RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 4

Ex_1 . • Consider
$$\varphi : \mathbb{C} \to \mathbb{C} \xrightarrow{\pi'} \mathbb{C}/\Lambda' . \quad \begin{array}{l} z \mapsto \alpha z \\ \qquad w \longmapsto [w] \end{array}$$
The map factors through $\mathbb{C}/\Lambda$ iff $\Lambda \subseteq \mathrm{Ker}(\varphi)$ (fundamental thm on homeom.)
This is indeed the case, under the hypothesis $\alpha\Lambda \leq \Lambda'$:

$$z \in \Lambda \quad \Rightarrow \quad \varphi(z) = [\alpha z] = 0 \quad \text{as} \quad \alpha z \in \Lambda'.$$

The induced map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is simply a rescaling in loc. coordinates, hence holomorphic.

Notice that φ is surjective. It is injective iff $\Lambda = \mathrm{Ker}(\varphi)$. We already know that $\Lambda \subseteq \mathrm{Ker}(\varphi)$. So

$$\varphi \text{ is bijective} \quad \text{iff} \quad \Lambda \supseteq \mathrm{Ker}(\varphi)$$
$$\text{iff} \quad \alpha\Lambda \geq \Lambda'$$

The (unique) inverse is automatically holomorphic, induced by the map
$$\mathbb{C} \to \mathbb{C} \to \mathbb{C}/\Lambda .$$
$$z \mapsto \frac{z}{\alpha}$$
$$w \longmapsto [w]$$

- Let $\Lambda = \{ n_1 \omega_1 + n_2 \omega_2 \mid n_1, n_2 \in \mathbb{Z} \}$. We have two possibilities:

  ① $\Im\left(\frac{\omega_2}{\omega_1}\right) > 0$

  ② $\Im\left(\frac{\omega_1}{\omega_2}\right) = \Im\left(\left(\frac{\omega_2}{\omega_1}\right)^{-1}\right) < 0$

It cannot be zero, since ω_1 and ω_2 are linearly indep. over $\mathbb{R}$. Set $\Lambda' = \alpha \Lambda$, where

$$\begin{cases} \alpha = \frac{1}{\omega_1} & \text{in } ① \\ \alpha = \frac{1}{\omega_2} & \text{in } ② \end{cases}$$

In any case, $\Lambda' = \mathbb{Z} + \tau \mathbb{Z}$ w/ $\Im(\tau) > 0$. Besides, from the previous point, $\mathbb{C}/\Lambda \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$.

- We want to show that $\exists \alpha \in \mathbb{C}^*$ s.t. $\alpha(\mathbb{Z}+\tau'\mathbb{Z}) = \mathbb{Z}+\tau\mathbb{Z}$. Suppose for a second that we have such α. Then

$$\alpha \in \mathbb{Z}+\tau\mathbb{Z} \implies \exists c,d \in \mathbb{Z} \text{ s.t. } \alpha = c\tau + d$$

$$\alpha\tau' \in \mathbb{Z}+\tau\mathbb{Z} \implies \exists a,b \in \mathbb{Z} \text{ s.t. } \alpha\tau' = a\tau + b$$

so that $\tau' = \frac{a\tau+b}{c\tau+d}$. Notice that we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$$

for mantain linear indep, so the determinant must be ± 1 (the only invertible elem in $\mathbb{Z}$). Notice that

$$\tau' = \frac{(a\tau+b)(c\bar{\tau}+d)}{|c\tau+d|^2} = \frac{a|\tau|^2 + bd + ad\tau + bc\bar{\tau}}{|c\tau+d|^2}$$

$$\Rightarrow \quad \operatorname{Im}(\tau') = \frac{ad-bc}{|c\tau+d|^2} \operatorname{Im}(\tau)$$

Hence, for $\tau' \in \mathbb{H}$, we must have $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = +1$.

this cannot be zero, since 1 and τ are lin. indep/$\mathbb{R}$.

So, we are tempted to define $\alpha = c\tau + d$. The above computations (reversed) show that indeed

$$\alpha(\mathbb{Z} + \tau'\mathbb{Z}) = \mathbb{Z} + \tau\mathbb{Z},$$

thus $T(\tau) \simeq T(\tau')$.

Ex 2. We can do this in $\mathbb{C}[z_0, \dots, z_n]$.

$(1 \Rightarrow 2)$. Suppose $F \in \mathbb{C}[z_0, \dots, z_n]$. Write F as a sum of monomials:

$$F = \sum_{k \geq 0} \sum_{k_0 + \dots + k_n = k} a_{k_0, \dots, k_n} z_0^{k_0} \cdots z_n^{k_n}$$

finite

Then $\forall \lambda \in \mathbb{C}^*$

$$F(\lambda z_0, \dots, \lambda z_n) = \sum_{k \geq 0} \sum_{k_0 + \dots + k_n = k} a_{k_0 \dots k_n} (\lambda z_0)^{k_0} \cdots (\lambda z_n)^{k_n}$$

$\|$

$$\stackrel{!}{=} \sum_{k\geq 0} \lambda^k \sum_{k_0+\cdots+k_n=k} a_{k_0\ldots k_n} z_0^{k_0}\cdots z_n^{k_n}$$

This equals $\lambda^d F(z_0,\ldots,z_n)$ in $\mathbb{C}[\lambda,z_0,\ldots,z_n]$ iff

- for $k\neq d$, $\displaystyle\sum_{k_0+\cdots+k_n=k} a_{k_0\ldots k_n} z_0^{k_0}\cdots z_n^{k_n} = 0$

$$\Leftrightarrow \quad a_{k_0\ldots k_n} = 0 \quad \forall k_0,\ldots,k_n \text{ st. } k_0+\cdots+k_n = k.$$

- for $k=d$, $\displaystyle F(z_0,\ldots,z_n) = \sum_{k_0+\cdots+k_n=d} a_{k_0\ldots k_n} z_0^{k_0}\cdots z_n^{k_n}.$

$(2 \Leftrightarrow 3)$. Suppose $F\in\mathbb{C}[z_0,\ldots,z_n]$. Write F as a sum of monomials:

$$F = \sum_{k\geq 0}\sum_{k_0+\cdots+k_n=k} a_{k_0,\ldots,k_n} z_0^{k_0}\cdots z_n^{k_n}$$

Define the Euler operator as

$$D = \sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i}$$

Then

$$\begin{aligned} DF(z_0,\ldots,z_n) &= \sum_{k\geq 0}\sum_{k_0+\cdots+k_n=k} a_{k_0,\ldots,k_n} D(z_0^{k_0}\cdots z_n^{k_n}) \\ &= \sum_{k\geq 0}\sum_{k_0+\cdots+k_n=k} \underbrace{(k_0+\cdots+k_n)}_{=k} a_{k_0\ldots k_n} z_0^{k_0}\cdots z_n^{k_n} \end{aligned}$$

$$\overset{!}{=} \sum_{k \geq 0} k \sum_{k_0 + \dots + k_n = k} a_{k_0 \dots k_n} z_0^{k_0} \cdots z_n^{k_n}.$$

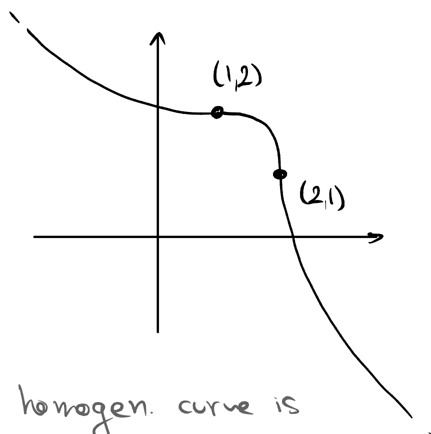
This equals $d \cdot F$ iff

- for $k \neq d$, $\displaystyle\sum_{k_0 + \dots + k_n = k} a_{k_0 \dots k_n} z_0^{k_0} \cdots z_n^{k_n} = 0$

  $\iff \quad a_{k_0 \dots k_n} = 0 \quad \forall k_0, \dots, k_n$ s.t. $k_0 + \dots + k_n = k$.

- for $k = d$, $\displaystyle F(z_0, \dots, z_n) = \sum_{k_0 + \dots + k_n = d} a_{k_0 \dots k_n} z_0^{k_0} \cdots z_n^{k_n}.$

Ex 3. The eqn $x^3 + y^3 = 9$ is the dehomogenisat. of an elliptic curve. In particular, we have the group law with the same defn as in the notes, and identity



$$O = [1 : -1 : 0]$$

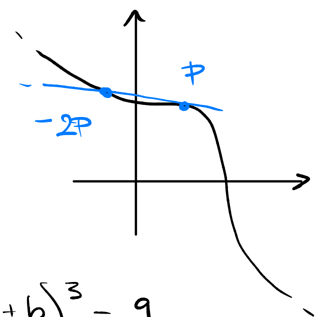
the homogen. curve is $F = z_0^3 + z_1^3 - 9 z_2^3 = 0$ and $O \in Z(F)$.

The idea is to use the group law to produce more rational solutions.

First, we start computing $-[(1,2)+(1,2)]$. By definition, this is the intersection of the cubic w/ the tangent at $(1,2)$.

We can do this for a generic point $P=(a,b)$ as follows.

The tangent at $P=(a,b)$ is $3(a^2x+b^2y-9)$. Notice that $a^3+b^3=9$, so the line can be written as



$$y = -\frac{a^2}{b^2}(x-a)+b.$$

The intersection w/ cubic is then

$$\begin{cases} y = -\frac{a^2}{b^2}(x-a)+b \\ x^3+y^3=9 \end{cases} \rightarrow \quad x^3+\left(-\frac{a^2}{b^2}(x-a)+b\right)^3 = 9$$

This is a cubic eqn with a double root at $x=a$. From the theory of cubic eqns, we find that the third root is at

$$x = \frac{a(a^3+2b^3)}{(a^3-b^3)} \quad \Rightarrow \quad y = -\frac{b(2a^3+b^3)}{a^3-b^3}$$

Hence: $$-2(a,b) = \left(\frac{a(a^3+2b^3)}{(a^3-b^3)}, -\frac{b(2a^3+b^3)}{a^3-b^3}\right).$$

Notice tha $(a,b)\in\mathbb{Q}^2 \Rightarrow -2(a,b)\in\mathbb{Q}^2$. Applying it to $(1,2)$, we find

$$\left(-\frac{17}{7}, \frac{20}{7}\right).$$

However, this is NOT a positive solution. The idea now is to consider the sum w/ the other solution

$$-\left[(2,1) + \left(-\frac{17}{7}, \frac{20}{7}\right)\right].$$

which is obtained as the intersection of the line passing through them and the cubic. The line passing through the points is

$$Z(13x + 31y - 57)$$

The new solution to the system

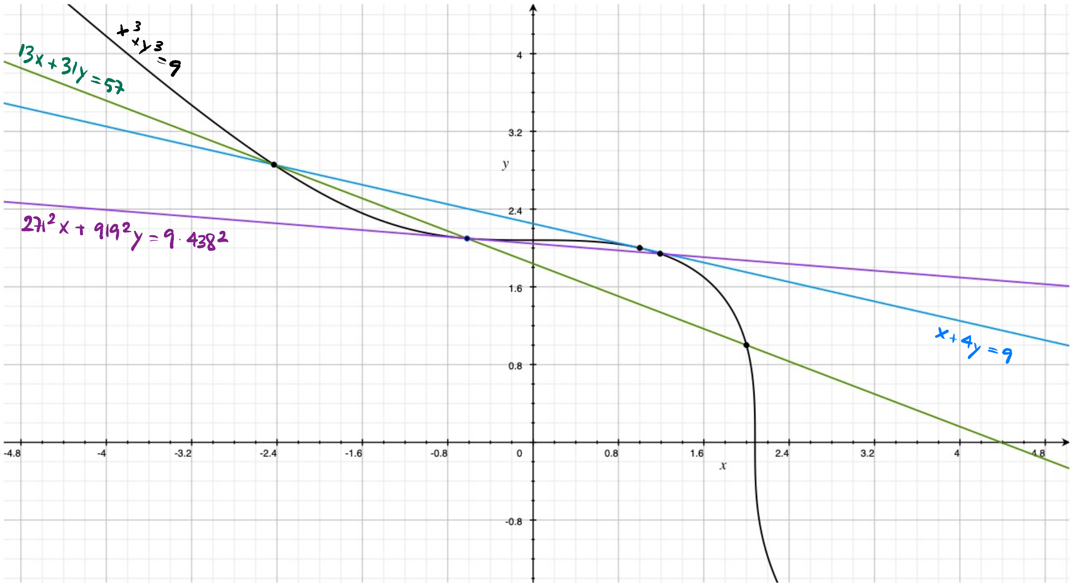
$$\begin{cases} 13x + 31y = 57 \\ x^3 + y^3 = 9 \end{cases} \quad \text{is} \quad Q = \left(-\frac{271}{438}, \frac{919}{438}\right)$$

Taking $-2Q$ gives

$$-2Q = \left(\frac{415\,280\,564\,497}{348\,671\,682\,660}, \frac{676\,702\,467\,503}{348\,671\,682\,660}\right)$$

which is now a positive rational solution!

Geometrically, the computation is pictured below.



x^3+y^3=9
13x+31y=57
271^2 x + 919^2 y = 9·438^2
x+4y=9
x
y
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8
-0.8