RIEMANN SURFACES - SPRING 2024

EXERCICES SHEET 8

Ex 1.

1) Let $\omega \in \Omega(\mathbb{P}^1)$. It consists of two elements:

$$f(z)\,dz \quad \text{on} \quad \mathbb{C} \cong \{[x:y] \mid y \neq 0\}$$

$$g(w)\,dw \quad \text{on} \quad \mathbb{C} \cong \{[x:y] \mid x \neq 0\}$$

with f, g holomorphic. The transition map is

$$\tau : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}, \qquad z \longmapsto w = 1/z.$$

Thus, f and g must satisfy

$$f(z) = g(1/z)\left(-1/z^2\right) \qquad \forall z \in \mathbb{C}\setminus\{0\}.$$

This is impossible unless $g \equiv 0$, since the LHS is holomorphic at $z=0$ while the RHS has a pole. This implies that $f \equiv 0$. We conclude that $\omega \equiv 0$.

2) On $\mathbb{C}/\Lambda$, we can write a holomorphic form as

$$\omega = f(z)\,dz \longrightarrow$$ NB: this follows from the shape of the local charts for $\mathbb{C}/\Lambda$ from Ex. Sheet 2.

with f a holomorphic, Λ-periodic function on $\mathbb{C}$. The only such function is a constant function.

3) The function $f: \mathbb{P}^1 \to \mathbb{P}^1$, $[z,w] \mapsto [z:w]$ is represented on the two charts of $\mathbb{P}^1$ by the functions

$$z \mapsto z \qquad \text{on} \qquad \{[z:w] \mid w \neq 0\}$$

$$w \mapsto \frac{1}{w} \qquad \text{on} \qquad \{[z:w] \mid z \neq 0\}$$

Hence, df is the meromorphic form

$$dz \qquad \text{on} \qquad \{[z:w] \mid w \neq 0\}$$

$$-\frac{1}{w^2}\, dw \qquad \text{on} \qquad \{[z:w] \mid z \neq 0\}$$

Notice that indeed on the overlap we have $1 = \left(-\frac{1}{(1/z)^2}\right) \cdot \left(-1/z^2\right)$. In particular, we see that df has a double pole at $\infty = [1:0]$ and no zeros:

$$\operatorname{div}(df) = -2[\infty] \quad \Rightarrow \quad \deg(\operatorname{div}(df)) = -2.$$

Ex 2. Fix $x \in X$, set $y = f(x)$. In local coordinates around x and y, we have

$$f: z \mapsto w = z^k, \qquad k = \mu_x(f).$$

On the other hand, let ω be any non-trivial meromorphic form on Y (so that $K_Y \sim \operatorname{div}(\omega)$). Locally around y, we can write $\omega = g(w)\, dw$. Let $f^*\omega$ be the meromorphic form on X defined locally around x as

$$
\begin{aligned}
f^*\omega &= g(f(z))\, df(z) \\
&\overset{!}{=} g(z^k)\, k\, z^{k-1}\, dz .
\end{aligned}
$$

Then $\operatorname{div}(f^*\omega) \sim K_X$. Moreover, if $\operatorname{ord}_y \omega = m$, i.e.

$$g(w) = a\, w^m + O(w^{m-1}), \qquad a \neq 0,$$

then

$$g(z^k)\, k z^{k-1} = ak\, z^{km+k-1} + O(z^{km+k}).$$

In other words, $\operatorname{ord}_x f^*\omega = k \cdot m + k - 1 = \mu_x(f)\operatorname{ord}_y(\omega) + (\mu_x(f) - 1)$.

All together, we find

$$
\begin{aligned}
K_X &\sim \sum_{x \in X} \operatorname{ord}_x(f^*\omega)\, [x] \\
&= \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \mu_x(f) \operatorname{ord}_y(\omega)\, [x] + \sum_{x \in X} (\mu_x(f) - 1) \\
&\sim f^* K_Y + R_f .
\end{aligned}
$$

Ex 3. Consider the case of du. This is simply the differential of the meromorphic map

$$
\begin{aligned}
u :\ X &\to \mathbb{P}^1 \\
[x:y:z] &\mapsto [x:z]
\end{aligned}
$$

Indeed, on the chart $U_z = \{z \neq 0\}$, it maps $[u:v:1]$ to u. In particular, du is a meromorphic form.

We now want to show that $f_u\, du = - f_v\, dv$ as meromorphic forms. Notice that the function $f : Z(f) \to \mathbb{C}$ is constant zero, hence its differential is constant zero. But

$$df = f_u\, du + f_v\, dv$$

hence the thesis. This extends to the point at ∞ as well.

Consider now a polynomial $p \in \mathbb{C}[u,v]$. The discussion above implies that

$$\omega_p = p(u,v) \frac{du}{f_v}$$

this because $\dfrac{du}{f_v} = - \dfrac{dv}{f_u}$ and f_u, f_v cannot vanish simultaneously by smoothness

↑

is holomorphic on the affine part $X \cap U_z$ (this is true for any degree). So we only need to check that ω_p is holomorphic at ∞ (i.e. $z=0$), provided that $\deg(p) \leq d-3$.

Consider $U = \{y \neq 0\}$, so that $X \cap U_y = Z(g)$,

$$g(s,t) = F(s,1,t)$$

On the intersection $X \cap U_y \cap U_z$, we have $u = \frac{s}{t}$ $v = \frac{1}{t}$; hence, on $X \cap U_y \cap U_z$:

$$\omega_p = p(u,v) \frac{du}{f_v(u,v)} = \frac{p\left(\frac{s}{t}, \frac{1}{t}\right)}{f_v\left(\frac{s}{t}, \frac{1}{t}\right)} \left(\frac{ds}{t} - \frac{1}{t^2} dt \right)$$

$$= t^{d-3-n} \frac{t^n p(\frac{s}{t}, \frac{1}{t})}{t^{d-1} f_v(\frac{s}{t}, \frac{1}{t})} (t\,ds - dt)$$

where we denoted by $n = \deg(p)$. By hypothesis, $n \leq d-3$, so t^{d-3-n} is polynomial in t. Similarly,

$$t^n p(\frac{s}{t}, \frac{1}{t}), \qquad t^{d-1} f_v(\frac{s}{t}, \frac{1}{t}), \qquad t\,ds - dt$$

are polynomial in t. So ω_p extends to $t=0$ (the point at ∞), provided that the denominator doesn't vanish at such points. This can be shown by looking at ω_p in $X \cap U_x \cap U_z$ and using the smoothness condition.

Notice that

$$\dim_{\mathbb{C}} \{ p \in \mathbb{C}[u,v] \mid \deg(p) \leq d-3 \} = \binom{d-3+2}{2} = \frac{(d-1)(d-2)}{2}$$

which is precisely the genus of X.

Ex 4.

① By hypothesis, $\mathcal{L}(D-[x])$ has codim $=1$, in $\mathcal{L}(D)$. In particular, we can fix a basis $f_1, \dots, f_n$ of $\mathcal{L}(D-[x])$ and extend it to a basis $f_0, f_1, \dots, f_n$ of $\mathcal{L}(D)$

by adding $f_0 \in \mathcal{L}(D) - \mathcal{L}(D-[x])$.

Then $\forall i \geq 1$

$$\operatorname{ord}_x(f_i) \geq -(n_x - 1) > n_x$$

If by contradiction $\operatorname{ord}_x(f_0) > n_x$, then we would have $f_0 \in \mathcal{L}(D-[x])$. Hence, $\operatorname{ord}_x(f_0) = n_x$.

② Suppose $\varphi_D(x) = \varphi_D(y)$. WLOG, we can suppose that φ_D is defined via the basis from the point above w.r.t. $x \in X$.

(if not, we can consider a change of basis which induces an automorphism of $\mathbb{P}^n$ and proceed from there).

In this case, we have

$$\varphi_D(x) = \left[1 : \frac{f_1(x)}{f_0(x)} : \cdots : \frac{f_1(x)}{f_0(x)}\right] = [1 : 0 : \cdots : 0]$$

have a
zero at x

since $\operatorname{ord}_x\left(\frac{f_i}{f_0}\right) = \operatorname{ord}_x(f_i) - \operatorname{ord}_x(f_0) > 0$.

Thus, $\varphi_D(y) = \varphi_D(x) = [1 : 0 : \cdots : 0]$ implies that

$$\operatorname{ord}_y(f_i) > \operatorname{ord}_y(f_0) \qquad \forall i \geq 1.$$

Since $|D|$ is free, we conclude that $f_1, \dots, f_n$ is a basis of $\mathcal{L}(D-[y])$.

This shows that

$$\varphi_D(x) = \varphi_D(y) \quad \text{iff} \quad \mathcal{L}(D-[x]) = \mathcal{L}(D-[y]).$$

It is easy to see that $\mathcal{L}(D-[x]) = \mathcal{L}(D-[y])$ iff

$$\mathcal{L}(D-[x]) = \mathcal{L}(D-[y]) = \mathcal{L}(D-[x]-[y]).$$

This concludes the proof of the first claim.

Now, since $|D|$ is free,

$$\ell(D-[x]) = \ell(D-[y]) = \ell(D)-1 \quad \forall x,y.$$

Thus, $\ell(D-[x]-[y])$ is either $\ell(D)-1$ or $\ell(D)-2$.

- If φ_D is injective, by the first part

$$\mathcal{L}(D-[x]-[y]) \subsetneq \mathcal{L}(D-[x]).$$

Hence, $\ell(D-[x]-[y]) = \ell(D)-2$.

- If $\ell(D-[x]-[y]) = \ell(D)-2$ $\forall x \neq y$, then the tower of inclusions

$$\mathcal{L}(D-[x]-[y]) \subseteq \mathcal{L}(D-[x]) \subseteq \mathcal{L}(D)$$

must be proper $\forall x \neq y$, hence φ_D is injective.

3 To have an embedding, we must check that $\forall \in X$, after choosing the basis from 1, $\exists i \geq 1$ s.t.

$$f_i = \frac{a}{z^{n_x-1}} + O(z^{-n_x+2}), \qquad a \neq 0.$$

This is because, if this is the case, near x we can write φ_D as

$$z \mapsto [1 : \cdots : a + O(z) : \cdots]$$

This is possible iff $\exists f \in \mathcal{L}(D-[x]) \setminus \mathcal{L}(D-2[x])$, thus concluding the proof.