RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 5

Ex 1. $f: X \to Y$ non-const holomorphic, X, Y cmct RSs.

- By Riemann-Hurwitz (RH):

$$2g_X - 2 = \underbrace{\deg(f)}_{\geq 1}(2g_Y - 2) + \underbrace{\sum_{x \in \mathrm{Ram}_f} (\mu_x(f) - 1)}_{\geq 0}$$

Hence, $2g_X - 2 \geq 2g_Y - 2$, which simplifies as $g_X \geq g_Y$.

- By RH

$$\sum_{x \in \mathrm{Ram}_f} (\mu_x(f) - 1) = 0.$$

Since the summands are positive, we conclude that $\mathrm{Ram}_f = \emptyset$, that is f is unramified.

- By RH,

$$2g_X - 2 = \deg(f)(-2) + 0 \quad \Rightarrow \quad g_X = 1 - \deg(f).$$

Since $\deg(f) \geq 1$ and $g_X \geq 0$, we conclude that

$$g_X = 0 \quad \& \quad \deg(f) = 1.$$

But $\deg(f) = 1$ implies that f is a biholormism, hence

$X \cong \mathbb{P}^1(\mathbb{C})$

- By RH (set $g_X = g_Y = g \geq 2$)

$$(2g-2) = \deg(f)\,(2g-2) + \sum_{x \in \mathrm{Ram}_f} (\mu_x(f) - 1)$$

$$\Rightarrow \quad \underbrace{(1-\deg(f))}_{\leq 0}\,\underbrace{(2g-2)}_{\geq 0} = \underbrace{\sum_{x \in \mathrm{Ram}_f} (\mu_x(f)-1)}_{\geq 0}$$

Thus, we conclude that $\deg(f) = 1$, hence X and Y are biholomorphic.

Ex 2. Looking at $z \in \mathbb{C}$ as $[z:1] \in \mathbb{P}^1(\mathbb{C})$, we find

$$F : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$
$$[z:w] \mapsto [z^3 : w^3 - z^2 w]$$

← NB: F is well-defined.

Indeed, in the chart $\{w \neq 0\}$, we find

$$F([z:w]) = F\left(\left[\frac{z}{w} : 1\right]\right) = \left[\frac{z^3}{w^3} : 1 - \frac{z^2}{w^2}\right] \stackrel{!}{=} \left[\frac{(z/w)^2}{1-(z/w)^2} : 1\right]$$

which is precisely the eqn for f.

As for the degree, notice that the eqn $f(z) = 1$ has 3 distinct solutions of mult. 1. Hence, $\deg f = 3$.

The ram points are the zeros of f' (+ possibly $\infty = [1:0]$). We have

$$f'(z) = \frac{z^2(3-z^2)}{(z^2-1)^2} = 0 \quad \rightsquigarrow \quad \begin{matrix} z = 0 \\ z = \pm\sqrt{3} \end{matrix}$$

Notice that $\mu_0(f) = 3$ and $\mu_{\pm\sqrt{3}}(f) = 2$.

$$f''(z) = -\frac{2z(z^2+3)}{(z^2-1)^3} \begin{cases} = 0 & \text{at } z = 0 \\ \neq 0 & \text{at } z = \pm\sqrt{3} \end{cases}$$

$$f'''(z) = \frac{6(z^4+6z^2+1)}{(z^2-1)^4} \neq 0 \quad \text{at } z = 0$$

To conclude, we need to check the point at ∞. To this end, we can consider F in the chart $\{z \neq 0\}$

$$F([z:w]) = F\left(\left[1 : \frac{w}{z}\right]\right) = \left[1 : \frac{w^3}{z^3} - \frac{w}{z}\right]$$

which is nothing but the fnct

$$g(w) = w^3 - w.$$

We have $g'(w)\big|_{w=0} = 3w^2 - 1\big|_{w=0} \neq 0$, so $\infty = [1:0]$ is not a ram. pnt. Notice that $g' = 0$ has $w = \pm\frac{1}{\sqrt{3}}$ as solution, in accordance w/ the computation for f.

To sum up:

- $\deg f = 3$
- $\mathrm{Ram}_f = \{ \underbrace{0 = [0:1]}_{\mu = 3}, \underbrace{\pm\sqrt{3} = [\pm\sqrt{3}:1]}_{\mu = 2} \}$

In this case, RH reads

$$\underbrace{(2 \cdot g_{\mathbb{P}^1(\mathbb{C})} - 2)}_{= -2} \overset{\checkmark}{=} \underbrace{3 \cdot (2 g_{\mathbb{P}^1(\mathbb{C})} - 1)}_{= -6} + \underbrace{((3-1) + (2-1) + (2-1))}_{= 4}$$

<u>Ex 3.</u> Consider $\mathcal{C} = Z(z_0^n + z_1^n - z_2^n) \subset \mathbb{P}^2(\mathbb{C})$ Fermat's curve. If $F, G, H \in \mathbb{C}[z, w]$ are non-constant, homog, coprime polys, satisfying $F^n + G^n - H^n = 0$, we deduce that

$$\varphi : \mathbb{P}^1(\mathbb{C}) \to \mathcal{C}, \quad [z:w] \mapsto [F(z,w) : G(z,w) : H(z,w)]$$

is well-defined.

It can be easily seen that φ is holomorphic. Moreover, φ has degree $d = \deg F = \deg G = \deg H$, since $\varphi([z:w]) = pt$ has solutions (counted w/ multiplicity). By RH:

$$(2 g_{\mathbb{P}^1(\mathbb{C})} - 2) \geq d \cdot (2 g_{\mathcal{C}} - 2).$$

Notice that $g_{\mathbb{P}^1} = 0$, $g_C = \frac{(n-1)(n-2)}{2}$ (by genus-deg. formula). Hence:

$$-2 \geq \underbrace{d}_{\geq 1}\left((n-1)(n-2) - 2 \right) \iff (n-1)(n-2) \leq 0$$

The polynomial $p(n) = (n-1)(n-2)$ take zero-values for $n = 1,2$ and is increasing for $n \geq 2$ w/ $p(3) = 2$.

Hence, $p(n) \leq 0$ has no solutions when $n \geq 2$.

Ex 4. By definition, $\mathrm{ord}_{z_0}(f) = \infty$ iff $f \equiv 0$.

Consider now f, g meromorphic in U. Then:

$$f(z) = a_m (z - z_0)^m + O((z-z_0)^{m-1}) \qquad a_m \neq 0$$

$$g(z) = b_n (z - z_0)^n + O((z-z_0)^{n-1}) \qquad b_n \neq 0$$

$$\hookrightarrow (f \cdot g)(z) = \underbrace{a_m \cdot b_n}_{\neq 0} (z - z_0)^{m+n} + O((z-z_0)^{m+n-1})$$

$$\hookrightarrow (f+g)(z) = \begin{cases} a_m (z-z_0)^m + \cdots & \text{if } m < n \\ b_n (z-z_0)^n + \cdots & \text{if } n < m \\ (a_m + b_m)(z-z_0)^m + \cdots & \text{if } m = n. \end{cases}$$

We deduce that:

$$\mathrm{ord}_{z_0}(f \cdot g) = m + n = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$$

$$\mathrm{ord}_{z_0}(f + g) \geq \min\{m, n\} = \min\{\mathrm{ord}_{z_0} f, \mathrm{ord}_{z_0} g\}.$$

The inequality is due to the fact that we can have $a_m + b_m = 0$, thus getting a higher order at z_0 for $f+g$.