ETHZ – Riemann Surfaces – Spring 2024 **Lecturer: Alessandro Giacchetto**
Exercise sheet 9

Exercise 1 (Weierstraß's gap theorem). *Let X be a compact Riemann surface of genus $g > 0$, and let $x \in X$ be a point. Show that there are precisely g integers (depending on x) of the form*

$$1 \leq k_1 < k_2 < \cdots < k_g < 2g \tag{1}$$

such that there exists no meromorphic function on X with a single pole of order k_i at x.

Exercise 2 (Compact Riemann surfaces are projective). *Show that every compact Riemann surface X admits an embedding in projective space.*

Exercise 3 (Holomorphic q-differentials). *For $q \in \mathbb{Z}$, a holomorphic differential of order q (or simply q-differential) on an open set $U \subset \mathbb{C}$ is an expression of the form*

$$f(z)\, dz^q\,. \tag{2}$$

As in the case of holomorphic functions or differentials (corresponding to the cases $q = 0$ and $q = 1$ respectively), there is a natural notion of transformation under a holomorphic change of coordinate $w = \tau(z)$: it is the one induced by $dw^q = (\tau'(z))^q\, dz^q$. Hence, we have a well-defined notion of holomorphic q-differentials on a Riemann surface.

- *Show that, for a compact Riemann surface X with canonical divisor K, the space $\mathcal{L}(qK)$ is isomorphic to the space of q-differential on X.*
- *Show that the dimension of $\mathcal{L}(qK)$ is given by the following table.*

Genus	*Order*	*Dimension*
$g = 0$	$q \leq 0$	$1 - 2q$
	$q > 0$	0
$g = 1$	q	1
$g \geq 2$	$q < 0$	0
	$q = 0$	1
	$q = 1$	g
	$q > 1$	$(2q-1)(g-1)$

Exercise 4. *A Riemann surface X is called hyperelliptic if and only if it admits a degree two map to $\mathbb{P}^1$.*

- *Show that every genus 0 Riemann surface is hyperelliptic.*
- *Show that every genus 1 Riemann surface is hyperelliptic.*
- *Show that every genus 2 Riemann surface is hyperelliptic.*