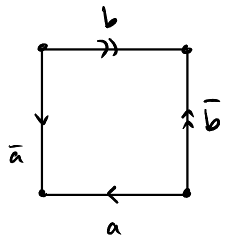
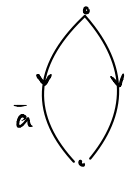
Riemann Surfaces - Spring 2024

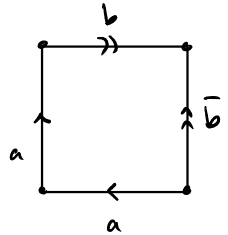
Exercices sheet 3

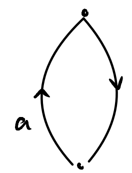
Ex_1.

$w_1 = a\bar{a}b\bar{b} =$ 

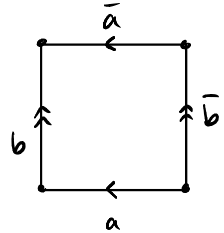
$\cong$ 

$\cong S^2$ (sphere)

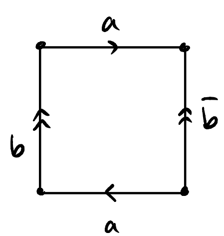
$w_2 = aab\bar{b} =$ 

$\cong$ 

$\cong P^2(\mathbb{R})$ (real proj plane)

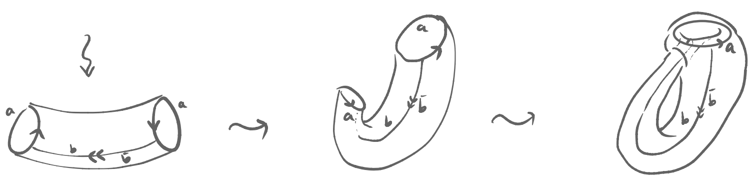
$w_3 = ab\bar{a}\bar{b} =$ 

$\cong T$ (torus)

$w_4 = aba\bar{b} =$ 

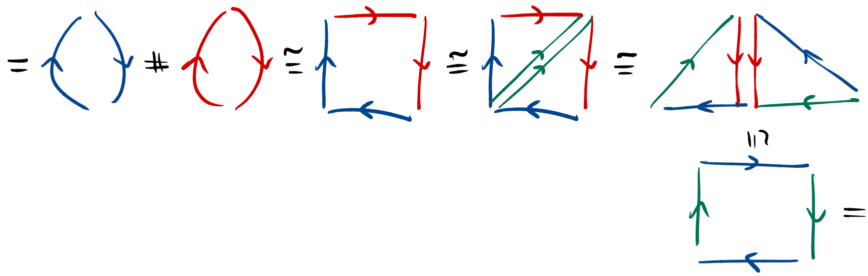
$\cong K$ (Klein bottle)

WHY?



Ex 2.

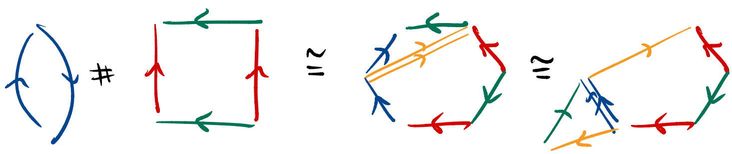
$\mathbb{P}^2(\mathbb{R})^{\#2} =$

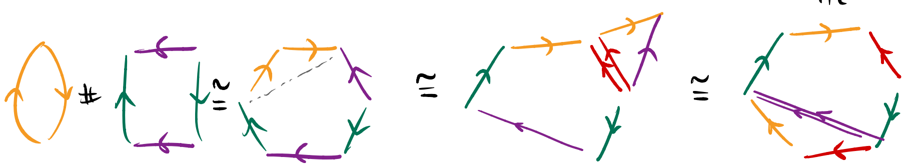


$= K$

Similarly:

$\mathbb{P}^2(\mathbb{R}) \# T =$

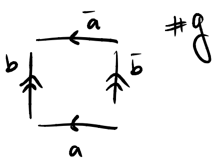




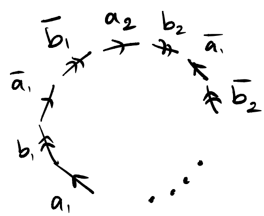
$\mathbb{P}^2(\mathbb{R}) \# K \cong \mathbb{P}^2(\mathbb{R})^{\#3}$

Ex3.

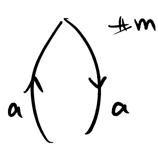
$T^{\# g} =$



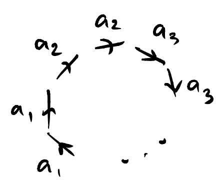
$=$



$\mathbb{P}^2(\mathbb{R})^{\# m} =$



$=$

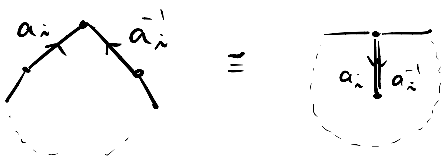


As for the fact that every identification polygon w (IP) is of that form, we proceed as follows.

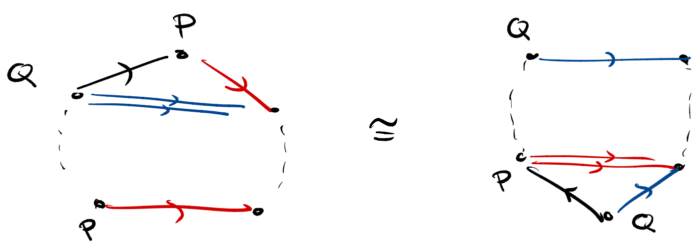
<u>SETUP</u> Let $A = \{a_1, \dots, a_n\}$, so that $A \cup \bar{A} = \{a_1, \dots, a_n, \bar{a}_1, \dots, \bar{a}_n\}$.

If a_i $\bar{a}_i$ appear, we say that a_i is of type I. If a_i or $\bar{a}_i$ appear twice, we say it is of type II.

<u>STEP 1</u>. We can remove adjacent type-I letters. In other words if w contains $a_i\bar{a}_i$ or $\bar{a}_i a_i$, by removing it we obtain an equivalent surface. This is just S^2 being the identity w.r.t. $\#$.

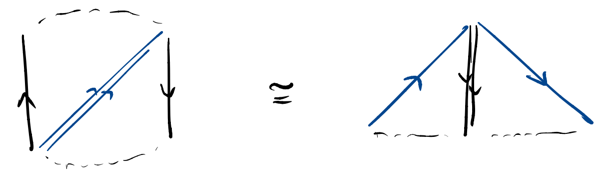


<u>STEP 2</u>. We can always obtain an equivalent IP where all vertices are identified. Indeed, if we have two vertices P and Q appearing p and q times after glueing, then moves like the one above reduces the appeareance to $p-1, q+1$



By repeating moves like above, we can reduce ourselves to the case with one vertex only.

STEP 3. We now want to make all type-II letters adjacent. This can be achieved with moves like

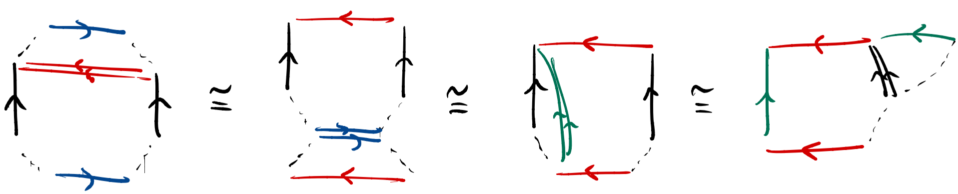


As a consequence, we find that

$$S_w \cong S_{w'} \# \mathbb{P}^2(\mathbb{R})^{\# m}$$

and w' has no adjacent type II letters.

STEP 4. We now want to make all type-I letters close to each other in a torus-like faschion. Again, this is achieved through the following sequence of moves:



As a consequence, we find that

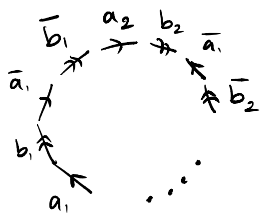
$$S_w \cong T^{\# g} \# \mathbb{P}^2(\mathbb{R})^{\# m}$$

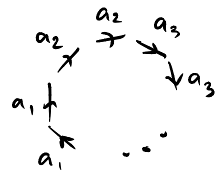
STEP 5. If $m=0$, we have $S_w \cong T^{\# g}$. If $m>0$, we can use $T \# \mathbb{P}^2(\mathbb{R}) \cong \mathbb{P}^2(\mathbb{R})^{\# 3}$ from Ex 3 to get

$$S_w \cong \mathbb{P}^2(\mathbb{R})^{\# (m+2g)}$$

In conclusion, every $\mathbb{P}$ is of the form $T^{\# g}$ or $\mathbb{P}^2(\mathbb{R})^{\# m}$.

Ex 4. Consider

$T^{\# g}$ = 

$\mathbb{P}^2(\mathbb{R})^{\# m}$ = 

After gueing, the boundary of the polygon is a good graph with

- 1 vertex, $2g$ edges, 1 face for $T^{\# g}$ $\Rightarrow$ $\chi(T^{\# g}) = 2-2g$
- " — ", m edges, " — " $\mathbb{P}^2(\mathbb{R})^{\# m}$ $\quad \chi(\mathbb{P}^2(\mathbb{R})^{\# m}) = 2-m$.