
Exercise 1. *Consider the meromorphic function $f(z) = \frac{z^3 - z^2}{z^2+1}$ on $\mathbb{C}$.*

(1) Using the identification $\mathbb{C} \subset \mathbb{P}^1, z \mapsto [z:1]$, on source and target, identify f as a map $\mathbb{P}^1 \to \mathbb{P}^1$ and write it in the form $[z:w] \mapsto [F(z,w) : G(z,w)]$ for homogeneous polynomials F, G of the same degree having no common factors.
(2) Compute $\operatorname{div}(f)$.

Exercise 2 (Abel's theorem 2.0)**.** *On a torus $T = \mathbb{C}/\Lambda$, consider the map*

$$\mathrm{A} \colon \mathrm{Div}(T) \longrightarrow T\,, \qquad \sum_{x \in T} n_x [x] \longmapsto \sum_{x \in T} n_x\, x\,. \tag{1}$$

The sum on the right-hand side is intended according to the group law on T. Show that D is principal if and only if $\deg(D) = 0$ and $\mathrm{A}(D) = [0]$. Conclude that two divisors $D, E \in \mathrm{Div}(X)$ are linearly equivalent if and only if $\deg(D) = \deg(E)$ and $\mathrm{A}(D) = \mathrm{A}(E)$.

Hint. *Use Abel's theorem for the torus and the existence of theta functions.*

Exercise 3 (Pullback of divisors)**.** *Let $\varphi \colon X \to Y$ be a holomorphic map between compact Riemann surfaces. For any point $y \in Y$, define the pullback*

$$\varphi^*[y] := \sum_{x \in \varphi^{-1}(y)} \mu_x(\varphi) \cdot [x]\,, \tag{2}$$

and extend it by linearity to a group morphism $\varphi^* \colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$. Prove that $\deg(\varphi^* D) = \deg(\varphi) \cdot \deg(D)$. What happens when you pullback principal divisors?

Exercise 4 (Complete linear systems)**.** *Let X be a compact Riemann surface, $D \in \mathrm{Div}(X)$. Define the complete linear system associated to D as*

$$|D| := \{\, E \in \mathrm{Div}(X) \mid E \geq 0 \text{ and } E \sim D \,\}\,. \tag{3}$$

Prove that the map

$$\mathbb{P}(\mathcal{L}(D)) \longrightarrow |D|\,, \qquad [f] \longmapsto \operatorname{div}(f) + D \tag{4}$$

is well-defined and is an isomorphism. Compute $|D|$ for:

- *$X = \mathbb{P}^1$ and $D = d[x]$;*
- *$X = \mathbb{C}/\Lambda$ and $D = [x]$.*