ETHZ – Toric Geometry – Spring 2025 **Organiser: Alessandro Giacchetto**
Exercise sheet 6

Exercise 1. *Prove that* $\mathrm{Cl}(\mathbb{P}^n) \cong \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Exercise 2. *Let P be a polytope with $0 \in \mathrm{int}(P)$ with associated fan Δ_P. Show that*

$$\psi_P \colon |\Delta_P| \longrightarrow \mathbb{R}, \qquad \psi_P(v) := \min_{u \in P} \langle u, v \rangle, \tag{1}$$

is a support function and that $[\psi_P] \neq 0$ in $\mathrm{Pic}(X_P)$. Use this to conclude that the fan obtained from the standard cube in $\mathbb{R}^3$ by replacing $(1,1,1)$ with $(1,2,3)$ is non-polytopal.

Exercise 3. *Let $X = \mathbb{P}^2$. Compute $H^p(X, \mathcal{O}_X)$ from the definition of sheaf cohomology, taking the affine cover of defined by the fan of $\mathbb{P}^2$ as an open cover.*

Exercise 4 (💀). *Let $X = \mathbb{P}^n$. For any $d \in \mathbb{Z}$, consider $D := dD_0$, where D_0 is the closure of the orbit associated with the ray generated by $e_0 = -(e_1 + \cdots + e_n)$.*

- *Prove that the support function ψ_D evaluates to zero on $e_1, \ldots, e_n$ and to m on e_0. Deduce that ψ_D is zero on the cone generated by $e_1, \ldots, e_n$, and is $m \langle e_i^*, \cdot \rangle$ on the cone generated by $e_0, \cdots, \widehat{e_i}, \cdots, e_n$.*
- *For $(u_1, \ldots, u_n) \in \mathbb{Z}^n$, prove that χ^u is the rational function*

$$\chi^u \colon [z_0, z_1, \cdots, z_n] \longmapsto \frac{z_1^{u_1} \cdots z_n^{u_n}}{z_0^{u_1 + \cdots + u_n}}. \tag{2}$$

- *For $d \geq 0$, verify that ψ_D is convex. Verify that*

$$P_D = \left\{ (u_1, \ldots, u_n) \in \mathbb{Z}^n \;\middle|\; u_i \geq 0 \text{ and } \sum_i u_i \leq d \right\}. \tag{3}$$

  Deduce that

$$H^p(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \mathbb{C}[z_1, \ldots, z_n]_d & \text{if } p = 0, \\ 0 & \text{else.} \end{cases} \tag{4}$$

  Here $\mathbb{C}[z_1, \cdots, z_n]_d$ denotes the space of polynomials of degree at most d.
- *For $d < 0$ verify that ψ_D is concave, so that $H^p_{Z(u)}(|\Delta|) = 0$ unless $Z(u) = 0$. Deduce that*

$$H^p(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} (z_1^{-1} \cdots z_n^{-1} \mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}])_d & \text{if } p = n, \\ 0 & \text{else.} \end{cases} \tag{5}$$