RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 6

Ex 1. Let φ be a meromorphic function on $\mathbb{P}^1$. If φ is constant ($\neq \infty$), then $\varphi([z:w]) = [\text{cnst} : 1]$ and we are good.

If φ is non-constant, we necessarily have at least a zero and a pole (since φ is surjective), and in finite numbers. Let

- $x_1, \dots, x_M$ be the zeros, of order $m_1, \dots, m_M$; set $x_i = [a_i : b_i]$
- $y_1, \dots, y_N$ be the poles, of order $n_1, \dots, n_N$; set $y_j = [c_j : d_j]$.

By design, all x_i and y_j are distinct, and

$$m_1 + \dots + m_M = n_1 + \dots + n_N = \deg(f) =: d.$$

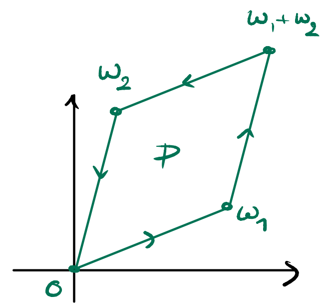
Consider now the meromorphic fnct

$$\psi([z:w]) = \left[\prod_{i=1}^{M} (z b_i - w a_i)^{m_i} : \prod_{j=1}^{N} (z d_j - w c_j)^{n_j} \right]$$

It has zeros of order m_i at x_i, and poles of order n_j at y_j. Hence, the meromorphic function φ/ψ has no zeros nor poles (since $\mathrm{ord}_x(\cdot)$ is a discrete valuation). We deduce that φ/ψ is a constant ($\neq 0$ and ∞), say $[p:q] \in \mathbb{P}^1$. Hence,

$$\varphi([z:w]) = \Big[\underbrace{p\cdot\prod_{i=1}^{M}(z b_i - w a_i)^{m_i}}_{=F(z,w)} : \underbrace{q\cdot\prod_{j=1}^{N}(z d_j - w c_j)^{n_j}}_{=G(z,w)}\Big]$$

Both F and G are homogeneous poly, of degree d, with no common factors.



Ex 2. By the residue thm,

$$\frac{1}{2\pi i}\oint_{\partial P} f(z)\,dz = \sum_{\substack{z_0 \text{ pole} \\ \text{in } P}} \operatorname*{Res}_{z=z_0} f(z)$$

On the other hand,

$$\frac{1}{2\pi i}\oint_{\partial P} f(z)\,dz = \frac{1}{2\pi i}\left(\int_0^{\omega_1} - \int_{\omega_2}^{\omega_1+\omega_2}\right) f(z)\,dz$$

$$+ \frac{1}{2\pi i}\left(\int_{\omega_1}^{\omega_1+\omega_2} - \int_0^{\omega_2}\right) f(z)\,dz$$

By Λ-periodicity, the two summands are zero. Hence:

$$\sum_{\substack{z_0 \text{ pole} \\ \text{in } P}} \operatorname*{Res}_{z=z_0} f(z) = 0.$$

Now assume f has a single pole at $z_0 \in P$ that is simple. We find

$$f(z) = a_{-1}\frac{1}{z-z_0} + O(1) \qquad \text{w/} \quad a_{-1} \neq 0.$$

Since there are no other poles we find $a_{-1} = 0$, which is a contraddiction.

Ex 3

i) Let K be a compact in $\mathbb{C} \setminus \Lambda$. If $z \in K$, then $|z| < R$ for some R big enough. The idea is then to split the series in two:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda^* \\ |\omega| \leq 2R}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) + \sum_{\substack{\omega \in \Lambda^* \\ |\omega| > 2R}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

The first sum is finite, and defines a continuous function on K. Hence, it is uniformly bounded:

$$\left| \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda^* \\ |\omega| \leq 2R}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \right| \leq A \qquad \forall z \in K$$

for some $A > 0$. The second sum is ifinite, and it can be bounded as

$$\left| \sum_{\substack{\omega \in \Lambda^* \\ |\omega| > 2R}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \right| \leq \sum_{\substack{\omega \in \Lambda^* \\ |\omega| > 2R}} \underbrace{\left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right|}$$

$$\leq \frac{R(2|\omega| + R)}{|\omega|^2 |\omega/2|^2} < \frac{10R}{|\omega|^3}$$

$$\leq B \sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^3}$$

where $B = 10R$. All together, we find that $\forall z \in K$

$$|\wp(z)| \leq A + B \sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^3}$$

for some $A, B > 0$. The Weirstrass M-test implies absolute and uniform convergence, provided that $\sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^3} < +\infty$.

To check that the sum is convergent, notice that the number of lattice points in the anullus $N \leq |z| < N+1$ grows linearly in N:

$$\#\{\omega \in \Lambda \mid N \leq |\omega| < N+1\} \leq cN \qquad \text{for some } c > 0.$$

Indeed, choose $r > 0$ s.t. the balls of radius r centered at points in Λ are disjoint. Notice that

$$\#\{\omega \in \Lambda \mid N \leq |\omega| < N+1\} \cdot \text{Area}(|z| < r)$$
$$\leq \text{Area}(N - r \leq |z| < N+1+r),$$

that is

$$\#\{\omega \in \Lambda \mid N \leq |\omega| < N+1\} \leq \frac{\pi\left((N+1+r)^2 - (N-r)^2\right)}{\pi r^2} \leq c \cdot N.$$

As a consequence,

$$\sum_{\omega \in \Lambda^*} \frac{1}{|\omega|^3} \leq \sum_{N=1}^{\infty} \frac{\#\{\omega \in \Lambda \mid N \leq |\omega| < N+1\}}{N^3}$$
$$\leq c \sum_{N=1}^{\infty} \frac{1}{N^2} = \frac{c\pi^2}{6} < \infty.$$

ii) By absoulte convergence,

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

which is clearly Λ-periodic. It is also odd, since

$$\begin{aligned}\wp'(-z) &= -2 \sum_{\omega \in \Lambda} \frac{1}{(-z-\omega)^3} \\ &= +2 \sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^3} \\ &= +2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3} = -\wp'(z).\end{aligned}$$

Similarly, $\wp$ is even.

To deduce Λ-periodicity of $\wp$, consider

$$f_\omega(z) := \wp(z+\omega) - \wp(z), \qquad \omega \in \Lambda.$$

Then $f'_\omega(z) = \wp'(z+\omega) - \wp'(z) = 0$ since $\wp'$ is Λ-periodic. Hence,

$$f_\omega(z) = \text{constant}.$$

We can determine the constant setting $z = -\frac{\omega}{2}$:

$$\text{constant} = f_\omega\left(-\frac{\omega}{2}\right) = \wp\left(\frac{\omega}{2}\right) - \wp\left(-\frac{\omega}{2}\right) = 0$$

since $\wp$ is even.

We conclude that $f_\omega \equiv 0$, i.e. $\wp$ is Λ-periodic.

iii) Notice that $\wp'$ has a triple pole of of order 3. Hence, $\wp'$ has 3 zeros, counted with multiplicity. Since $\wp'$ is odd, it is easy to check that $\left[\frac{\omega_1}{2}\right], \left[\frac{\omega_2}{2}\right], \left[\frac{\omega_1+\omega_2}{2}\right]$ are zeros:

$$\wp'\left(\frac{n_1\omega_1 + n_2\omega_2}{2}\right) = \wp'\left(\frac{n_1\omega_1 + n_2\omega_2}{2} - n_1\omega_1 - n_2\omega_2\right)$$
$$= \wp'\left(-\frac{n_1\omega_1 + n_2\omega_2}{2}\right)$$
$$= -\wp'\left(\frac{n_1\omega_1 + n_2\omega_2}{2}\right)$$

Hence, $\left[\frac{\omega_1}{2}\right], \left[\frac{\omega_2}{2}\right], \left[\frac{\omega_1+\omega_2}{2}\right]$ are zeros. Since they are distinct in $\mathbb{C}/\Lambda$, there cannot be more zeros. Moreover, they are necessarily simple.

iv) A Λ-periodic meromorphic function w/ double poles on Λ has Laurent expansion

$$f(z) = \frac{a}{z^2} + O(z^{-1}) \quad \text{for some } a \in \mathbb{C}.$$

Hence, $f(z) - a\wp(z) = O(z^{-1})$, that is, it is Λ-periodic w/ at most a simple pole on Λ. By the the Weierstraß gap thm, it is constant: $f(z) - a\wp(z) = b \in \mathbb{C}$.

v) We have $\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$, which evaluates zero at $z=0$. Hence, $c=0$:

$$\wp(z) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$$

Thus:

$$\wp'(z) = -\frac{2}{z^3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + O(z^5)$$

We deduce that

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{2}{5} g_2 \frac{1}{z^2} - \frac{4}{7} g_3 + O(z)$$

$$(\wp(z))^3 = \frac{1}{z^6} + \frac{3}{20} g_2 \frac{1}{z^2} + \frac{3}{28} g_3 + O(z)$$

Summing up:

$$(\wp'(z))^2 - \left[4(\wp(z))^3 - g_2\,\wp(z) - g_3\right] = O(z)$$

We deduce that the above combination is holomorphic on $\mathbb{C}/\Lambda$ and vanishes at $[0]$, hence it is constant zero.

Ex 4. We know that E_Λ is smooth, provided that

$$S := \left\{ p \in \mathbb{C}^3 \setminus \{0\} \;\middle|\; F(p) = \frac{\partial F}{\partial z_0}(p) = \frac{\partial F}{\partial z_1}(p) = \frac{\partial F}{\partial z_2}(p) = 0 \right\} = \emptyset$$

where $F = z_1^2 z_2 - z_0^3 + \frac{g_2}{4} z_0 z_2^2 + \frac{g_3}{4} z_2^3$.

From the theory of (depressed) cubic eqns, we have that

$$x^3 - \frac{g_2}{4}x - \frac{g_3}{4}$$

has three distinct roots, provided that $-4\left(-\frac{g_2}{4}\right)^3 - 27\left(-\frac{g_3}{4}\right)^2 \neq 0$.

This is equivalent to $\Delta \neq 0$. Hence, we can write

$$F = z_1^2 z_2 - (z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2) \quad \text{w/ } \alpha_i \text{ distinct.}$$

Suppose now by contraddiction that $\exists\, p = [a:b:c] \in S$.

Notice that $\frac{\partial F}{\partial z_1}(p) = 2bc = 0 \Rightarrow b = 0$ or $c = 0$

If $c=0$, then $F(p) = -a^3 = 0$. Hence, $p = [0:1:0]$. But then

$$\frac{\partial F}{\partial z_2}(p) = z_1^2 + \alpha_1(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2) + \alpha_2(z_0 - \alpha_1 z_2)(z_0 - \alpha_3 z_2) + \alpha_3(z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2) \Big|_{[0:1:0]}$$

$= 1 \neq 0$, contraddiction.

If $b=0$, then $F(p) = (a - \alpha_1 c)(a - \alpha_2 c)(a - \alpha_3 c) = 0$, which gives $p = [\alpha_i : 0 : 1]$ for $i = 1, 2, 3$. But then

$$\frac{\partial F}{\partial z_0}(p) = -(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2) - (z_0 - \alpha_1 z_2)(z_0 - \alpha_3 z_2) - (z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2) \Big|_{p = [\alpha_i : 0 : 1]}$$

$= -(\alpha_i - \alpha_j)(\alpha_i - \alpha_k) \neq 0$, contraddiction.

w/ $\{\alpha_i, \alpha_j, \alpha_k\} = \{\alpha_1, \alpha_2, \alpha_3\}$.

We conclude that $S = \emptyset$, i.e. E_λ is smooth.

It is easy to check that Φ is holomorphic and non-costant. To prove it is a biholomorphism, we only need to show that $\deg \Phi = 1$.

Let us look at $O = [0:1:0]$. It's fibre is simply $\Phi^{-1}(O) = [0]$. To compute the multiplicity, consider the chart $z_1 \neq 0$, so that setting

$$x = \frac{z_0}{z_1}, \qquad y = \frac{z_2}{z_1},$$

the map Φ is given by $(x, y) = \left(\frac{\wp(z)}{2\wp'(z)}, \frac{1}{2\wp'(z)} \right)$

The curve E_Λ on this chart is $Z\left(y - x^3 + \frac{g_2}{4} x y^2 + \frac{g_3}{4} y^3\right)$ whose derivative in y does not vanish at $(0,0)$. Hence, it is locally the graph of a function $y(x)$: $\exists U \subseteq \mathbb{C}^2$ neigh of $(0,0)$ and $V \subseteq \mathbb{C}$ neigh. of 0 s.t.

$$\begin{aligned} V &\xrightarrow{\cong} Z\left(y - x^3 + \tfrac{g_2}{4} x y^2 + \tfrac{g_3}{4} y^3\right) \cap U \\ x &\longmapsto (x, y(x)) \end{aligned}$$

Thus, multiplicity of Φ at $[0]$ is equal to the order of the zero of $x = \frac{\wp(z)}{2\wp'(z)}$ at $z=0$. We know that $\wp$ has a simple pole and $\wp'$ has a triple pole, hence the multiplicity is 1. We conclude that $\deg \Phi = 1$.