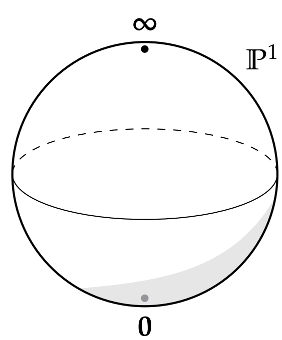
Exercise 1. *Verify the orbit–cone correspondence for $\mathbb{P}^1 \times \mathbb{P}^1$.*

Exercise 2. *The goal of this exercise is to recover the fan of $\mathbb{P}^2$ from its toric structure. More precisely, write down:*

- *The embedded torus $T \subset \mathbb{P}^2$ and its action on $\mathbb{P}^2$.*
- *The isomorphism between $\mathbb{Z}^2$ and the character and cocharacter lattices.*
- *The one-to-one correspondence between the limit points of one-parameter subgroups and the cones of the fan associated with $\mathbb{P}^2$.*

Exercise 3. *The goal of this exercise is to understand rational functions on the projective line $\mathbb{P}^1$.*

- *Convince yourself that all rational functions on $\mathbb{P}^1$ are of the form*
$$f([z,w]) = \frac{p(z,w)}{q(z,w)}, \qquad [z,w] \in \mathbb{P}^1, \tag{1}$$
where $p, q \in \mathbb{C}[z,w]$ are homogeneous *polynomials* of the same degree *and* with no common factors.
- *To the rational function f as above, associate the function*
$$F \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad F([z,w]) = [p(z,w), q(z,w)], \tag{2}$$
with p and q as above. Convince yourself that this defines a one-to-one correspondence. In particular, zeros of f correspond to points in $F^{-1}([0:1]) = p^{-1}(0,0)$, and poles of f correspond to points in $F^{-1}([1:0]) = q^{-1}(0,0)$. For this reason, it is customary to denote $\mathbf{0} = [0:1]$ and $\boldsymbol{\infty} = [1:0]$ in $\mathbb{P}^1$, see figure below.



Recall that the Weil group of X is defined as the free abelian group generated by codimension-1 irreducible subvarieties of X. If X is one-dimensional, then the Weil group is simply the free abelian group generated by the points of X. Explicitly, for X one-dimensional:
$$\mathrm{WDiv}(X) := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{Z}, \text{ finitely many } a_x \text{ are non-zero} \right\}. \tag{3}$$
Recall that $\mathrm{PDiv}(X)$ is the subgroup of divisors associated with global rational functions on X, called principal divisors. The divisor of a rational function f, denoted $\mathrm{div}(f)$, is the formal sum

of zeros minus poles, counted with multiplicity. The quotient $\mathrm{WDiv}(X)/\mathrm{PDiv}(X)$ is called the class group, denoted $\mathrm{Cl}(X)$.

Exercise 4. *The goal of this exercise is to compute the class group of $\mathbb{P}^1$. We proceed in three steps.*

- *For f as in equation (1), write down $\mathrm{div}(f)$. (Hint: use the fundamental theorem of algebra.)*
- *Consider the map*

$$\deg\colon \mathrm{WDiv}(\mathbb{P}^1) \longrightarrow \mathbb{Z}, \qquad \deg\left(\sum_{p\in\mathbb{P}^1} a_p p\right) = \sum_{p\in\mathbb{P}^1} a_p, \tag{4}$$

  called the degree map*. It is clearly surjective. Its kernel is called the group of degree-zero Weil divisors. Show that $\mathrm{PDiv}(\mathbb{P}^1) \subseteq \ker(\deg)$, i.e., every principal divisor has degree zero.*
- *Show that the converse is also true on $\mathbb{P}^1$: every degree-zero Weil divisor is principal. Conclude that*

$$\mathrm{Cl}(\mathbb{P}^1) = \frac{\mathrm{WDiv}(\mathbb{P}^1)}{\mathrm{PDiv}(\mathbb{P}^1)} \cong \mathbb{Z}. \tag{5}$$