RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 10

Ex 1. Start with an arbitrary base $\eta_1, \dots, \eta_g$ of $\Omega(X)$. The Riemann bilinear relations imply that the A-periods matrix is invertible.

$$A = \left(\oint_{a_i} \eta_j \right)_{i,j=1,\dots,g} \in GL(g, \mathbb{C}).$$

We then define $\omega_i = \sum_{j=1}^{g} (A^{-1})_{ji} \eta_j$. We claim that the A-periods matrix of $(\omega_i)_{i=1}^{g}$ is the identity:

$$\oint_{a_i} \omega_j = \sum_{k=1}^{g} (A^{-1})_{kj} \underbrace{\oint_{a_i} \eta_k}_{= A_{ik}} = \sum_{k=1}^{g} A_{ik} (A^{-1})_{kj} = \delta_{ij}$$

Considering now the matrix of B-periods, $\tau = \left(\oint_{b_i} \omega_j \right)$, the symmetry follows from the Riemann bilinear identity w/ $\omega = \omega_i$, $\eta = \omega_j$:

$$0 = \int_X \omega_i \wedge \omega_j = \sum_{k=1}^{g} \left(\underbrace{\oint_{a_k} \omega_i}_{= \delta_{ki}} \underbrace{\oint_{b_k} \omega_j}_{= \tau_{kj}} - \underbrace{\oint_{a_k} \omega_j}_{= \delta_{kj}} \underbrace{\oint_{b} \omega_i}_{= \tau_{ki}} \right)$$

$$= (\tau_{ij} - \tau_{ji}).$$

As for the imaginary part statement, it follows from Riemann's inequality w/ $\omega = \sum_{i=1}^{g} x_i \omega_i$, $x_i \in \mathbb{R}$:

$$0 \geq \Im\Big(\sum_{i=1}^{g} \underbrace{\Big(\oint_{a_i} \sum_{j=1}^{g} x_j \omega_j\Big)}_{= x_i} \underbrace{\overline{\Big(\oint_{b_i} \sum_{k=1}^{g} x_k \omega_k\Big)}}_{= \sum_{k=1}^{g} x_k \overline{\tau}_{i,k}} \Big)$$

$$= \Im\Big(\sum_{i,k=1}^{g} x_i \overline{\tau}_{i,k} x_k \Big),$$

which is equivalent to $x\, \Im(\tau)\, x \geq 0 \quad \forall x \in \mathbb{R}^g$.

Ex 2. On a torus $X = \mathbb{C}/\Lambda$, the period lattice is precisely Λ. This is because a basis of $H_1(X, \mathbb{Z})$ are the cycles defined by $a = [0,1]$ and $b = [0, \tau]$ and a basis of $\Omega(X)$ is dz, so that

$$A = \oint_a dz = 1, \qquad B = \oint_b dz = \tau.$$

Take now $D = [\![p]\!] - [\![q]\!]$ w/ $[p], [q] \in X$ two distinct points. The existence of a meromorphic function w/ a single simple pole is equivalent to asking if D is principal. By Abel's thm, this is true iff $AJ_0(D) \equiv 0 \pmod{\Lambda}$. Choosing $[0]$ as base-point, we see that

$$AJ_0(D) = \int_0^p dz - \int_0^q dz \equiv p - q \mod \Lambda$$

Since $[p] \neq [q]$ in X, $AJ_0(D) \not\equiv 0 \mod \Lambda$. Hence D is not principal.

Ex 3. Choose $Y = \mathbb{P}^1$, $y_1 = \underline{0}$, $y_2 = \underline{\infty}$. Computing ~~a~~ the ~~H~~ $H_{0,\mathfrak{z}_0}((d),(d))$ is the same as computing holomorphic maps

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

of $\deg = d$ w/ $f^{-1}(\underline{0}) = \{x_0\}$ of multiplicity d and $f^{-1}(\underline{\infty}) = \{x_\infty\}$ of mult. d. This is the same as computing meromorphic fncts w/ a single zero at $x_0 = [a:b]$ of order d, and a single pole at $x_\infty = [\alpha:\beta]$ of order d. There is a unique such f given by

$$f : [z_0 : z_1] \longmapsto [(bz_0 - az_1)^d : (\beta z_0 - \alpha z_1)^d]$$

However, all such maps are isomorphic. Indeed, given

$$f : [z_0 : z_1] \longmapsto [(bz_0 - az_1)^d : (\beta z_0 - \alpha z_1)^d]$$

$$f' : [z_0 : z_1] \longmapsto [(b'z_0 - a'z_1)^d : (\beta' z_0 - \alpha' z_1)^d]$$

we define the biholomorphic map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ (a Möbius transf.)

$$\varphi : [z_0 : z_1] \longmapsto \left[\frac{\alpha a' - a\alpha'}{\beta' a' - b'\alpha'} z_0 - \frac{\beta a' - b\alpha'}{\beta' a' - b' a'} z_1 : \frac{\alpha b' - a\beta'}{\beta' a' - b'\alpha'} z_0 - \frac{\beta b' - b\beta'}{\beta a' - b'\alpha'} z_1 \right]$$

It is easy to check that φ is invertible and $f = f' \circ \varphi$.

To conclude, we have to compute $\mathrm{Aut}(f)$ for a specific f in its iso class. We can choose $x_0 = \underline{0}$ & $x_\infty = \underline{\infty}$, so that

$$f : [z_0, z_1] \mapsto [z_0^d ; z_1^d].$$

Then arguing as above we see that the automorphisms of f are

$$\varphi : [z_0 : z_1] \mapsto [e^{\frac{2\pi i}{d} k} z_0 : z_1], \qquad k = 0, \dots, d-1.$$

Thus $Aut(f) \cong \mathbb{Z}/_{d\mathbb{Z}}$ and

$$H_{0 \xrightarrow{d} 0}((d), (d)) = \frac{1}{d}$$

<u>Ex 4.</u> Take $\begin{cases} x = 3(1\,2) + 5(1\,2\,3) \\ y = 4(1\,3) - 6(1\,2\,3) \end{cases} \in \mathbb{C}[S_3]$

Then

$$x + y = 3(1\,2) + 4(1\,3) - (1\,2\,3)$$

$$\begin{aligned} x \cdot y &= 12(1\,2)\cdot(1\,3) + 20(1\,2\,3)(1\,3) - 18(1\,2)(1\,2\,3) \\ &\qquad - 30(1\,2\,3)^2 \\ &= 12(1\,2\,3) + 20(1\,2) - 18(1\,3) - 30(1\,3) \\ &= 12(1\,2\,3) + 20(1\,2) - 48(1\,3) \end{aligned}$$

As for C_μ w/ $\mu = (2,1) \vdash 3$, the only permutations of 3 elements of cycle type $(2,1)$ are

$$\mathcal{C}_\mu = \{(1\,2), (2\,3), (3\,1)\}$$

Thus:

$$C_\mu = (1\ 2) + (2\ 3) + (3\ 1).$$

<u>Ex 5.</u> We have $C_{(3)} = (1\ 2\ 3) + (1\ 3\ 2)$ in $\mathbb{C}[S_3]$. Thus

$$\begin{aligned} H^\bullet_{0\xrightarrow{3}0}((3),(3)) &= \frac{1}{3!}[\mathrm{id}]\, C_{(3)}^2 \\ &= \frac{1}{6}[\mathrm{id}]\Big((1\ 2\ 3)^2 + (1\ 2\ 3)(1\ 3\ 2) + (1\ 3\ 2)(1\ 2\ 3) \\ &\qquad + (1\ 3\ 2)^2\Big) \\ &= \frac{1}{6}[\mathrm{id}]\Big((1\ 3\ 1) + \underbrace{\mathrm{id} + \mathrm{id}}_{=2\,\mathrm{id}} + (1\ 2\ 3)\Big) \\ &= \frac{2}{6} = \frac{1}{3} \end{aligned}$$

More generally, in $\mathbb{C}[S_d]$ we have that $C_{(d)}$ is a sum of $(d-1)!$ elements (called cyclic permutations), which can be written as

$$\sigma = (\tau(1)\ \tau(2)\ \cdots\ \tau(d-1)\ d), \qquad \tau \in S_{d-1}$$

For instance, in S_4:

$(1\ 2\ 3\ 4)$	for $\tau = \mathrm{id}$	$(2\ 3\ 1\ 4)$	for $\tau = (1\ 2\ 3)$
$(2\ 1\ 3\ 4)$	$(1\ 2)$	$(3\ 1\ 2\ 4)$	for $\tau = (1\ 3\ 2)$
$(1\ 3\ 2\ 4)$	$(2\ 3)$		
$(3\ 2\ 1\ 4)$	$(3\ 1)$		

When computing $C_{(d)}^2$, we obtain all possible products of 2 elements. To get the identity, we have to combine a $\sigma \in \mathcal{C}_{(d)}$ w/ its inverse, which is the unique element in $\mathcal{C}_{(d)}$ w/ inverted order. As there are $(d-1)!$ elements in $C_{(d)}$:

$$C_{(d)}^2 = (d-1)!\ \mathrm{id} + \dots$$

$$\Downarrow$$

$$H^{\bullet}_{0 \xrightarrow{d} 0}((d),(d)) = \frac{(d-1)!}{d!} = \frac{1}{d}.$$

We can now argue that all covers in $H^{\bullet}_{0 \xrightarrow{d} 0}((d),(d))$ are connected. Indeed, as all branch points are fully ramified, the source must be connected. Hence:

$$H_{0 \xrightarrow{d} 0}((d),(d)) = H^{\bullet}_{0 \xrightarrow{d} 0}((d),(d)) = \frac{1}{d}$$

in accordance w/ Ex 3.

Ex 6. In $\mathbb{C}[S_4]$:

$$C_{(3,1)} = (1\ 2\ 3) + (1\ 3\ 1) + (1\ 2\ 4) + (1\ 4\ 2) + (1\ 3\ 4) + (1\ 4\ 3) + (2\ 3\ 4) + (2\ 4\ 3)$$

$$C_{(2,2)} = (1\ 2)(3\ 4) + (1\ 3)(2\ 4) + (1\ 4)(2\ 3)$$

Thus:

$$C_{(2,2)}^2 = 3\,\mathrm{id} + 2(1\ 2)(3\ 4) + 2(1\ 3)(2\ 4) + 2(1\ 4)(2\ 3)$$

and we immidiately deduce that $C_{(3,1)} \cdot C_{(2,2)}^2$ does not contain the identity, since the inverse of a $\sigma \in S_4$ of cycle type $(3,1)$ has cycle type $(3,1)$.