RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 2

<u>Ex 1</u>. The line passing through N and (x_0, y_0) is of the form

$$y = mx + q \rightsquigarrow \begin{cases} \pm 1 = q \\ y_0 = mx_0 + q \end{cases} \qquad \begin{cases} q = \pm 1 \\ m = \dfrac{y_0 \mp 1}{x_0} \end{cases}$$

This assumes $(x_0, y_0) \neq N$, so $x_0 \neq 0$. We now want to solve

$$\begin{cases} y = 0 \\ y = \dfrac{y_0 \mp 1}{x_0} x \pm 1 \end{cases} \Rightarrow \quad (x, y) = \left(\frac{x_0}{1 \mp y_0}, 0 \right)$$

In other words:

$$\varphi_N(x_0, y_0) = \frac{x_0}{1 - y_0} \quad \cdots \quad \varphi_S(x_0, y_0) = \frac{x_0}{1 + y_0}$$

We want to compute now φ_N^{-1}. Consider the following line:

$$y = mx + q \quad \text{s.t.} \quad \begin{cases} \pm 1 = q \\ 0 = mt + q \end{cases} \Rightarrow \begin{cases} q = \pm 1 \\ m = \mp \dfrac{1}{t} \end{cases}$$

So we have to solve

$$\begin{cases} y = \pm \left(1 - \frac{1}{t} x \right) \\ x^2 + y^2 = 1 \end{cases} \rightsquigarrow \quad x^2 + \left(1 - \frac{x}{t} \right)^2 = 1$$

The equation gives

$$\left(\left(1+\frac{1}{t^2}\right)x-\frac{2}{t}\right)x=0 \xRightarrow[\;]{\text{discard } x=0,\,y=1} (t^2+1)x=2t \Rightarrow x=\frac{2t}{t^2+1}$$

Thus, $y=\pm\left(1-\frac{2}{t^2+1}\right)=\pm\frac{t^2-1}{t^2+1}$. In other words,

$$\varphi_N^{-1}(t)=\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right) \qquad \dots \qquad \varphi_S^{-1}(t)=\left(\frac{2t}{t^2+1},-\frac{t^2-1}{t^2+1}\right)$$

We want to prove that φ_N and φ_N^{-1} are continuous.
Note that the extension of φ_N to $\mathbb{R}^2\setminus\{y=0\}$, that is

$$\tilde{\varphi}_N:\mathbb{R}^2\setminus\{y=0\}\to\mathbb{R},\quad (x,y)\mapsto\frac{x}{1-y}$$

is continuous. Thus, $\forall V\subseteq\mathbb{R}$ open, $\tilde{\varphi}_N^{-1}(V)$ is open in $\mathbb{R}^2\setminus\{y=0\}$. But

$$\varphi_N^{-1}(V)=\tilde{\varphi}_N^{-1}(V)\cap U_N$$

which is indeed open in U_N. This proves that φ_N is cont.
A similar argument holds for φ_N^{-1}. Same for φ_S and φ_S^{-1}.
The transition fnct on U_{NS} is

$$\varphi_{NS}(t)=\varphi_N\circ\varphi_S^{-1}(t)=\varphi_N\left(\frac{2t}{t^2+1},-\frac{t^2-1}{t^2+1}\right)$$

$$=\frac{\frac{2t}{t^2+1}}{1+\frac{t^2-1}{t^2+1}}=\frac{2t}{2t^2}=\frac{1}{t}$$

which is smooth in the domain $\varphi_N(U_N\cap U_S)=\mathbb{R}\setminus\{0\}$.

Ex 2. Take Γ_f as unique cover. Then

$$\varphi : \Gamma_f \to U \subseteq \mathbb{R}^n \qquad \text{(the projection)}$$
$$(x,y) \mapsto x$$

is invertible, with inverse $x \mapsto (x, f(x)) \in \Gamma_f$. Both φ and φ^{-1} are continuous, since

$$\forall A \subseteq U \text{ open}, \quad \varphi^{-1}(A) = (A \times \mathbb{R}^m) \cap \Gamma_f \text{ open}$$

Conversely, take $A \times B$ open in $U \times \mathbb{R}^m$. Then $\varphi(A \times B) = A$, which is open.

As for the trasition fnct, $\varphi \circ \varphi^{-1} = \mathrm{id}$ on U is smooth.

NB. We didn't even needed f smooth. In general, every atlas with a single chart is a smooth mnfld.

Ex 3. Take $[z_0 : z_1] \in \mathbb{P}^1_{\mathbb{C}}$. Then, if $z_0 \neq 0$

$$[z_0 : z_1] = [z_0 \bar{z}_0 : z_1 \bar{z}_0]$$
$$= [|z_0|^2 : z_1 \bar{z}_0]$$
$$= [y_0 : y_1 + i y_2] \quad \text{for } y_i \in \mathbb{R},\ y_0 > 0$$

If $z_0 = 0$, we would have the same w/ $y_0 = 0$.

Since we cannot have $y_0 = y_1 = y_2 = 0$, we find

$$y_0^2 + y_1^2 + y_2^2 = r \neq 0$$

$$[z_0 : z_1] = \left[\frac{y_0}{r} : \frac{y_1}{r} + i \frac{y_2}{r} \right]$$

$$= [x_0 : x_1 + i x_2], \qquad x_i \in \mathbb{R}, \quad x_0 \geq 0$$

$$x_0^2 + x_1^2 + x_2^2 = 1$$

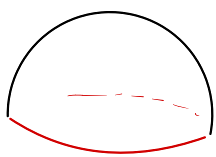
Now notice that if $z_0 \neq 0$, the point on

$$H = \{ (x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1, \ x_0 \geq 0 \}$$

is uniquely defined. In contrast, if $z_0 = 0$ then

$$[0 : x_1 + i x_2] = [0 : 1] \quad \forall (x_1, x_2), \text{ scaling by } x_1 + i x_2.$$

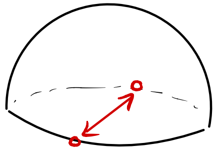
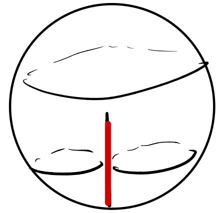
Thus, the whole boundary is identified to a pnt

$$\mathbb{P}^1_{\mathbb{C}} \cong$$  $$\cong S^2$$

Ex 4 The same argument for $\mathbb{P}^2_{\mathbb{R}}$ gives

$$[0 : x_1 : x_2] = [0 : -x_1 : -x_2].$$

Here the unique scaling is by ± 1. Thus

$\mathbb{P}^2_{\mathbb{R}} \cong$  $\cong$ 

<u>Ex 5.</u> Write $z = x + iy$, $\omega_i = \alpha_i + i\beta_i$ $(i = 1, 2)$. Then we can always change variables $(x, y) \mapsto (u, v)$ so that

$$z = \omega_1 u + \omega_2 v$$

Indeed, the change of variables corresponds to the system

$$x = \alpha_1 u + \alpha_2 v$$

$$y = \beta_1 u + \beta_2 v$$

which has a solution (u, v) by linear indep. Thus, given z, we define

ceiling fnct.

$$\Delta = \lfloor u \rfloor \omega_1 + \lfloor v \rfloor \omega_2 \in \Lambda$$

and set $z_0 = z - \Delta$. Then $z_0 \in P$. We also deduce that, topologically, $T \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1$.

The function $\pi|_P$ is injective in the interior of P. At the sides, we have

$$u\omega_1 \equiv u\omega_1 + \omega \qquad \forall u \in [0, 1]$$

$$v\omega_2 \equiv \omega_1 + v\omega_2 \qquad \forall v \in [0, 1]$$

In particular, all vertices of P are identified.

Take now $r < \min\{|w_1|, |w_2|\}$. Then $\forall z \in \mathbb{C}$, $\pi|_{B_r(z)}$ is one-to-one. Thus, we can take

$$\{ U_z := \pi(B_r(z)),\ \varphi_z := (\pi|_{U_z})^{-1} \}_{z\in\mathbb{C}}$$

as atlas for T. Then, by φ_z are homeomorphisms by design. Besides, the transition maps are given by

$$\varphi_{z_1 z_2}(z) = z + \Omega, \qquad \Omega = (\lfloor u_2 \rfloor - \lfloor u_1 \rfloor)\omega_1 + (\lfloor v_2 \rfloor - \lfloor v_1 \rfloor)\omega_2$$

for $z_i = u_i \omega_1 + v_i \omega_2$.