RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 1

Ex 1. Take $h = \Delta x \in \mathbb{R}$.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i v(x+\Delta x, y)}{\Delta x}$$

$$= u_x + i v_x$$

Take now $h = i\Delta y$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i v(x, y+\Delta y)}{i\Delta y}$$

$$= -i u_y + v_y$$

As the limits coincide, we find $u_x = v_y$ & $v_x = -u_y$.

Ex 2. Parametrise γ_r as

$$\gamma_r : [0,1] \to \mathbb{C}, \qquad \gamma_r(t) = r e^{2\pi i t}$$

Then

$$\frac{1}{2\pi i} \oint_{\gamma_r} z^{n-1} dz = \frac{1}{2\pi i} \int_0^1 r^{n-1} e^{2\pi i (n-1) t} (r 2\pi i) e^{2\pi i t} dt$$

$$= r^n \int_0^1 e^{2\pi i n t} dt$$

If $n=0 \Rightarrow r^0 \int_0^1 dt = 1.$

If $n \neq 0 \Rightarrow \frac{r^n}{2\pi i n} e^{2\pi i n t} \Big|_{t=0}^{1} = 0.$

Ex 3. Simple pole $\Rightarrow f(z) = \sum_{n=-1}^{\infty} a_n (z-z_0)^n$. Thus

$$\lim_{|z| \to z_0} (z-z_0) f(z) = \lim_{|z| \to z_0} \sum_{n=-1}^{\infty} a_n (z-z_0)^{n+1} = a_{-1} = \operatorname{Res}_{z=z_0} f(z).$$

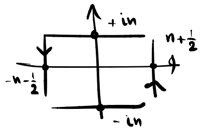
Ex 4. For $\Gamma = \gamma \cup \gamma_1 \cup \dots \cup \gamma_N$ as in the figure, we have

$$\oint_\Gamma f(z)\, dz = 0$$

as Γ is contractictible within the holomorphicity domain. Thus

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \frac{1}{2\pi i} \sum_{i=1}^{N} \underbrace{\oint_{-\gamma_i} f(z)\, dz}_{}$$

$$= \operatorname{Res}_{z=z_i} f(z) \quad \text{by Cauchy}$$

Ex 5. For $n \geq 0$, take $\gamma_n =$ 

. Notice that

$$\cos(\pi z) = \cos(\pi x)\cosh(\pi y) - i \sin(\pi x)\sinh(\pi y)$$

$$\Rightarrow |\cos(\pi z)|^2 = \cos^2(\pi x)\cosh^2(\pi y) + \sin^2(\pi x)\sinh^2(\pi y)$$

$$
\begin{aligned}
&= \cos^2(\pi x)\left(\cosh^2(\pi y) - \sinh^2(\pi y)\right) \\
&\qquad + \left(\sin^2(\pi x) + \cos^2(\pi x)\right)\sinh^2(\pi y) \\
&= \cos^2(\pi x) + \sinh^2(\pi y)
\end{aligned}
$$

and similarly $|\sin(\pi z)|^2 = \sin^2(\pi x) + \sinh^2(\pi y)$. Thus:

$$
|\cot(\pi z)|^2 = \frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\sin^2(\pi x) + \sinh^2(\pi y)} .
$$

Along the vertic. sides of γ_n, i.e. $z = \pm\left(n + \frac{1}{2}\right) + y$,

$$
|\cot(\pi z)|^2 = \frac{\sinh^2(\pi y)}{1 + \sinh^2(\pi y)} = 1 - \frac{1}{1 + \sinh^2(\pi y)} \leq 1.
$$

and along the horiz. sides, i.e. $z = x \pm in$,

$$
|\cot(\pi z)|^2 \leq \frac{1 + \sinh^2(\pi n)}{\sinh^2(\pi n)} = \coth^2(\pi n) \leq \sqrt{2}
$$

Thus, overall along γ_n, $|\cot(\pi z)| \leq 2$. Besides, $\frac{1}{|z^{2k}|} \leq \frac{1}{n^{2k}}$

$$
\left| \frac{1}{2\pi i} \oint_{\gamma_n} \underbrace{\frac{\pi}{z^{2k}} \cot(\pi z)}_{= f(z)} dz \right| \leq \frac{1}{2\pi} \cdot \frac{\pi}{n^{2k}} \cdot 2 \cdot \ell(\gamma_n)
$$

$$
= \frac{1}{n^{2k}} 2(2n + 2n+1) = 2 \cdot \frac{4n+1}{n^{2k}} \xrightarrow[n\to\infty]{} 0
$$

Thus, $\frac{1}{2\pi i} \oint_{\gamma_n} f(z)\,dz \xrightarrow[n\to\infty]{} 0$.

On the other hand, the integrand has poles at $z=0$ and $z = \pm m$, $m > 0$ integer. Besides,

• at $z=0$, $$f(z) = \frac{\pi}{z^{2k}}\left(\sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m} B_{2m} \pi^{2m-1} z^{2m-1}}{(2m)!}\right)$$

$$\Rightarrow \quad \operatorname*{Res}_{z=0} f(z) = (-1)^k \frac{B_{2k} (2\pi)^{2k}}{(2k)!}$$

• at $z=\pm m$, the pole is simple:

$$\operatorname*{Res}_{z=m} f(z) = \lim_{z\to m} (z-m) \frac{\pi}{z^{2k}} \cot(\pi z)$$

$$= \lim_{z\to 0} z \cdot \frac{\pi}{(z+m)^{2k}} \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{m^{2k}}$$

Thus, by residue thm

$$\frac{1}{2\pi i} \oint_{\gamma_n} f(z)\,dz = (-1)^k \frac{B_{2k}(2\pi)^{2k}}{(2k)!} + \sum_{\substack{m=-n,\dots,+n \\ m\neq 0}} \frac{1}{m^{2k}}$$

$$= (-1)^k \frac{B_{2k}(2\pi)^{2k}}{(2k)!} + 2\sum_{m=1}^{n} \frac{1}{m^{2k}}$$

In the limit $n\to\infty$, we find

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2\cdot(2k)!}.$$

The same strategy for odd values would give $0=0$.