

$$\begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\quad \phi \quad} E_\Lambda \\ [z] &\longmapsto [\wp(z) : 2\wp'(z) : 1] \end{aligned} \tag{5}$$

is well-defined (here we set $\phi([0]) = [0:1:0]$), and show that it defines a biholomorphism. Conclude that all elliptic curves are biholomorphic to complex tori.