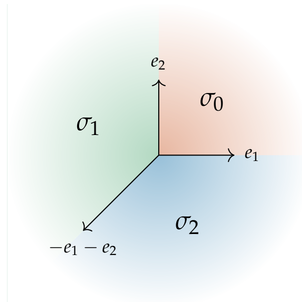
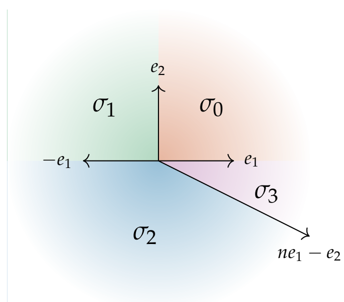
Exercise 1. *Work out the details of the construction of $\mathbb{P}^2$ from the following fan in $\mathbb{R}^2 \supset \mathbb{Z}^2$.*



More precisely, the fan consists of the two-dimensional cones $\sigma_0, \sigma_1, \sigma_2$, the one-dimensional cones given by their faces, namely $\sigma_0 \cap \sigma_1$, $\sigma_0 \cap \sigma_2$, $\sigma_1 \cap \sigma_2$, and the zero-dimensional cone, the origin, which is the triple intersection $\sigma_0 \cap \sigma_1 \cap \sigma_2 = \{ 0 \}$.

Exercise 2. *Fix a non-negative integer $n \geq 0$, and consider the fan Δ_n in $\mathbb{R}^2 \supset \mathbb{Z}^2$ represented as follows.*



As above, the fan consists of the two-dimensional cones $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ and all their intersections. Compute the associated variety X_{Δ_n}, known as the n-th Hirzebruch surface, denoted Σ_n. Convince yourself that the resulting variety is

$$\Sigma_n \cong \frac{(\mathbb{C}^2 \setminus \{ 0 \}) \times (\mathbb{C}^2 \setminus \{ 0 \})}{\mathbb{C}^* \times \mathbb{C}^*}, \tag{1}$$

where the action of $\mathbb{C}^* \times \mathbb{C}^*$ is given by

$$(\lambda, \mu) \cdot (z_1, z_2, w_1, w_2) = (\lambda z_1, \lambda z_2, \mu w_1, \lambda^n \mu w_2). \tag{2}$$

Comment: From the above representation, one can deduce that Σ_n is the total space of the projectivisation of the rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathbb{P}^1$, where the action of $\mathbb{C}^*_\lambda$ gives the total space of $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathbb{P}^1$ and the action of $\mathbb{C}^*_\mu$ projectivise the fibres.

Exercise 3. *Let Δ be a fan in $\mathbb{R}^n$ with lattice N, and Δ' be a fan in $\mathbb{R}^{n'}$ with lattice N'. Define*

$$\Delta \times \Delta' = \left\{ \sigma \times \sigma' \;\middle|\; \sigma \in \Delta, \sigma' \in \Delta' \right\}. \tag{3}$$

Convince yourself that $\Delta \times \Delta'$ is a fan in $\mathbb{R}^{n+n'}$ with lattice $N \oplus N'$, and that the associated variety is $X_{\Delta \times \Delta'} \cong X_\Delta \times X_{\Delta'}$. Use this to give a short proof that the fan Δ_0 from exercise 2 gives $\mathbb{P}^1 \times \mathbb{P}^1$.