RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 7

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<u>Ex 1</u>. The meromorphic fnct $f(z) = \frac{z^3 - z}{z^2+1}$ on $\mathbb{C}$ corresponds to the holomorphic fnct

$$\hat{f} : \mathbb{P}^1 \to \mathbb{P}^1, \qquad [z:w] \longmapsto [x:y] = [\underbrace{z^3 - zw^2}_{:=F(z,w)} : \underbrace{z^2 w + w^3}_{:=G(z,w)}]$$

Indeed, on the charts $\{w \neq 0\} \subset \mathbb{P}^1$ (domain) and $\{y \neq 0\} \subset \mathbb{P}^1$ (codomain) we find

$$\hat{f}([z:1]) = [z^3 - z : z^2+1] = \left[\frac{z^3-z}{z^2+1} : 1\right] = [f(z) : 1]$$

In particular, we see that $\hat{f}$ has 3 simple zeros (the zeros of $F$) an 3 simple poles (the zeros of $G$):

$$F(z,w) = z(z+w)(z-w) \qquad \alpha_1 = [0:1],\ \alpha_2 = [1:-1],\ \alpha_3 = [1:1]$$

$$\leadsto$$

$$G(z,w) = w(z+iw)(z-iw) \qquad \beta_1 = [1:0],\ \beta_2 = [i:-1],\ \beta_3 = [i:1]$$

Thus,

$$\mathrm{div}(\hat{f}) = (\alpha_1 + \alpha_2 + \alpha_3) - (\beta_1 + \beta_2 + \beta_3)$$

<u>Ex 2</u>.

($\Rightarrow$) Let $D = \mathrm{div}(f)$ be principal. Then $\deg(D) = 0$ by the "#zeros = #poles" thm. Moreover, by Abel's thm,

$$A(D) = \sum_{x \in T} \mathrm{ord}_x(f)\, x = [0]$$

$(\Leftarrow)$. Let $D = \sum_{i=1}^{M} m_i [x_i] - \sum_{j=1}^{N} n_j [y_j]$ w/ $m_1, \dots, m_M, n_1, \dots, n_N \geq 1$

and $x_1, \dots, x_M, y_1, \dots, y_N$ distinct. The $\deg(D) = 0$ condition is

$$\sum_{i=1}^{M} m_i = \sum_{j=1}^{N} n_j$$

and the $A(D) = 0$ condition is

$$\sum_{i=1}^{M} m_i x_i = \sum_{j=1}^{N} n_j y_j \mod \Lambda$$

To simplify the analysis, suppose WLOG that $\Lambda = \mathbb{Z} + \tau \mathbb{Z}$. It is easy to see that there exists representatives $x_i, y_j \in \mathbb{C}$ s.t.

$$[x_i] = x_i, \quad [y_j] = y_j, \quad \sum_{i=1}^{M} m_i x_i = \sum_{j=1}^{N} n_j y_j$$

In this case, consider the ratio of theta functions

$$f(z) = \frac{\prod_{i=1}^{M} \vartheta_{(x_i)}(z;\tau)^{m_i}}{\prod_{j=1}^{N} \vartheta_{(y_j)}(z;\tau)^{n_j}}$$

It is meromorphic, $\Lambda$-periodic, w/ zeros at $[x_i] = x_i$ of order $m_i$ and poles at $[y_j] = y_j$ of order $n_j$. Hence, $\mathrm{div}(f) = D$.

Ex3. Let $D = \sum_{y \in Y} n_y [y] \in \mathrm{Div}(Y)$. Then

$$\varphi^* D = \sum_{y \in Y} n_y \sum_{x \in \varphi^{-1}(y)} \mu_x(\varphi) [x].$$

Thus:

$$\deg(\varphi^* D) = \sum_{y \in Y} n_y \underbrace{\sum_{x \in \varphi^{-1}(y)} \mu_x(\varphi)}_{= \deg(\varphi)}$$

$$= \deg(\varphi) \underbrace{\sum_{y \in Y} n_y}_{= \deg(D)}$$

$$= \deg(\varphi) \cdot \deg(D).$$

If $D = \mathrm{div}(f)$ is principal, $\varphi^* D$ is principal as well. Indeed, consider $g = f \circ \varphi$, which is a meromorphic function on $X$. Then

$x \in X$ is a zero/pole of $g$ $\iff$ $\varphi(x)$ is a zero/pole of $f$

and $\mathrm{ord}_x(g) = \mathrm{ord}_{\varphi(x)}(f) \cdot \mu_x(\varphi)$. Thus:

$$\varphi^* D = \sum_{y \in Y} \mathrm{ord}_y(f) \sum_{x \in \varphi^{-1}(y)} \mu_x(\varphi) [x].$$

$$= \sum_{y \in Y} \sum_{x \in \varphi^{-1}(y)} \mathrm{ord}_x(g) [x]$$

$$= \sum_{x \in X} \mathrm{ord}_x(g) [x] = \mathrm{div}(g).$$

Ex 4. The map is well-defined, as $\mathrm{div}(\lambda f) = \mathrm{div}(f)$ $\forall \lambda \in \mathbb{C}^*$. To prove is an iso, we show injectivity / surjectivity.

- INJ. Let $\mathrm{div}(f) + D = \mathrm{div}(g) + D$. Then $\mathrm{div}\left(\frac{f}{g}\right) = 0$. Thus, $\frac{f}{g} = \lambda \in \mathbb{C}^*$. In other words, $[f] = [g]$ in $\mathbb{P}(\mathcal{L}(D))$.

- SURJ. Take $E \sim D$, $E \geq 0$. Then $E - D = \mathrm{div}(f)$ and $f \in \mathcal{L}(D)$, since $E$ is positive.

In our examples:

- For $X = \mathbb{P}^1$, $D = d[x]$, we know that

$$\mathcal{L}(D) = \begin{cases} \{0\} & \text{if } d < 0 \\ \mathbb{C}[z,w]_d & \text{if } d \geq 0 \end{cases}$$

Thus,

$$|D| = \begin{cases} \emptyset & \text{if } d < 0 \\ \mathbb{P}(\mathbb{C}[z,w]_d) \cong \mathbb{P}^d & \text{if } d \geq 0 \end{cases}$$

- For $X = \mathbb{C}/\Lambda$, $D = [x]$, we first claim that $\mathcal{L}(D) = \mathcal{O}_X \cong \mathbb{C}$

Indeed, $f \in \mathcal{L}(D) \setminus \{0\}$ iff

$$\mathrm{div}(f) + [x] \geq 0$$

This means that $f$ has at most a simple pole at $x$ and no other poles. Meromorphic functions w/ a single simple pole on a torus do not exist. Hence, $f$ is holomorphic. As a consequence:

$$\mathbb{P}(\mathcal{L}(D)) = \{[1]\} \cong \{D\} = |D|.$$