ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Toric Geometry

Participants' notes

Spring 2025

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Abstract

Toric varieties provide a rich class of examples in algebraic geometry that bridge combinatorics and geometry, making them an ideal starting point for exploring the interplay between these fields. We will introduce toric varieties, study their combinatorial and abstract structures, and examine their basic geometry, including singularities, Picard groups, and cohomology.

Contents

Chapter 1

Introduction

1.1 Organisation of the seminar

Toric geometry is a rich and well-structured branch of algebraic geometry that studies varieties constructed from combinatorial data. It provides a bridge between algebraic geometry and polyhedral geometry, allowing for deep insights into both subjects. The key objects of study in toric geometry are toric varieties, which are varieties containing an algebraic torus $(\mathbb{C}^*)^n$ as a dense open subset, with an extended action of the torus on the entire variety.

A major advantage of toric geometry is its explicit and combinatorial nature. Many geometric properties of toric varieties, such as singularities, divisors, and intersection theory, can be understood purely in terms of convex geometry. This makes toric varieties an important tool in areas such as integer programming and mirror symmetry. Additionally, toric varieties often provide concrete examples and intuition for more abstract concepts in algebraic geometry.

Main references

[1] J.-P. Brasselet, *Introduction to Toric Varieties* (Publicações Matemáticas). IMPA, 2008.

[2] W. Fulton, *Introduction to Toric Varieties* (Annals of Mathematics Studies 131). Princeton University Press, 1993.

[3] D. A. Cox, J. B. Little, H. K. Schenck, *Toric Varieties* (Graduate Studies in Mathematics 124). American Mathematical Society, 2011.

Guidelines for seminar talks and note-taking

Each student will take on one or two responsibilities: giving a seminar talk and/or taking notes while preparing questions. Both roles are essential for the success of the seminar and contribute to a deeper understanding of the material.

Giving a seminar talk. A good seminar talk is structured, engaging, and clear. When preparing your presentation, start by identifying the key ideas: What are the main concepts? Why are they important? How do they fit into the broader context of toric geometry? Before diving into technical proofs, take some time to provide intuition and motivation.

Here are a few practical tips:

- Organisation: Plan your talk with a clear structure—begin with an overview, introduce necessary definitions, state the main results, and then explain proofs or computations step by step.
- Examples: Illustrate abstract ideas with concrete examples. Toric geometry is highly visual, so well-chosen diagrams or computations can greatly aid understanding.
- Notation and clarity: Be mindful of notation and avoid overwhelming the audience with too many symbols at once. Whenever possible, explain formulas in words.
- Time management: Practice beforehand to ensure your talk fits within the allotted time. If a proof is too long, highlight only the key steps and refer to written notes for details.
- Engagement: Encourage questions and interaction. If a concept is tricky, take a moment to check if everyone is following before moving on.

For additional tips on giving a seminar talk, see the advice provided by Johannes Schmitt (link here).

Taking notes and preparing questions. Note-taking is not just about transcribing the lecture; it should provide a structured and useful resource for review. Good notes should capture the essential ideas, definitions, and results while filtering out unnecessary details.

Consider the following strategies:

- Structure: Follow the logical flow of the talk. Separate definitions, theorems, and proofs clearly, and include brief explanations of why each result matters.
- Diagrams and examples: Since toric geometry has a strong combinatorial aspect, sketches of cones, fans, or polytopes can make the notes much clearer.
- Clarifications: If something was unclear during the talk, try to rephrase it more clearly in the notes.
- Conciseness: Avoid writing every word verbatim. Instead, focus on capturing the main points succinctly.

For additional tips on mathematical writing, see the Appendix or the advices provided by Keith Conrad (link here).

Notes written by other students should not be modified; any necessary corrections should be left to the organiser and the student responsible for that section of the notes.

Students responsible for note-taking should also prepare questions based on the talk. The aim of this assignment is two-fold. The first goal is to help the audience by being "their voice", asking questions that many students might have but may not have the courage to ask. Questions that you might consider silly—such as *Can you explain this point again?*—are encouraged. The second goal is to help develop critical thinking while reading and listening. In this case, more motivational questions—such as *What is the motivation behind this definition?*—are encouraged.

The student responsible for questions does not have to prepare all questions in advance. Spontaneous questions during the exposition are even more welcome. It is also worth explaining what this role is not for: it is not aimed at interrogating the speaker nor at showing off.

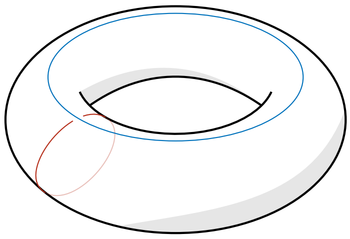
Tentative schedule

Here is the tentative schedule for the seminar, including topics, assignments, and references.

Date	Topics	Presenter	Notes	References
27 Feb	Cones, faces Monoids, algebraic varieties	Jonas Carl	Janine	[1, §1.1-1.2], [2, §1.1-1.2] [1, §1.3-2.2], [2, §1.1-1.2]
6 Mar	Affine toric varieties	Yu-Yuan Elisa M.	Sirawit	[1, §2.3], [2, §1.3] [1, §2.3], [2, §1.3]
13 Mar	From fans to toric varieties	Zheming Marco	Ilan	[1, §3.1-3.2], [2, §1.4] [1, §3.2-3.3], [2, §1.4]
20 Mar	Orbits	Noah Alex	Elisa L.	[1, §4.1-4.2], [2, §3.1] [1, §4.2], [2, §3.1]
27 Mar	Smooth, complete, projective From toric varieties to fans	Sirawit Lizanne	Shengyang	[1, §3.4], [2, §2.1] [1, §4.3], [2, §2.3]
3 Apr	Toric varieties from polytopes Divisors	Rasmus Davide	Noah	[1, §4.3], [2, §2.3] [1, §4.4], [2, §1.5]
10 Apr	Divisors (continued) Sheaf cohomology	Elisa L. Janine	Alex	[3, §4.2] Notes
17 Apr	Cohomology and Picard groups Cohomology of line bundles	Shengyang Ilan	Lizanne	Notes [2, §3.4-3.5]
8 May	Resolution of singularities	Jonas Yu-Yuan	Rasmus	[3, §1.11]
15 May	Chow groups Riemann–Roch and Pick’s formula	Elisa M. Zheming	Davide	[2, §5.2] [2, §5.3]
22 May	Stanley’s theorem	Carl Marco	Alessandro	[2, §5.6]

1.2 What is a toric variety?

Before giving the definition of a toric variety, let us first ask: *What is a torus?* In algebraic topology, a torus is a 2-dimensional surface with the shape of a doughnut.



Mathematically, it is the topological space $S^1 \times S^1$. More generally, an n-dimensional torus, according to a topologist, is the space $(S^1)^n$. The key feature of such a space is that it carries a natural group structure inherited from the natural group structure on the circle.

However, the notion of a torus in algebraic geometry is slightly different, albeit related. To understand the definition of an algebraic torus, let us start with the familiar equation defining a circle S^1 inside the two-dimensional real space:

$$x^2 + y^2 = 1, \qquad (x, y) \in \mathbb{R}^2. \tag{1.1}$$

Over the complex numbers, allowing $(x, y) \in \mathbb{C}^2$, something remarkable happens: the above equation can be factorized as $(x + \mathrm{i}y)(x - \mathrm{i}y) = 1$. By changing variables to $z = x + \mathrm{i}y$ and $w = x - \mathrm{i}y$, we obtain

$$zw = 1, \qquad (z, w) \in \mathbb{C}^2. \tag{1.2}$$

It is not hard to see that this equation describes points in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Indeed, z can take any non-zero value in $\mathbb{C}$, and then w is uniquely determined as z^{-1}. In other words:

$\mathbb{C}^*$ *is the algebraic equivalent of* S^1.

In higher dimensions, we consider the n-fold product $(\mathbb{C}^*)^n$, which carries a natural multiplicative group structure $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$:

$$(z_1, \ldots, z_n) \cdot (w_1, \ldots, w_n) = (z_1 w_1, \ldots, z_n w_n), \tag{1.3}$$

where the component-wise product $z_i w_i$ is the usual multiplication of complex numbers.

With this motivation in place, we are now ready to define an algebraic torus.

Definition 1.1 *An (algebraic)* ***torus*** *is an affine variety T isomorphic to $(\mathbb{C}^*)^n$, where T inherits the group structure from this isomorphism.*

The coordinate ring of $T = (\mathbb{C}^*)^n$ is given by the ring of Laurent polynomials. Indeed, from Equation (1.2) we deduce

$$\mathbb{C}[T] = \frac{\mathbb{C}[z_1, w_1, \dots, z_n, w_n]}{(z_1 w_1 - 1, \dots, z_n w_n - 1)} \cong \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]. \tag{1.4}$$

This fact will be the starting point for the combinatorial description of toric varieties, which we can now define.

Definition 1.2 *A* ***toric variety*** *is a (normal) algebraic variety X containing a torus T as a dense open subset, such that the action*[1] *of T on itself extends to an action on all of X.*

The torus $(\mathbb{C}^*)^n$ is the simplest example of a toric variety.

By 'filling in' the origin, we can see that the affine space $\mathbb{C}^n$ is another example of a toric variety. In this case, the torus action extends by setting

$$(z_1, \dots, z_n) \cdot (x_1, \dots, x_n) = (z_1 x_1, \dots, z_n x_n) \tag{1.6}$$

for all $(z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ and $(x_1, \dots, x_n) \in \mathbb{C}^n$.

Another example is the projective space $\mathbb{P}^n$, where the action is given by

$$(z_1, \dots, z_n) \cdot [x_0, x_1, \dots, x_n] = [x_0, z_1 x_1, \dots, z_n x_n] \tag{1.7}$$

for all $(z_1, \dots, z_n) \in (\mathbb{C}^*)^n$ and $[x_0, x_1, \dots, x_n] \in \mathbb{P}^n$.

Toric varieties can also be viewed as highly symmetric spaces. Unlike general algebraic varieties, which may have complicated or limited symmetry,

[1] Recall the definition of a group action. Let G be a group and let X be a set. A map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned} \tag{1.5}$$

is called **action** of G on X if

- for every $x \in X$, $e_G \cdot x = x$, where e_G is the neutral element of G, and
- for every $g, h \in G$ and for every $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$.

toric varieties are explicitly built around the action of a torus, making their geometry more accessible and amenable to a combinatorial description.

However, after this definition, the combinatorial nature of toric varieties is not yet apparent. The next section aims to clarify this connection, which will be explored in detail in the following lectures.

1.3 The combinatorial nature of toric geometry

We have already noticed that the coordinate ring of a torus coincides with the ring of Laurent polynomials. In other words, the 'nice' algebraic functions from a torus $T \cong (\mathbb{C}^*)^n$ to $\mathbb{C}$ are the polynomial functions in $z_1, \dots, z_n$ and $z_1^{-1}, \dots, z_n^{-1}$. Among such functions, we are particularly interested in those that take values in $\mathbb{C}^* \subset \mathbb{C}$ and respect the group structure.

Definition 1.3 *A **character** of a torus T is a morphism $\chi \colon T \to \mathbb{C}^*$ that is also a group homomorphism. Characters form a subgroup of the coordinate ring of T:*

$$M = \mathrm{Hom}_{\mathrm{gp}}(T, \mathbb{C}^*). \tag{1.8}$$

The subscript stands for 'group', highlighting the fact that we consider group homomorphisms.

It is not hard to see that the characters of a torus form a lattice, which is a free abelian group of finite rank.

Lemma 1.4 *The characters of a torus T form a lattice M, which is called the **character lattice**. Under the isomorphism $T \cong (\mathbb{C}^*)^n$, we have $M \cong \mathbb{Z}^n$.*

Proof Consider $T = (\mathbb{C}^*)^n$. We have already established that a morphism to $\mathbb{C}$ is a Laurent polynomial. If we restrict ourselves to morphisms to $\mathbb{C}^*$, we are led to consider only Laurent monomials:

$$(z_1, \dots, z_n) \longmapsto c \cdot z_1^{a_1} \cdots z_n^{a_n} \tag{1.9}$$

for some $c \in \mathbb{C}^*$ and $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Among such morphisms, those that also respect the group structure must have $c = 1$, as can be easily seen by considering the identity element $(z_1, \dots, z_n) = (1, \dots, 1)$. Thus, a character of $T = (\mathbb{C}^*)^n$ is given by

$$\chi(z_1, \dots, z_n) = z_1^{a_1} \cdots z_n^{a_n} \tag{1.10}$$

for $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. In other words, we can identify the set of characters with $\mathbb{Z}^n$. It is straightforward to verify that the multiplicative group structure on the set of characters corresponds to the additive structure of $\mathbb{Z}^n$:

$$\left(z_1^{a_1} \cdots z_n^{a_n}\right) \cdot \left(z_1^{b_1} \cdots z_n^{b_n}\right) = z_1^{a_1+b_1} \cdots z_n^{a_n+b_n}. \tag{1.11}$$

Finally, if T is an abstract torus that is isomorphic to $(\mathbb{C}^*)^n$, we find that its group of characters is naturally isomorphic to $\mathbb{Z}^n$. $\square$

In plain English:

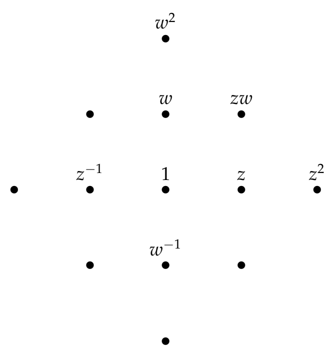
The character lattice of a torus
corresponds to
the group of monic Laurent monomials.

This feature immediately leads us to the combinatorial aspects of lattice points. Indeed, our goal is to construct varieties that contain a dense open torus with an action that respects the action of the torus on itself. At the algebraic level, this corresponds to constructing subrings of the ring of Laurent polynomials that preserve the lattice structure. This can be achieved by taking cones, as the following examples illustrate. More precise definitions will follow in the upcoming lectures.

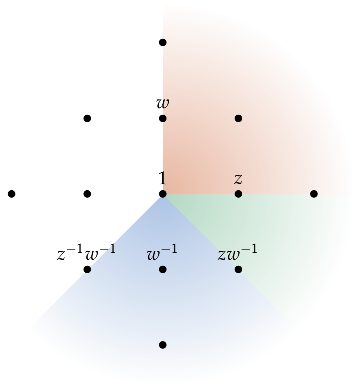
As first example, consider $n = 2$. The coordinate ring of $\mathbb{C}^* \times \mathbb{C}^*$ is $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$, which contains the lattice of characters of monic Laurent monomials through the correspondence

$$z^a w^b \quad \longleftrightarrow \quad (a, b) \in \mathbb{Z}^2. \tag{1.12}$$

We can visualize this lattice as follows.



As explained above, the main idea is to construct varieties by taking subrings of the ring of Laurent polynomials that are compatible with the lattice structure. For instance, consider the three subrings corresponding to the cones depicted below.



By examining all the monomials contained in the red cone, we can immediately see that we obtain the subring $\mathbb{C}[z, w]$. This is nothing but the coordinate ring of the variety $\mathbb{C}^2$, which is indeed toric.

Now, consider the green cone. In this case, the subring that contains all monomials inside this cone is $\mathbb{C}[z, zw^{-1}]$. By setting $\zeta = zw^{-1}$, we find that this ring is equivalent to $\mathbb{C}[z, \zeta]$, which corresponds again to the variety $\mathbb{C}^2$.

To conclude, consider the blue cone. In this case, the corresponding ring is not $\mathbb{C}[zw^{-1}, z^{-1}w^{-1}]$, as one might initially think. Indeed, the monomial y^{-1} is not contained in this ring, and we are forced to include it by hand, considering the ring $\mathbb{C}[zw^{-1}, z^{-1}w^{-1}, w^{-1}]$ instead. Setting $\xi_1 = xy^{-1}$, $\xi_2 = y^{-1}$, and $xi_3 = x^{-1}y^{-1}$, we realise that the relation $\xi_2^2 = \xi_1\xi_3$ holds. In other words, the ring associated with the blue cone is given by

$$\frac{\mathbb{C}[\xi_1, \xi_2, \xi_3]}{\langle \xi_2^2 - \xi_1\xi_2 \rangle}. \tag{1.13}$$

The variety whose coordinate ring is the one above is known as the 'double cone', which can be visualized as the set of points (x_1, x_2, x_3) in three-dimensional space $\mathbb{C}^3$ satisfying the equation $x_2^2 = x_1x_2$.

This is another (and much less trivial) example of a toric variety, containing the torus $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$ given by

$$t_1 = \frac{x_1}{x_2}, \qquad t_2 = \frac{1}{x_2}. \tag{1.14}$$

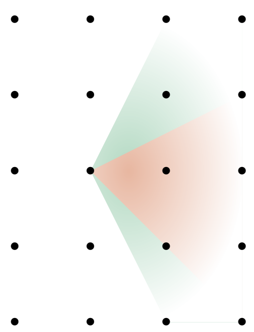
To summarise, we have the following assignments:

$$\begin{aligned}
&\text{red cone} &&\longrightarrow && \mathbb{C}[z,w] &&\longrightarrow && \mathbb{C}^2,\\
&\text{green cone} &&\longrightarrow && \mathbb{C}[z,\zeta] &&\longrightarrow && \mathbb{C}^2,\\
&\text{blue cone} &&\longrightarrow && \frac{\mathbb{C}[\xi_1,\xi_2,\xi_3]}{\langle \xi_2^2-\xi_1\xi_2\rangle} &&\longrightarrow && \text{double cone.}
\end{aligned}$$

These examples prompt two main questions.

1. Given several cones, what is the relation between the associated varieties?
2. In general, varieties are constructed by gluing together algebraic varieties. Can we construct toric varieties by gluing together cones?

Let us answer the first question. Consider the two cones below.



The red cone is contained in the green cone, which means that the red ring is contained in the green one. However, after taking the associated varieties, we are forced to reverse the inclusion: the green variety is contained in the red one. This is somewhat counter-intuitive, as it would be more desirable to have a construction that preserves inclusions.

A workaround is provided by taking dual cones. We will define dual cones in the next lecture, but the key idea is that they consist of linear functionals that are non-negative on the original cone. By working with dual cones, we obtain a construction of the form:

$$\begin{array}{ccccccc}
\sigma & \longrightarrow & \check{\sigma} & \longrightarrow & R_\sigma & \longrightarrow & X_\sigma \\
\text{cone} & & \text{dual cone} & & \text{algebra} & & \text{variety}
\end{array} \tag{1.15}$$

that systematically preserves inclusions. In other words, if $\sigma_1 \subseteq \sigma_2$, then $X_{\sigma_1} \subseteq X_{\sigma_2}$.

Let us now move to the second question, namely, the construction of toric varieties by gluing together cones. Consider the one-dimensional example of the lattice $\mathbb{Z}$ of monic Laurent monomials inside $\mathbb{C}[\zeta^{\pm 1}]$. We define three cones as follows:

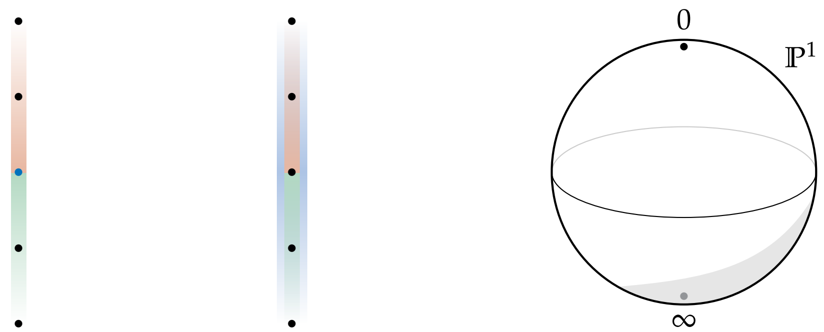
- The red cone of non-negative integers $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.
- The green cone of non-positive integers $-\mathbb{Z}_{\geq 0} = \{0, -1, -2, \dots\}$.
- The blue degenerate cone consisting of only the origin $\{0\}$.

One can interpret this assignment as two cones, red and green, glued together at their intersection, the blue cone. By taking duals, we obtain:

- The dual of the red cone, which consists of the non-negative integers.
- The dual of the green cone, which consists of the non-positive integers.
- The dual of the blue cone, which consists of all integers, $\mathbb{Z}$.

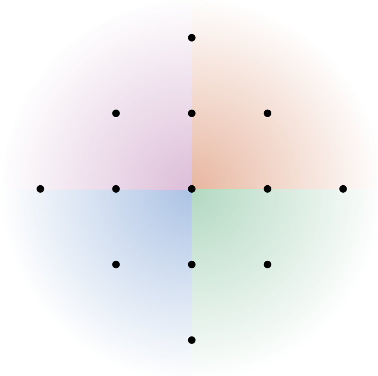
This construction gives rise to the rings $\mathbb{C}[z]$ for the dual red cone, $\mathbb{C}[z^{-1}]$ for the dual green cone, and $\mathbb{C}[z^{\pm 1}]$ for the dual blue cone. Taking the associated varieties, we obtain two copies of the affine line $\mathbb{C}$ (red and green), glued together along one copy of $\mathbb{C}^*$ (blue). This is nothing but the projective line:

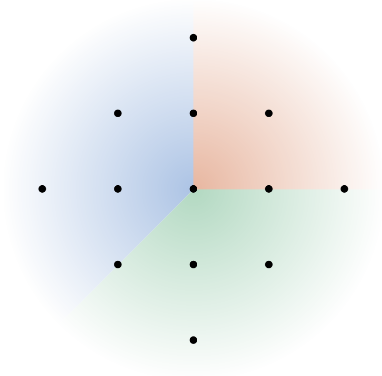
$$\mathbb{P}^1 = \frac{\mathbb{C} \sqcup \mathbb{C}}{\text{glued along } \mathbb{C}^*}. \tag{1.16}$$



The example illustrates the power of this combinatorial perspective: all the information about $\mathbb{P}^1$ and its toric structure is encoded in three simple cones.

As an exercise for the reader, try to determine the (two-dimensional) toric varieties associated with the following sets of two-dimensional cones, glued along their intersections.





To conclude, let us explain why this construction always produces a toric variety. Any collection of cones built out of the lattice $\mathbb{Z}^n$ must include the origin, which corresponds to the most degenerate cone. This cone is associated with the constant monic Laurent monomial 1. Its dual cone is the entire ring $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, which defines an n-dimensional algebraic torus. As a result, the variety obtained from this construction always contains an open dense torus—precisely the defining feature of a toric variety!

Question 1.5 *Is there any connection between algebraic tori and quotients of affine space by lattices?*

In algebraic geometry, quotients of affine space $\mathbb{C}^g$ by a lattice Λ of rank $2g$ (admitting a Riemann form) are called abelian varieties. As topological spaces, they are $2g$-dimensional topological tori, that is, they are homeomorphic to $(S^1)^{2g}$. The case $g = 1$ corresponds to elliptic curves. At first glance, there is no direct connection between abelian varieties and algebraic tori. However, when considering families of abelian varieties that degenerate, one is naturally led to the concept of semiabelian varieties—which are extensions of abelian varieties by an algebraic torus.

Question 1.6 *Can two isomorphic varieties admit non-isomorphic toric structures?*

The answer is no: if two toric varieties X and Y are isomorphic as abstract algebraic varieties, then they are also isomorphic in the category of toric varieties. The proof is rather intricate and relies on the so-called Cox construction [4].

Chapter 2

Combinatorics of toric varieties

Recall that in the last chapter we talked about how we can construct toric varieties from cones. In this chapter, we will make this construction precise. The process follows these steps: we begin with a cone and its dual, construct a monoid and an algebra from the dual cone, and finally obtain an algebraic variety.

Written by Janine
Talks by Jonas and Carl

$$\begin{array}{ccccccccc} \sigma & \longrightarrow & \check{\sigma} & \longrightarrow & S_\sigma & \longrightarrow & R_\sigma & \longrightarrow & X_\sigma \\ \text{cone} & & \text{dual cone} & & \text{monoid} & & \text{algebra} & & \text{variety} \end{array} \tag{2.1}$$

2.1 Convex geometry

We begin with the definition of a polyhedral cone. Intuitively, a polyhedral cone is a cone in a real vector space that is cut out by finitely many hyperplanes.

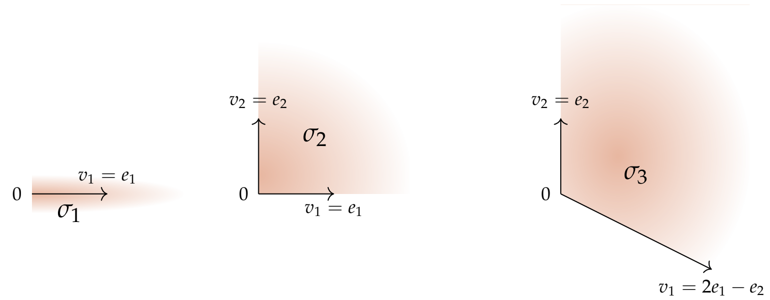
Definition 2.1 *Let V be an n-dimensional real vector space (often, assumed to be $\mathbb{R}^n$). Let $\{ v_1, \ldots, v_r \} \subset V$ be a finite set. Then*

$$\sigma = \operatorname{Cone}(v_1, \ldots, v_r) := \left\{ \sum_{i=1}^{r} c_i v_i \;\middle|\; c_i \geq 0 \right\} \tag{2.2}$$

is called a ***polyhedral cone*** *generated by the vector $v_1, \ldots, v_r$. We define the* ***dimension****, denoted $\dim(\sigma)$, as the vector-space dimension of the smallest linear space in V containing σ.*

Notice that a cone is defined as a subset of a real vector space that is closed under positive scalar multiplication. In these notes, we will simply refer to 'polyhedral cones' as 'cones'.

Example 2.2 *Let us look at some first examples in* $\mathbb{R}^2$.



The three cones σ_1, σ_2 *and* σ_3 *have dimension* 1, 2, *and* 2, *respectively.*

As hinted in the introduction, we are interested in subalgebras of the algebra of Laurent polynomials. The latter naturally arises as the coordinate ring of an algebraic torus and contains the lattice of characters via the correspondence $\mathbb{Z}^n \ni (a_1, \ldots, a_n) \leftrightarrow z_1^{a_1} \cdots z_n^{a_n} \in \mathbb{C}[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]$. Subalgebras that are compatible with this lattice structure can be realised through cones with generators in a lattice.

Definition 2.3 *Let N be a lattice of rank n and set $N_{\mathbb{R}} := N \otimes \mathbb{R}$, which is a real vector space of dimension n with naturally contains the lattice N. A cone σ in $N_{\mathbb{R}}$ is called a* ***lattice*** *(or* ***rational)*** ***cone*** *if one can choose the generators of σ in N. In this case, we will say that σ is a lattice cone in $N_{\mathbb{R}}$.*

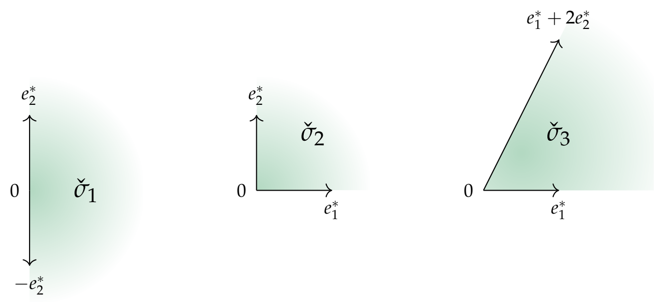
As we would like the assignment $\sigma \mapsto X_\sigma$ to be inclusion-preserving, we are naturally led to consider dual cones. To formalise this, let us first establish some notation. Let V^* be the dual space of V and $\langle \cdot, \cdot \rangle \colon V^* \times V \to \mathbb{R}$ the dual pairing.

Definition 2.4 *Given a cone σ in V, we define*

$$\check{\sigma} := \{ \lambda \in V^* \mid \langle \lambda, v \rangle \geq 0 \quad \forall v \in \sigma \} \tag{2.3}$$

to be the ***dual cone****.*

Example 2.5 *We continue with the examples from Example 2.2, for which the dual cones are as follows.*



The name 'dual cone' is justified because it is, in fact, a cone. For a proof, see [2, Section 1.2].

Lemma 2.6 *Let σ be a cone in V. Then $\check{\sigma}$ is also a cone in V^*. Furthermore, if σ is a lattice $N_{\mathbb{R}}$, then $\check{\sigma}$ is a lattice cone with respect to the* ***dual lattice*** *$M := \mathrm{Hom}_{\mathrm{gp}}(N, \mathbb{Z})$ in $M_{\mathbb{R}} := M \otimes \mathbb{R} = N_{\mathbb{R}}^*$.*

As expected, the assignment $\sigma \mapsto \check{\sigma}$ is reflexive. It is a direct consequence of the following fact. Let C be a non-empty, open, convex set in V and $x \notin C$. Then there is a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$.

Corollary 2.7 *Let σ be a cone. Then $(\check{\sigma})^\vee = \sigma$.*

Building on the idea that polyhedral cones are cut out by hyperplanes, we introduce the concept of faces.

Definition 2.8 *Let σ be a cone in V and let $\lambda \in \check{\sigma} \subseteq V^*$. Then*

$$\tau := \sigma \cap \lambda^\perp \tag{2.4}$$

is called a ***face*** *of σ. We write $\tau \preceq \sigma$. Notice that every cone is a face of itself, by taking $\lambda = 0$. We write $\tau \prec \sigma$ to denote a proper faces.*

Example 2.9 *In Example 2.2, the trivial cone $\{0\}$ is a face of σ_1; the cone σ_1 is a face of σ_2.*

We now list some properties of faces:

Lemma 2.10 *Let σ be a cone in V. The following holds.*

1. *Every face is a cone.*
2. *Any intersection of faces is a face.*
3. *Faces of faces are faces.*

Proof We will prove Item 2. The other properties follow from similar arguments. To this end, it suffices to prove the following relation:

$$\bigcap_{i\in I}(\sigma \cap \lambda_i^{\perp}) = \sigma \cap \left(\sum_{i\in I} \lambda_i^{\perp}\right) \tag{2.5}$$

for any finite index set I. We have $v \in \sigma \cap (\sum_i \lambda_i^{\perp})$ if and only if $\langle \sum_i \lambda_i, v\rangle = \sum_i \langle \lambda_i, v\rangle = 0$. Since $\langle \lambda_i, v\rangle \geq 0$ for all $i \in I$, we have $\sum_i \langle \lambda_i, v\rangle = 0$ if and only if $\langle \lambda_i, v\rangle = 0$ for all $i \in I$. This shows Equation (2.5). □

Taking duals transform sums into intersections and is inclusion-reversing.

Lemma 2.11 *Let $\sigma, \sigma_1, \sigma_2$ be cones in V.*

- *If $\sigma = \sigma_1 + \sigma_2$ then $\check{\sigma} = \check{\sigma}_1 \cap \check{\sigma}_2$.*
- *If $\tau \preceq \sigma$, then $\check{\sigma} \subseteq \check{\tau}$.*

The first point of the lemma is quite useful to compute dual cones. The second point, instead, can be made more explicit with the following result.

Lemma 2.12 *Let $\tau = \sigma \cap \lambda^{\perp}$ be a face of σ in V. Then $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}\langle -\lambda\rangle$.*

Proof On the one hand $(\check{\tau})^{\vee} = \tau$. On the other hand

$$\big(\check{\sigma} + \mathbb{R}_{\geq 0}\langle -\lambda\rangle\big)^{\vee} = \sigma \cap \big(\mathbb{R}_{\geq 0}\langle -\lambda\rangle\big)^{\vee} = \sigma \cap \lambda^{\perp} = \tau. \tag{2.6}$$

Here we used the fact that if $v \in \sigma$ then $\langle -\lambda, v\rangle \geq 0$ if and only if $\langle \lambda, v\rangle \leq 0$ if and only if $\langle \lambda, v\rangle = 0$. □

Faces of a cone and its dual satisfy a natural duality result.

Proposition 2.13 *Let τ be a face of σ in V. Then $\tau^* := \check{\sigma} \cap \tau^{\perp}$ is a face of $\check{\sigma}$. The map $\tau \mapsto \tau^*$ provides an order-reversing bijection:*

$$\{\, \textit{faces of } \sigma \,\} \xleftrightarrow{1:1} \{\, \textit{faces of } \check{\sigma} \,\}\,. \tag{2.7}$$

Moreover, it satisfies the dimension formula $\dim(\tau) + \dim(\tau^*) = n$*, where* $n = \dim(V)$*.*

Example 2.14 *We consider $\tau = \mathrm{Cone}(e_2)$ as a face of σ_2 (see Example 2.2). The covector $\lambda = e_1^* \in \check{\sigma}_2$ satisfies $\tau = \sigma_2 \cap \lambda^{\perp}$ and we have $\check{\tau} = \check{\sigma}_1 + \mathbb{R}_{\geq 0}\langle -\lambda\rangle$.*

We conclude with the notion of relative interior, which expresses the intuitive idea of interior of a cone even if the dimension of the cone is strictly smaller than that of the ambient space.

Definition 2.15 *The **relative interior** of a cone σ in V is the interior of σ in $\mathbb{R}\langle\sigma\rangle$, the smallest linear space it spans in V.*

Example 2.16 *The relative interior of the cone σ generated by e_1 in $\mathbb{R}^2$ is $\sigma \setminus \{ 0 \}$, the positive real x-axis. Notice that the interior of σ as a topological subspace of $\mathbb{R}^2$ is empty.*

2.2 Monoids and algebras from cones

2.2.1 Monoids

Given a cone σ in $N_{\mathbb{R}}$, we can form the set of lattice points inside the cone: $\sigma \cap N$. A natural question arises: how much of the group structure of N is retained within this intersection? In general, $\sigma \cap N$ is not closed under inverses, meaning it lacks the full structure of a group. However, it is still closed under addition, forming a weaker algebraic structure known as a monoid. This motivates the following definition.

Definition 2.17 *A **monoid** is a non-empty set S with an associative binary operation $+\colon S \times S \to S$ that is commutative, has a zero element, and satisfies the simplification law:*

$$s + t = s' + t \qquad \Longrightarrow \qquad s = s' \qquad \text{for all } s, s', t \in S. \tag{2.8}$$

As anticipated, our first example of a monoid is directly related to cones.

Lemma 2.18 *If σ is a cone in $N_{\mathbb{R}}$, then $\sigma \cap N$ is a monoid.*

Proof If $v, u \in \sigma \cap N$, then $v + u \in \sigma \cap N$ and the rest is easily verified. □

In practice, we are most interested in monoids that can be described by a finite set of generators.

Definition 2.19 *A monoid S is **finitely generated** if there exist $s_1, \dots, s_k \in S$ such that any element $s \in S$ can be written as $s = a_1 s_1 + \cdots + a_k s_k$ with $a_i \in \mathbb{Z}_{\geq 0}$. In this case, we write $S = \mathbb{Z}_{\geq 0}\langle s_1, \dots, s_k\rangle$.*

The following lemma will be useful in the next section.

Lemma 2.20 (Gordan's Lemma) *If σ is a lattice cone in $N_{\mathbb{R}}$, then $\sigma \cap N$ is a finitely generated monoid.*

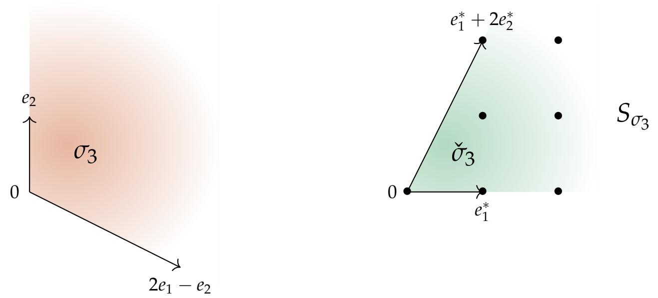
Proof Let $v_1, \ldots, v_r \in N$ be a set of lattice generating vectors for the cone σ. The set $K := \{ \sum_{i=1}^{r} c_i v_i \mid 0 \leq c_i \leq 1 \}$ is compact and N is discrete, therefore $K \cap N$ is a finite set. We claim that it generates $\sigma \cap N$. Indeed, every $v \in \sigma \cap N$ can be written as $v = \sum_i (a_i + c_i) v_i$ where $a_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq c_i \leq 1$. Each v_i and the sum $\sum_i c_i v_i$ belong to $K \cap N$, hence the claim. $\square$

Definition 2.21 *For a cone σ in $N_{\mathbb{R}}$, we denote*

$$S_\sigma := \check{\sigma} \cap M \tag{2.9}$$

where $M = \mathrm{Hom}_{\mathrm{gp}}(N, \mathbb{Z})$ is the dual lattice.

Example 2.22 *In $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$, consider again σ_3 from Example 2.2.*



On the right, the lattice points in $S_\sigma = \check{\sigma} \cap M$ are marked with $\bullet$. Notice that S_σ is not generated solely by the vectors e_1^* and $e_1^* + 2e_2^*$. To obtain a set of generators, one has to add $e_1^* + e_2^*$. Then, S_σ is generated by $e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*$. This is in line with the proof of Gordan's Lemma.

We conclude with a natural characterisation of monoids associated with faces.

Lemma 2.23 *Let σ be a lattice cone in $N_{\mathbb{R}}$ and $\tau = \sigma \cap \lambda^\perp$ a face of σ, with $\lambda \in S_\sigma = \check{\sigma} \cap M$. Then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}\langle -\lambda \rangle$.*

Proof This is a direct consequence of $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}\langle -\lambda \rangle$, see Lemma 2.12. $\square$

2.2.2 Algebras

In the introduction, we have seen how the character lattice of an abstract torus is naturally contained in the coordinate ring of the torus, which is identified with the algebra of Laurent polynomials. Here we are going to invert the logic: starting from a lattice, we construct an associated abstract algebra of Laurent polynomials. This will enable us to associate an algebra to each rational cone.

Let M be a lattice of rank n, and define the algebra (called the group algebra of M over $\mathbb{C}$)

$$\mathbb{C}[M] := \left\{ \sum_{m \in M} c_m z^m \;\middle|\; c_m \in \mathbb{C}, \text{finitely many } c_m \text{ are non-zero} \right\}, \quad (2.10)$$

with the natural addition and multiplication defined by linear extension of $z^{m_1} z^{m_2} := z^{m_1+m_2}$. If M is identified with $\mathbb{Z}^n$, then $\mathbb{C}[M]$ is naturally identified with $\mathbb{C}[z^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, the algebra of Laurent polynomials. For this reason, we refer to elements of $\mathbb{C}[M]$ as Laurent polynomials in M.

Definition 2.24 *The **support** of a Laurent polynomial in M, $f = \sum_{m \in M} c_m z^m$, is defined as*

$$\operatorname{supp}(f) := \{ m \in M \mid c_m \neq 0 \}. \quad (2.11)$$

In other words, the support of f encodes all monomials that appear in f. The following proposition is a direct consequence of Gordan's Lemma 2.20.

Proposition 2.25 *Let σ be a lattice cone in $N_{\mathbb{R}}$. Let $M = \mathrm{Hom}_{\mathrm{gp}}(N, \mathbb{Z})$ be the dual lattice. The following*

$$R_\sigma := \mathbb{C}[S_\sigma] := \{ f \in \mathbb{C}[M] \mid \operatorname{supp}(f) \subseteq S_\sigma \} \quad (2.12)$$

is a finitely generated $\mathbb{C}$-algebra.

In what follows, we will use interchangeably the notation R_σ and $\mathbb{C}[S_\sigma]$, with the latter used when emphasising the role of the monoid.

By unpacking the definition of support, it follows that, given a set of generators $s_1, \ldots, s_k$ of S_σ, then the associated algebra is simply

$$\mathbb{C}[S_\sigma] = \mathbb{C}\left[z^{s_1}, \ldots, z^{s_k}\right] \subseteq \mathbb{C}[M]. \quad (2.13)$$

This fact explains the notation $\mathbb{C}[S_\sigma]$. In particular, given an isomorphism $M \cong \mathbb{Z}^n$ so that $s_i \cong (s_{i,1}, \ldots, s_{i,n}) \in \mathbb{Z}^n$, we then have

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}\left[z_1^{s_{1,1}} \cdots z_n^{s_{1,n}}, \ldots, z_1^{s_{k,1}} \cdots z_n^{s_{k,n}}\right] \subseteq \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]. \quad (2.14)$$

We will come back to this observation in the next section.

2.3 Affine toric varieties

So far, we have seen how to associate a finitely generated algebra R_σ to a lattice cone σ in $N_{\mathbb{R}}$. The next step is to associate an affine algebraic set to R_σ, thereby constructing an affine toric variety. Before doing so, let us recall some basic notions and facts from algebraic geometry.

2.3.1 All you need to know from algebraic geometry

A key principle in algebraic geometry is that polynomial equations define geometric objects. More precisely, let $\mathbb{C}[\xi_1, \ldots, \xi_k]$ be the polynomial ring in k variables.

Definition 2.26 *Let $E = \{ f_1, \ldots, f_r \} \subset \mathbb{C}[\xi_1, \ldots, \xi_k]$ be a finite collection of polynomials. The **affine algebraic set** defined by E is*

$$\mathsf{V}(E) := \left\{ x \in \mathbb{C}^k \;\middle|\; f_1(x) = \cdots = f_r(x) = 0 \right\} \subseteq \mathbb{C}^k. \tag{2.15}$$

*In other words, affine algebraic sets are precisely those subsets of $\mathbb{C}^k$ that arise as the vanishing locus of polynomials. An affine algebraic set is called an **affine algebraic variety** if it is irreducible, i.e. it is not the union of two proper affine algebraic sets.*

The correspondence between algebra and geometry is contravariant: if we enlarge the set of defining polynomials, the solution set becomes smaller. That is, if $E \subseteq F$, then $\mathsf{V}(E) \supseteq \mathsf{V}(F)$. Moreover, since the set of polynomials defining an algebraic set is invariant under multiplication, we can replace E with its two-sided ideal $\langle E \rangle$. This means that to any ideal of the polynomial ring, we can associate the affine algebraic set that is the vanishing locus of the polynomials in that ideal.

Just as polynomials define geometric objects, a geometric object determines a set of polynomials that vanish on it.

Definition 2.27 *Let $X \subseteq \mathbb{C}^k$. The **vanishing ideal** of X is the set of all polynomials that vanish on X:*

$$\mathsf{I}(X) := \{ f \in \mathbb{C}[\xi_1, \ldots, \xi_k] \mid f|_X = 0 \} . \tag{2.16}$$

This assignment reverses the previous one: whereas $\mathsf{V}(E)$ turns a set of polynomials into a geometric object, $\mathsf{I}(X)$ turns a geometric object into an algebraic structure. In particular, we have $X \subseteq Y$ if and only if $\mathsf{I}(X) \supseteq \mathsf{I}(Y)$, reflecting the dual nature of this assignment.

Example 2.28 *Consider a single point $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$. The set of polynomials vanishing at x is precisely the ideal*

$$\mathsf{I}(\{ x \}) = \langle \xi_1 - x_1, \ldots, \xi_k - x_k \rangle, \tag{2.17}$$

which is known as the maximal ideal[1] *corresponding to x. In what follows, it will be denoted as $\mathfrak{m}_x$.*

A fundamental result in algebraic geometry is Hilbert's Nullstellensatz (German for "zero-locus-theorem"), which establishes a deep link between points in affine space and maximal ideals of the polynomial ring.

Theorem 2.29 (Nullstellensatz) *Every maximal ideal in $\mathbb{C}[\xi_1, \ldots, \xi_k]$ is of the form $\mathfrak{m}_x$ for some $x \in \mathbb{C}^k$. In other words, we get a one-to-one correspondence between points $\mathbb{C}^k$ and maximal ideals in $\mathbb{C}[\xi_1, \ldots, \xi_k]$.*

This means that studying maximal ideals in the polynomial ring is equivalent to studying points in affine space. A direct consequence of Hilbert's Nullstellensatz is an expression for the affine algebraic set defined by an ideal in purely commutative algebra terms.

Corollary 2.30 *Let I be an ideal in $\mathbb{C}[\xi_1, \ldots, \xi_k]$. Then the affine algebraic set it defines is precisely*

$$\mathsf{V}(I) = \{\, x \in \mathbb{C}^k \mid I \subseteq \mathfrak{m}_x \,\}\,. \tag{2.18}$$

To systematically study functions on an algebraic set, we introduce the notion of a coordinate ring. It is the algebro-geometric equivalent to the ring of continuous functions to $\mathbb{R}$ on a topological space, or the ring of differentiable function to $\mathbb{R}$ on a smooth manifold.

Definition 2.31 *Given an ideal I in $\mathbb{C}[\xi_1, \ldots, \xi_k]$, let $V = \mathsf{V}(I)$. The* ***coordinate ring*** *of the affine algebraic set V is defined as the quotient*

$$R_V := \frac{\mathbb{C}[\xi_1, \ldots, \xi_k]}{I_V}, \tag{2.19}$$

where $I_V := \mathsf{I}(\mathsf{V}(I))$ is the vanishing ideal[2] *of $\mathsf{V}(I)$. The coordinate ring is a finitely generated $\mathbb{C}$-algebra generated by the equivalence classes $\xi_i + I_V$ of the coordinate functions.*

Example 2.32 *If $I = \{\, 0 \,\}$ in $\mathbb{C}[\xi_1, \ldots, \xi_k]$ is the trivial ideal, then $\mathsf{V}(I) = \mathbb{C}^k$, $R_V = \mathbb{C}[\xi_1, \ldots, \xi_k]$. In other words, the coordinate ring of the affine space is the whole polynomial algebra.*

[1] Recall that an ideal $\mathfrak{m}$ of ring R is called maximal if there are no other ideals contained between $\mathfrak{m}$ and R. Also, a proper ideal $\mathfrak{p}$ is called prime if the following property holds: if a and b are two elements of R such that their product ab is $\mathfrak{p}$, then a is in $\mathfrak{p}$ or b is in $\mathfrak{p}$.

[2] The ideal $\mathsf{I}(\mathsf{V}(I))$ is called the radical ideal of I. Intuitively, it is obtained by taking all roots of elements of I. It contains I, and in general it might be bigger.

As a second example, consider $I = \langle \xi_1^2 \rangle$ *in* $\mathbb{C}[\xi_1, \xi_2]$. *Then*

$$\mathsf{V}(I) = \{ (x_1, x_2) \in \mathbb{C}^2 \mid x_1 = 0 \} \tag{2.20}$$

is a coordinate line isomorphic to $\mathbb{C}$, *and* $R_V = \mathbb{C}[\xi_1, \xi_2]/\langle \xi_1 \rangle \cong \mathbb{C}[\xi_2]$. *Notice that in this case* $\mathsf{I}(\mathsf{V}(I)) = \langle \xi_1 \rangle$ *is strictly bigger than* I. *The example also motivates the name 'radical ideal'.*

For a third example, consider $I = \langle \xi_1 \xi_2 - 1 \rangle$ *in* $\mathbb{C}[\xi_1, \xi_2]$. *Then*

$$\mathsf{V}(I) = \{ (x_1, x_2) \in \mathbb{C}^2 \mid x_1 x_2 = 1 \} \tag{2.21}$$

is an algebraic torus isomorphic to $\mathbb{C}^*$ *and* $R_V = \mathbb{C}[\xi_1, \xi_2]/\langle \xi_1 \xi_2 - 1 \rangle \cong \mathbb{C}[\xi^{\pm 1}]$. *In other words, the coordinate ring of an algebraic torus is the algebra of Laurent polynomials, as anticipated in the introduction.*

We can now give a more general version of Hilbert's Nullstellensatz. The key idea is that every affine algebraic set V can be reconstructed from its coordinate ring R_V. The set of maximal ideals of R_V, known as the maximal spectrum, encodes this correspondence.

Theorem 2.33 *Let* V *be an affine algebraic set. There is a one-to-one correspondence between points in* V *and maximal ideals in its coordinate ring:*

$$V \overset{1:1}{\longleftrightarrow} \{ \mathfrak{m} \subseteq R_V \mid \mathfrak{m} \text{ is a maximal ideal} \} =: \operatorname{Specm}(R_V). \tag{2.22}$$

We call $\operatorname{Specm}(R_V)$ *the* ***maximal spectrum*** *of* R_V.

This correspondence is not just a set-theoretic bijection—it actually respects topological structures. By equipping both sides with the Zariski topology[3], one can prove that $V \cong \operatorname{Specm}(R_V)$ is a homeomorphism of topological spaces. More abstractly, there is an equivalence between the category of affine varieties and (the opposite of) that of finitely generated $\mathbb{C}$-algebras that are integral domains. The integrality corresponds to irreducibility.

Remark 2.34 *Every finitely generated* $\mathbb{C}$*-algebra* R *gives rise to an affine variety by writing* $R \cong \mathbb{C}[\xi_1, \ldots, \xi_k]/I$. *This identification allows us to recover the geometry of an affine algebraic set purely from its algebraic structure, establishing a fundamental bridge between algebra and geometry.*

To conclude, we collect below some definitions and facts that will be useful throughout the course.

[3]The Zariski topology is define on $\mathbb{C}^k$ by declaring all sets of the form $\mathsf{V}(I)$ for some ideal I to be closed. An analogous definition holds for the spectrum of a ring.

- **Irreducibility.** Let V be an affine algebraic set. The following are equivalent:
  - The affine algebraic set V is irreducible.
  - The vanishing ideal $\mathsf{I}(V)$ is a prime ideal.
  - The coordinate ring R_V is an integral domain.
- **Dimension.** Let R be a ring. We say that a chain of prime ideals of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ has length n. We define the dimension of a R to be the supremum of the lengths of all chains of prime ideals in R. The dimension of an affine algebraic variety $V = \mathrm{Specm}(R)$ is defined as the dimension of R. If $U \subseteq V$ is an open dense subset of dimension n, then V has dimension n as well.

2.3.2 Affine toric varieties: definition and examples

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Talks by Yu-Yuan and Elisa M.

Recall that for a lattice cone σ, we defined the finitely generated monoid and the finitely generated $\mathbb{C}$-algebra as

$$S_\sigma = \check{\sigma} \cap M \qquad \text{and} \qquad R_\sigma = \{\, f \in \mathbb{C}[M] \mid \mathrm{supp}(f) \subseteq S_\sigma \,\}, \tag{2.23}$$

respectively. We now define the affine variety associated with σ by taking its maximal spectrum, completing the following assignment.

$$\begin{array}{ccccccccc} \sigma & \longrightarrow & \check{\sigma} & \longrightarrow & S_\sigma & \longrightarrow & R_\sigma & \longrightarrow & X_\sigma \\ \text{cone} & & \text{dual cone} & & \text{monoid} & & \text{algebra} & & \text{variety} \end{array}$$

For reasons that will become clear shortly, we restrict our attention to strictly convex cones.

Definition 2.35 *A cone σ in V is said to be **strictly convex** if it does not contain a full line through the origin. That is, there exists no $v \in V \setminus \{\, 0 \,\}$ such that $\mathbb{R}\langle v \rangle \subseteq \sigma$.*

Notice that $\{\, 0 \,\}$ is always a face of a strictly convex cone. A useful characterisation of strict convexity is expressed in terms of duals. The proof is left as an exercise.

Lemma 2.36 *A cone $\sigma \subseteq V$ is strictly convex if and only if $\check{\sigma} \subseteq V^*$ has full dimension.*

This property of the dual cone will be useful later. We are now ready to define the affine toric variety associated with σ.

Definition 2.37 *Let σ be a strictly convex lattice cone in $N_{\mathbb{R}}$. The associated affine variety is defined as*

$$X_\sigma := \operatorname{Specm}(R_\sigma). \tag{2.24}$$

Since the algebra R_σ is finitely generated, X_σ is indeed an affine algebraic set. To justify calling it a 'variety', we should check its irreducibility (cf. Definition 2.26).

Lemma 2.38 *Let σ be a strictly convex lattice cone in $N_{\mathbb{R}}$. The affine algebraic set X_σ is irreducible. Thus, X_σ is an affine algebraic variety.*

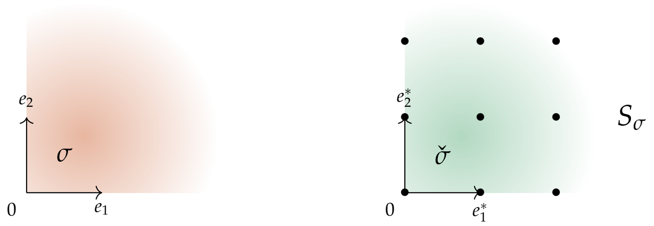
Proof It suffices to check that R_σ is an integral domain. This follows from the simplification law for multiplication in R_σ, which in turn follows from the simplification law in the monoid S_σ. □

Let us analyse some examples. We recall the main picture, starting from a lattice cone σ in $N_{\mathbb{R}}$, let $M = \operatorname{Hom}_{\mathrm{gp}}(N, \mathbb{Z})$ be the dual lattice. The lattice M naturally sits inside the algebra of Laurent polynomials in M as monic monomials:

$$M \longrightarrow \mathbb{C}[M], \qquad m \longmapsto z^m. \tag{2.25}$$

By this assignment, a system of generators for S_σ corresponds to a system of $\mathbb{C}$-algebra generators for R_σ. Let us see this in practice with two examples.

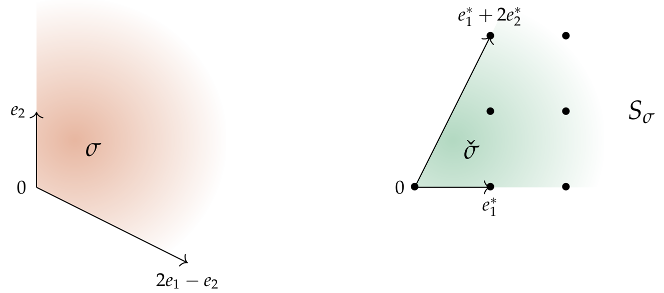
Example 2.39 *Let $\sigma = \mathrm{Cone}(e_1, e_2)$ in $\mathbb{R}^2$ with standard lattice.*



We see that S_σ is generated by $e_1^* \cong (1,0)$ and $e_2^* \cong (0,1)$. The identifications follow from the natural identification $\mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \cong \mathbb{Z}^2$. Therefore, $R_\sigma = \mathbb{C}[z_1, z_2]$ and so $X_\sigma \cong \mathbb{C}^2$.

The example can be generalised by taking the cone σ generated by any generators of a lattice N. The resulting variety is simply $\mathbb{C}^n$.

Example 2.40 *Consider the cone* $\sigma = \mathrm{Cone}(e_2, 2e_1 - e_2)$ *in* $\mathbb{R}^2$ *with standard lattice* $\mathbb{Z}^2$*, cf. Example 2.2.*



A generating set of S_σ *is* s_1, s_2, s_3 *with*

$$s_1 = e_1^* \cong (1,0), \qquad s_2 = e_1^* + e_2^* \cong (1,1), \qquad s_3 = e_1^* + 2e_2^* \cong (1,2). \tag{2.26}$$

Again, the identifications follow from the natural identification $\mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \cong \mathbb{Z}^2$*. Thus, the monic monomials associated with the generators are* z_1*,* z_1z_2*, and* $z_1z_2^2$ *respectively, so that*

$$R_\sigma = \mathbb{C}\left[z_1, z_1z_2, z_1z_2^2\right] \cong \frac{\mathbb{C}[\xi_1, \xi_2, \xi_3]}{\langle \xi_2^2 - \xi_1\xi_3 \rangle}. \tag{2.27}$$

The last equality follows from setting $\xi_1 = z_1$*,* $\xi_2 = z_1z_2$*,* $\xi_3 = z_1z_2^2$*, which must satisfy the relation* $\xi_2^2 - \xi_1\xi_3$*. Hence, the quotient by the ideal* $I_\sigma = \langle \xi_2^2 - \xi_1\xi_3 \rangle$*. As a consequence, we deduce that*

$$X_\sigma \cong \mathsf{V}(I_\sigma) = \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 \;\middle|\; x_2^2 - x_1x_3 = 0 \right\} \subset \mathbb{C}^3. \tag{2.28}$$

In other words, the variety X_σ *is a double cone. See Figure 2.1 for an illustration of its real points.*

The key takeaway from the above example is that we can describe the variety X_σ by examining the generators of the monoid S_σ and the relations they satisfy. This also motivates the notation $\mathbb{C}[S_\sigma]$ in place of R_σ. However, there is some flexibility in the choice of generators, as illustrated in the following example.

Example 2.41 *Consider the cone* $\sigma = \{0\}$ *in* $\mathbb{R}^n$ *with standard lattice* $\mathbb{Z}^n$*. We see that* $\check{\sigma}$ *is* $(\mathbb{R}^n)^*$*, so* $S_\sigma = \mathbb{Z}^n$*. There are several possible choices of generators for* $\mathbb{Z}^n$*. Let us analyse two natural ones.*

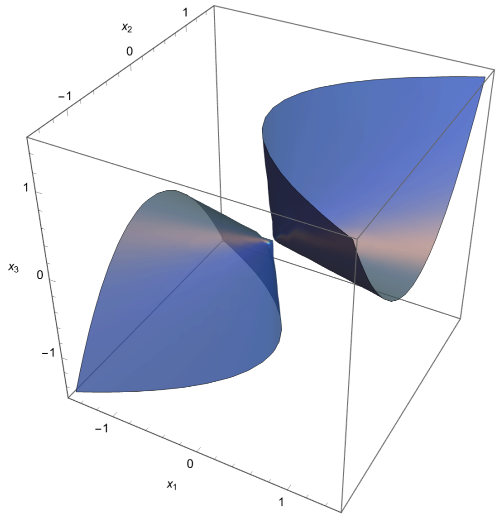


Figure 2.1: The real points of the double cone $x_2^2 = x_1x_3$.

One possibility is to pick $(e_1^*, \ldots, e_n^*, -e_1^*, \ldots, -e_n^*)$. With this choice, we obtain

$$R_\sigma = \mathbb{C}\left[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}\right] \cong \frac{\mathbb{C}[\xi_1, \ldots, \xi_{2n}]}{\langle \xi_1\xi_{n+1} - 1, \ldots, \xi_n\xi_{2n} - 1\rangle}. \tag{2.29}$$

In other words, we identify the algebra R_σ with the quotient of the polynomial algebra in $2n$ variables by the ideal $I_\sigma = \langle \xi_1\xi_{n+1} - 1, \ldots, \xi_n\xi_{2n} - 1\rangle$. Hence, one can identify X_σ with

$$X_\sigma \cong \mathsf{V}(I_\sigma) = \{\, (x_1, \ldots, x_{2n}) \mid x_1x_{n+1} = \cdots = x_nx_{2n} = 1 \,\} \subset \mathbb{C}^{2n}. \tag{2.30}$$

On the other hand, one can also choose $(e_1^*, \ldots, e_n^*, -e_1^* - e_2^* - \cdots - e_n^*)$ as generators of $\mathbb{Z}^n$. With this choice, we obtain the identification

$$R_\sigma = \mathbb{C}\left[z_1, \ldots, z_n, (z_1 \cdots z_n)^{-1}\right] \cong \frac{\mathbb{C}[\eta_1, \ldots, \eta_n, \eta_{n+1}]}{\langle \eta_1 \cdots \eta_{n+1} - 1\rangle}, \tag{2.31}$$

where the corresponding ideal is $J_\sigma = \langle \eta_1 \cdots \eta_{n+1} - 1\rangle$. In this case, we obtain a different representation of X_σ:

$$X_\sigma \cong \mathsf{V}(J_\sigma) = \{\, (y_1, \ldots, y_{n+1}) \mid y_1 \cdots y_{n+1} = 1 \,\} \subset \mathbb{C}^{n+1}. \tag{2.32}$$

In both cases, the resulting affine variety is nothing but an n-dimensional algebraic torus $(\mathbb{C}^*)^n$, as one can check from the isomorphisms

$$\begin{aligned} (\mathbb{C}^*)^n &\longrightarrow \mathsf{V}(I_\sigma), & (t_1, \ldots, t_n) &\longmapsto (t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}), \\ (\mathbb{C}^*)^n &\longrightarrow \mathsf{V}(J_\sigma), & (t_1, \ldots, t_n) &\longmapsto (t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}). \end{aligned} \tag{2.33}$$

The above example highlights how we may have multiple ways to represent R_σ, namely:

$$R_\sigma \cong \frac{\mathbb{C}[\xi_1, \ldots, \xi_k]}{I_\sigma} \cong \frac{\mathbb{C}[\eta_1, \ldots, \eta_\ell]}{J_\sigma}. \tag{2.34}$$

Although $\mathsf{V}(I_\sigma)$ may not be equal to $\mathsf{V}(J_\sigma)$, they are homeomorphic to the same variety: the spectrum X_σ of the algebra R_σ.

We now aim to understand the structure of the ideal associated with cones. Observe that, in general, if we denote a generating set of the monoid S_σ by $s_1, \ldots, s_k$, then any relation among these generators can be written as $\sum_{i=1}^k c_i s_i = 0$ for some $c_i \in \mathbb{Z}$. By separating the positive and negative scalars c_i, we can rewrite the above relation as

$$\sum_{i=1}^{k} a_i s_i - \sum_{i=1}^{k} b_i s_i = 0 \tag{2.35}$$

for some $a_i, b_i \in \mathbb{Z}_{\geq 0}$. As a consequence, the algebra associated with the cone σ is given by

$$R_\sigma = \mathbb{C}\left[z^{m_1}, \ldots, z^{m_k}\right] \cong \frac{\mathbb{C}[\xi_1, \ldots, \xi_k]}{I_\sigma}, \tag{2.36}$$

where I_σ is generated by relations of the form

$$\prod_{i=1}^{k} \xi_i^{a_i} - \prod_{i=1}^{k} \xi_i^{b_i} = 0 \tag{2.37}$$

with $a_i, b_i \in \mathbb{Z}_{\geq 0}$ corresponding to the linear relations in Equation (2.35).

Definition 2.42 *Relations of the form*

$$\prod_{i=1}^{k} \xi_i^{a_i} - \prod_{i=1}^{k} \xi_i^{b_i} = 0 \tag{2.38}$$

*with $a_i, b_i \in \mathbb{Z}_{\geq 0}$ are called **binomial**[4] **relations**. A prime ideal of the polynomial algebra $\mathbb{C}[\xi_1, \ldots, \xi_k]$ generated by a set of binomial relations is called a **toric ideal**.*

It can be shown that the ideal associated with a cone σ (for any choice of generators of the monoid S_σ) is indeed a toric ideal.

We are now ready to state the main result of this section: the affine algebraic variety associated with a strictly convex lattice cone is a toric variety. That is, it contains a dense open torus with an action that extends the torus action on itself (cf. Definition 1.2).

[4]From the Latin *bi-* and *nomen*, which means "of two terms".

Theorem 2.43 (Affine toric varieties from cones) *Let σ be a strictly convex lattice cone in $N_{\mathbb{R}}$. Then, the affine variety X_σ contains a torus $T \cong (\mathbb{C}^*)^n$ as a Zariski open dense subset, such that the action of T on itself extends to an action on all of X_σ. In particular, X_σ has dimension n.*

Proof Fix a minimal set of generators $m_1, \ldots, m_n$ of the dual lattice M, which provide an identification $M \cong \mathbb{Z}^n$. Let $s_1, \ldots, s_k$ be a system of generators of S_σ which, under the identification of the lattice, can be written as

$$s_i = \sum_{j=1}^{n} s_{i,j} m_j \cong (s_{i,1}, \ldots, s_{i,n}) \in \mathbb{Z}^n. \tag{2.39}$$

Let $X_\sigma \cong \mathsf{V}(I_\sigma) \subseteq \mathbb{C}^k$ be a representation in $\mathbb{C}^k$, obtained through the usual process of generating binomial relations from $s_1, \ldots, s_k$.

Consider the function from the n-dimensional algebraic torus to the affine variety $\mathsf{V}(I_\sigma)$ defined by

$$\iota\colon (\mathbb{C}^*)^n \longrightarrow \mathsf{V}(I_\sigma), \qquad t = (t_1, \ldots, t_n) \longmapsto (t^{s_1}, \ldots, t^{s_k}), \tag{2.40}$$

where $t^{s_i} := t_1^{s_{i,1}} \cdots t_n^{s_{i,n}} \in \mathbb{C}^*$. We claim that ι is an embedding, i.e., a bijection onto its image.

First, we check that ι is well-defined, meaning that its image lies in $\mathsf{V}(I_\sigma)$. Suppose that $\sum_{i=1}^k a_i s_i = \sum_{i=1}^k b_i s_i$ is a relation among the generators. Then:

$$\begin{aligned} \iota_1(t)^{a_1} \cdots \iota_k(t)^{a_k} &= (t_1^{s_{1,1}} \cdots t_n^{s_{1,n}})^{a_1} \cdots (t_1^{s_{k,1}} \cdots t_n^{s_{k,n}})^{a_k} \\ &= t_1^{\sum_i a_i s_{i,1}} \cdots t_n^{\sum_i a_i s_{i,n}} \\ &= t_1^{\sum_i b_i s_{i,1}} \cdots t_n^{\sum_i b_i s_{i,n}} \\ &= \cdots = \iota_1(t)^{b_1} \cdots \iota_k(t)^{b_k}. \end{aligned} \tag{2.41}$$

This proves that $\iota(t)$ satisfies the binomial relations, meaning that $\iota(t) \in \mathsf{V}(I_\sigma)$.

Next, we show injectivity and surjectivity onto the image. To do so, we define auxiliary functions from $\mathsf{V}(I_\sigma)$ to $\mathbb{C}$. Let $s \in S_\sigma$ be such that all points $s + m_l$ also belong to S_σ for all $l = 1, \ldots, n$, where m_l are the fixed generators of M. The existence of such an element follows from the strict convexity of σ, which is equivalent to the full-dimensionality of $\check{\sigma}$ (see Remark 2.44 for a counter-example). Since $s \in S_\sigma$ and $s + m_l \in S_\sigma$, we can write them in terms of the monoid generators:

$$s = \sum_{i=1}^{k} c_{0,i} s_i, \qquad s + m_l = \sum_{i=1}^{k} c_{l,i} s_i, \tag{2.42}$$

for some $c_{l,i} \in \mathbb{Z}_{\geq 0}$. These coefficients satisfy the relations

$$\sum_{i=1}^{k} s_{i,j}(c_{l,i} - c_{0,i}) = \delta_{j,l} \tag{2.43}$$

for all $j, l = 1, \dots, n$. Using these coefficients, we define the auxiliary functions

$$f_l \colon \mathsf{V}(I_\sigma) \longrightarrow \mathbb{C}, \qquad x = (x_1, \dots, x_k) \longmapsto x_1^{c_{l,1}} \cdots x_k^{c_{l,k}} \tag{2.44}$$

for $l = 0, 1, \dots, n$.

Using the relations (2.43), a direct computation verifies that

$$\frac{f_j(\iota(t))}{f_0(\iota(t))} = t_j. \tag{2.45}$$

This proves injectivity since $(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0})$ serves as a left inverse of ι.

Next, we establish surjectivity onto its image. Clearly, the image of ι is contained in $T := \mathsf{V}(I_\sigma) \cap (\mathbb{C}^*)^k$. Let us show that any point in $x \in T$ has a preimage. Set

$$t = \left(\frac{f_1(x)}{f_0(x)}, \dots, \frac{f_n(x)}{f_0(x)} \right). \tag{2.46}$$

It is straightforward to check that $t \in (\mathbb{C}^*)^n$ and that $\iota(t) = x$. This proves the claim, with T being the image of ι. In particular, $T \cong (\mathbb{C}^*)^n$ is an n-dimensional algebraic torus in $\mathsf{V}(I_\sigma)$.

Let us now prove that T is open and dense in $\mathsf{V}(I_\sigma)$. For openness: since $\mathsf{V}(I_\sigma) \setminus (\mathbb{C}^*)^k$ is closed in the Zariski topology (recall that $(\mathbb{C}^*)^k$ is the complement of an affine variety in $\mathbb{C}^k$, hence it is open), it follows that $\mathsf{V}(I_\sigma) \cap (\mathbb{C}^*)^k$ is open in $\mathsf{V}(I_\sigma)$. For density: since $(\mathbb{C}^*)^k$ is dense in $\mathbb{C}^k$, it follows that $\mathsf{V}(I_\sigma) \cap (\mathbb{C}^*)^k$ is dense in $\mathsf{V}(I_\sigma)$. This also confirms that the affine variety $\mathsf{V}(I_\sigma)$ has dimension n, as it contains an n-dimensional torus as an open dense subset.

Finally, to define an action of the torus on $\mathsf{V}(I_\sigma)$ that extends the action of the torus on itself, we set

$$(\mathbb{C}^*)^n \times \mathsf{V}(I_\sigma) \longrightarrow \mathsf{V}(I_\sigma), \qquad t \cdot x := (t^{s_1} x_1, \dots, t^{s_k} x_k). \tag{2.47}$$

Here T is identified with $(\mathbb{C}^*)^n$ via ι. It is straightforward to check that this action is well-defined and extends the torus action on itself.

According to Definition 1.2, one should check that the variety X_σ is normal. We will postpone this check to Section 3.2 (where we will also discuss the definition of normality). $\square$

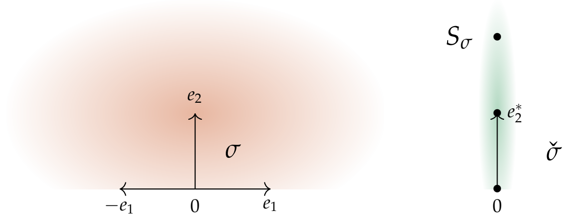
More abstractly, the torus T in X_σ is nothing but

$$T = \mathrm{Specm}(\mathbb{C}[M]) \subseteq X_\sigma, \tag{2.48}$$

with the inclusion that follows from the inclusion $R_\sigma \subseteq \mathbb{C}[M]$ as $\mathbb{C}$-algebras. The definition of the torus in the above proof is nothing but the inclusion $T \subseteq X_\sigma$ in coordinates, after a (non-canonical) choice of isomorphism $M \cong \mathbb{Z}^n$ and generators $s_1, \dots, s_k$ for the cone σ.

Remark 2.44 *The assumption of strict convexity played a crucial role in the proof. Specifically, we relied on the existence of an element $s \in S_\sigma$ such that $s + m_l$ still belong to S_σ for all generators m_l of the lattice M. This property follows from the characterisation of strict convexity in terms of the full-dimensionality of the dual.*

To see why this condition is necessary, consider the cone $\sigma = \mathrm{Cone}(e_1, -e_1, e_2)$ in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. In this case, there is no element $s \in S_\sigma$ satisfying the required condition, demonstrating that strict convexity is essential.



We conclude with three examples.

Example 2.45 *Consider Examples 2.39 and 2.40, namely the two cones $\sigma_1 = \mathrm{Cone}(e_1, e_2)$ and $\sigma_2 = \mathrm{Cone}(e_2, 2e_1 - e_2)$ in $\mathbb{R}^2$ with standard lattice. We know that the associated varieties are*

$$X_{\sigma_1} \cong \mathbb{C}^2, \qquad X_{\sigma_2} \cong \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 \;\middle|\; x_2^2 - x_1 x_3 = 0 \right\} \subset \mathbb{C}^3, \tag{2.49}$$

namely the complex plane and the double cone. Since both cones are in $\mathbb{R}^2$, we should be able to find an open dense 2-dimensional torus in both X_{σ_1} and X_{σ_1}. The first case is quite simple:

$$\iota_1 \colon (\mathbb{C}^*)^2 \hookrightarrow \mathbb{C}^2, \qquad (t_1, t_2) \longmapsto (t_1, t_2). \tag{2.50}$$

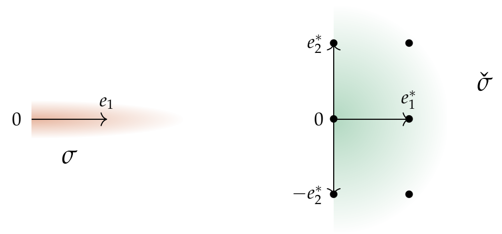
As for the double cone, recall our choice of generators for the monoid: $s_1 = e_1^*$, $s_2 = e_1^* + e_2^*$, $s_3 = e_1^* + 2e_2^*$. Thus, the construction from Theorem 2.43, and more precisely Equation (2.40)*, tells us that the embedding of the torus is given by*

$$\begin{aligned} \iota_2 \colon (\mathbb{C}^*)^2 &\hookrightarrow \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 \;\middle|\; x_2^2 - x_1 x_3 = 0 \right\}, \\ (t_1, t_2) &\longmapsto (t_1, t_1 t_2, t_1 t_2^2). \end{aligned} \tag{2.51}$$

The real points of the double cone together with the torus embedding can be visualised interactively at this link.

Another consequence of Theorem 2.43 is that the dimension of X_σ is always equal to the dimension of the ambient space of the cone σ, regardless of the dimension of σ itself. The next example illustrates how the dimension of the cone influences the construction of the variety.

Example 2.46 *Let* $\sigma = \mathrm{Cone}(e_1)$ *in* $\mathbb{R}^2$ *with standard lattice* $\mathbb{Z}^2$.



We see that $S_\sigma = \mathbb{Z}_{\geq 0}\langle e_1^*, e_2^*, -e_2^*\rangle$. *Thus, the algebra and variety associated with the cone are given by* $R_\sigma = \mathbb{C}[z_1, z_2, z_2^{-1}] \cong \mathbb{C}[\xi_1, \xi_2, \xi_3]/\langle \xi_2\xi_3 - 1\rangle$ *and* $X_\sigma \cong \mathbb{C} \times \mathbb{C}^*$ *with* $\dim X_\sigma = 2$ *even though* $\dim \sigma = 1$.

The example can be generalised in higher dimension as follows. Let σ *be the cone in* $\mathbb{R}^n$ *with standard lattice* $\mathbb{Z}^n$ *generated by* $e_1, \ldots, e_r$ *for* $r \leq n$. *Then* S_σ *is generated by* $e_1^*, \ldots, e_r^*, \pm e_{r+1}^*, \ldots, \pm e_n^*$. *Now, we have*

$$\begin{aligned} R_\sigma &= \mathbb{C}\left[z_1, \ldots, z_r, z_{r+1}^{\pm 1}, \ldots, z_n^{\pm 1}\right] \\ &\cong \frac{\mathbb{C}[\xi_1, \ldots, \xi_r, \eta_1, \cdots \eta_{2(n-r)}]}{\langle \eta_1\eta_{n-r+1} - 1, \ldots, \eta_{n-r}\eta_{2(n-r)}\rangle}, \end{aligned} \tag{2.52}$$

and so $X_\sigma = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$.

More generally, one can prove that if σ is a r-dimensional, strictly convex lattice cone in $N_\mathbb{R}$ with N of rank n, then

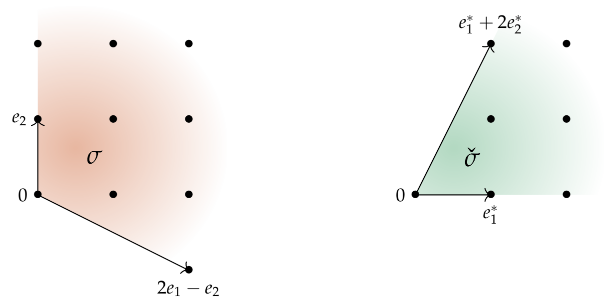
$$X_\sigma \cong X_{\sigma'} \times (\mathbb{C}^*)^{n-r}, \tag{2.53}$$

where σ' is the cone σ considered within the r-dimensional vector space it spans. In particular, $X_{\sigma'}$ is r-dimensional. In plain English:

The codimension of σ *determines*
the number of factors of $\mathbb{C}^*$ *appearing in* X_σ.

To conclude, the next example highlights the crucial role of the lattice. Specifically, the same cone, when viewed inside different lattices, can yield different affine varieties. This is expected, as the monoid S_σ depends on the choice of the lattice in which σ resides.

Example 2.47 *Let $\sigma = \mathrm{Cone}(e_2, 2e_1 - e_2)$ in $\mathbb{R}^2$ with the standard lattice $N := \mathbb{Z}^2$ (cf. Example 2.2). This gives $M = \mathrm{Hom}_{\mathrm{gp}}(N, \mathbb{Z}) = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*$, which is the standard lattice in the dual space.*

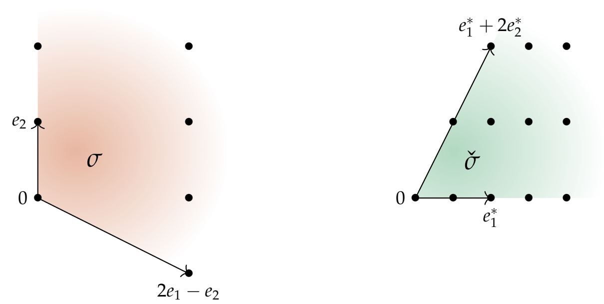


We have that $S_\sigma = \mathbb{Z}_{\geq 0}\langle e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^* \rangle$, hence

$$R_\sigma = \mathbb{C}\left[z_1, z_1 z_2, z_1 z_2^2\right] \cong \frac{\mathbb{C}[\xi_1, \xi_2, \xi_3]}{\langle \xi_2^2 - \xi_1 \xi_3 \rangle}. \tag{2.54}$$

Therefore, $X_\sigma \cong \mathsf{V}(\langle \xi_2^2 - \xi_1\xi_3 \rangle)$ is a double cone (cf. Example 2.40).

However, consider a different lattice $N' := \mathbb{Z}(2e_1) \oplus \mathbb{Z}e_2$. This gives $M' = \mathrm{Hom}_{\mathrm{gp}}(N', \mathbb{Z}) = \mathbb{Z}(\frac{1}{2}e_1^*) \oplus \mathbb{Z}e_2^*$ as dual lattice. Let us denote the same cone σ by σ' in this case.



We have that $S_{\sigma'}$ is generated by only two elements of the lattice M', namely $\frac{1}{2}e_1^*$ and $\frac{1}{2}e_1^* + e_2^*$. Thus, $R_{\sigma'} = \mathbb{C}[z_1, z_1z_2] \cong \mathbb{C}[\eta_1, \eta_2]$. Therefore, $X_{\sigma'} \cong \mathbb{C}^2$ is simply the complex plane.

This means that, even if σ and σ' coincide as cones, X_σ might differ from $X_{\sigma'}$ due to the different lattice structure!

In plain English:

The lattice matters in the construction of the variety associated with a cone.

The example above also shows that the schematic assignment $\sigma \to \cdots \to X_\sigma$ is not, strictly speaking, complete. We can fix it by including the datum of the lattice, which is now going to make the variety X_σ well-defined.

$$
\begin{array}{ccccc}
(\sigma, N) \longrightarrow & (\check{\sigma}, M) \longrightarrow & S_\sigma \longrightarrow & R_\sigma \longrightarrow & X_\sigma \\
\text{cone, lattice} & \text{dual cone, dual lattice} & \text{monoid} & \text{algebra} & \text{variety}
\end{array}
$$

2.4 Toric varieties

Written by Ilan
Talks by Zheming and Marco

In last lecture, we saw in Theorem 2.43 how to construct an affine toric variety X_σ from a strictly convex lattice cone σ. More precisely, the affine variety X_σ contains a torus as an open dense subset, such that the action of the torus extends to the whole variety. However, we know that more general varieties can be constructed from affine ones by glueing. This is the goal of today's lecture. In particular, we will introduce what we call fans, that is a collection of cones that satisfies certain conditions, allowing us to "glue" the corresponding affine varieties in such a way that preserves the torus action.

2.4.1 Fans

Let us start off with the formal definition of a fan.

Definition 2.48 *A **fan** Δ in $N_\mathbb{R}$ is a finite collection of strongly convex lattice cones such that:*

1. *Every face of a cone in Δ is a cone in Δ.*
2. *If σ, σ' are cones in Δ, then $\sigma \cap \sigma'$ is a common face of σ and σ'.*

Recall that if τ is a face of σ, we have the following inclusions throughout our construction.

$$\begin{aligned} \tau &\hookrightarrow \sigma \\ \check{\tau} &\hookleftarrow \check{\sigma} \\ S_\tau &\hookleftarrow S_\sigma \\ R_\tau &\hookleftarrow R_\sigma \\ X_\tau &\hookrightarrow X_\sigma \end{aligned}$$

Therefore, a face of a cone will result in an affine subvariety. If this face is endowed with inclusions into several cones, we will obtain ones into several varieties. This is the reason behind both properties of Definition 2.48: we would like to keep track of inclusions of faces. Therefore, the idea here is to “glue” affine varieties “along” those obtained from the faces of their corresponding cones. Let us now exemplify this.

Example 2.49 *Consider in $\mathbb{R}$ with standard lattice $\mathbb{Z}$ the cones $\sigma_0 = \operatorname{Cone}(e_1)$, $\sigma_1 = \operatorname{Cone}(-e_1)$, and their intersection $\tau := \sigma_0 \cap \sigma_1$. They can be represented, together with their duals, as follows.*

Then $\Delta = \{ \sigma_0, \sigma_1, \tau \}$ forms a fan. We get the corresponding monoids:

$$S_{\sigma_0} = \mathbb{Z}_{\geq 0}\langle e_1^* \rangle, \quad S_{\sigma_1} = \mathbb{Z}_{\geq 0}\langle -e_1^* \rangle, \quad S_\tau = \mathbb{Z}_{\geq 0}\langle e_1^*, -e_1^* \rangle, \tag{2.55}$$

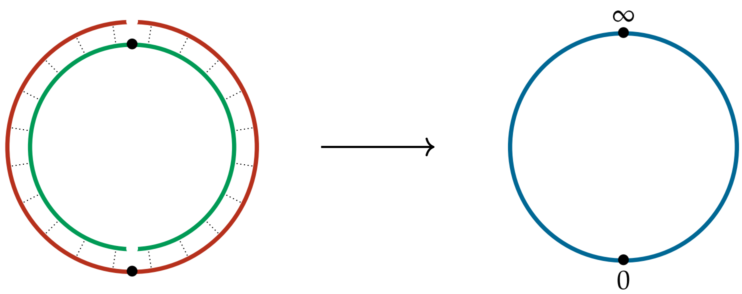
and the corresponding varieties:

$$X_{\sigma_0} = \mathbb{C}_{(z)}, \qquad X_{\sigma_1} = \mathbb{C}_{(z^{-1})}, \qquad X_\tau = \mathbb{C}^*_{(z)}, \tag{2.56}$$

where the indices indicate the coordinate used. We also have the inclusions

$$\begin{aligned} X_\tau &\hookrightarrow X_{\sigma_0} \\ z &\longmapsto z \end{aligned}, \qquad \begin{aligned} X_\tau &\hookrightarrow X_{\sigma_1} \\ z &\longmapsto z^{-1} \end{aligned}. \tag{2.57}$$

Topologically, this is the same as taking $\mathbb{C}^*$ and adding to it a point at zero and a point at infinity. This turns out to be exactly the construction of $\mathbb{P}^1$, visually represented as follows.



We can see here both lines given by $\mathbb{C}_{(z)}$ (in red) and $\mathbb{C}_{(z^{-1})}$ (in green) that are being identified whenever the points are non-zero, yielding $\mathbb{P}^1$. More formally, if we denote by $[x_0, x_1]$ the homogeneous coordinates on the projective line, we have the following coordinate charts that make a covering of it:

$$U_0 := \left\{ [x_0, x_1] \in \mathbb{P}^1 \;\middle|\; x_0 \neq 0 \right\}, \quad U_1 := \left\{ [x_0, x_1] \in \mathbb{P}^1 \;\middle|\; x_1 \neq 0 \right\}. \tag{2.58}$$

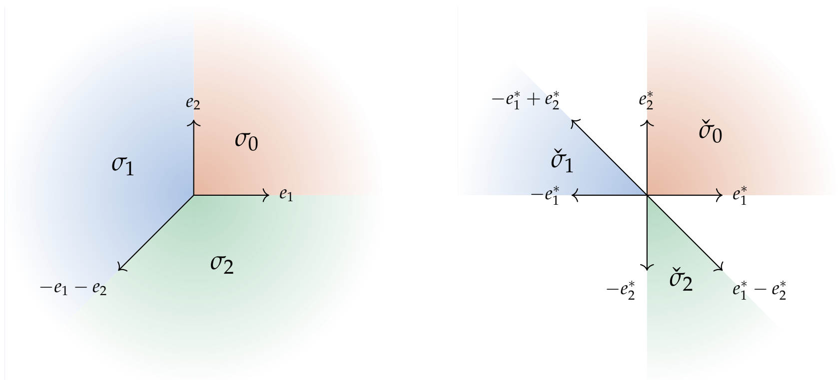
So U_0 is isomorphic to X_{σ_0} (via $[x_0, x_1] \mapsto \frac{x_1}{x_0} =: z$) and U_1 is isomorphic to X_{σ_1} (via $[x_0, x_1] \mapsto \frac{x_0}{x_1} = z^{-1}$). Moreover $U_0 \cap U_1$ is isomorphic to X_τ (via both maps). Finally, $\mathbb{P}^1$ is obtained by glueing elements in $U_0 \cap U_1$ from U_0 and U_1 when seen in $U_0 \sqcup U_1$.

We may do the same with $\mathbb{P}^2$, using a slightly more elaborate fan.

Example 2.50 *Consider the fan in $\mathbb{R}^2$ with standard lattice defined by*

$$\begin{aligned} \sigma_0 &= \mathrm{Cone}(e_1, e_2), & \tau_0 &= \mathrm{Cone}(-e_1 - e_2) = \sigma_1 \cap \sigma_2, \\ \sigma_1 &= \mathrm{Cone}(e_2, -e_1 - e_2), & \tau_1 &= \mathrm{Cone}(e_1) = \sigma_1 \cap \sigma_2, \\ \sigma_2 &= \mathrm{Cone}(e_1, -e_1 - e_2), & \tau_2 &= \mathrm{Cone}(e_2) = \sigma_0 \cap \sigma_1, \end{aligned} \tag{2.59}$$

and their triple intersection $v = \{ 0 \} = \sigma_1 \cap \sigma_2 \cap \sigma_3$. The fan and the duals of its top-dimensional cones can be visualised as follows.



We then have:

$$S_{\sigma_0} = \mathbb{Z}_{\geq 0}\langle e_1^*, e_2^* \rangle, \quad S_{\sigma_1} = \mathbb{Z}_{\geq 0}\langle -e_1^*, -e_1^* + e_2^* \rangle, \quad S_{\sigma_2} = \mathbb{Z}_{\geq 0}\langle -e_2^*, e_1^* - e_2^* \rangle. \tag{2.60}$$

Thus, the associated algebras and varieties

$$\begin{aligned} R_{\sigma_0} &= \mathbb{C}[z_1, z_2], & R_{\sigma_1} &= \mathbb{C}[z_1^{-1}, z_1^{-1}z_2], & R_{\sigma_2} &= \mathbb{C}[z_2^{-1}, z_1 z_2^{-1}], \\ X_{\sigma_0} &= \mathbb{C}^2_{(z_1,z_2)}, & X_{\sigma_1} &= \mathbb{C}^2_{(z_1^{-1},z_1^{-1}z_2)}, & X_{\sigma_2} &= \mathbb{C}^2_{(z_2^{-1},z_1 z_2^{-1})}. \end{aligned} \tag{2.61}$$

The intersection $\tau_2 = \sigma_0 \cap \sigma_1$ gives

$$S_{\tau_2} = \mathbb{Z}_{\geq 0}\langle e_1^*, -e_1^*, e_2^* \rangle, \quad R_{\tau_2} = \mathbb{C}[z_1, z_1^{-1}, z_2], \quad X_{\tau_2} = \mathbb{C}^*_{(z_1)} \times \mathbb{C}_{(z_2)}, \tag{2.62}$$

yielding the maps and the glueing (also known as push-out).

$$\begin{tikzcd} X_{\tau_2} \arrow[r, "(z_1,z_2)\mapsto(z_1,z_2)"] \arrow[d, "(z_1,z_2)\mapsto(z_1^{-1},z_1^{-1}z_2)"'] & X_{\sigma_0} \arrow[d] \\ X_{\sigma_1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{P}^2 \setminus \{ [0:0:1] \} \end{tikzcd} \tag{2.63}$$

The glueing of the other 2-dimensional cones along their 1-dimensional faces gives in a similar way $\mathbb{P}^2$ minus a point. Furthermore, taking into account the glueing along the triple intersection $v = \sigma_0 \cap \sigma_1 \cap \sigma_2$ yields the projective plane $\mathbb{P}^2$.

We would like to formalise the above glueing process. To this end, let us first understand the relationship between the affine variety of a cone and that of one of its faces.

Lemma 2.51 *Let σ be a cone in $N_{\mathbb{R}}$ and τ one of its faces. Then*

$$X_\tau = X_\sigma \setminus \mathrm{V}(f). \tag{2.64}$$

where $f \in R_\sigma$ is determined by τ.

Proof Let $\tau = \sigma \cap \lambda^\perp$ for $\lambda \in S_\sigma$. We know that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}\langle -\lambda \rangle$, cf. Lemma 2.23. If we write $S_\sigma = \mathbb{Z}_{\geq 0}\langle s_1, \ldots, s_k \rangle$, we may assume that $\lambda = s_k$ and write $-\lambda = s_{k+1}$, so $S_\tau = \mathbb{Z}_{\geq 0}\langle s_1, \ldots, s_k, s_{k+1} \rangle$. This gives the varieties

$$\begin{aligned} X_\tau &\subseteq \mathbb{C}^k_{(x_1,\ldots,x_k)} = \mathrm{Specm}(\mathbb{C}[\xi_1, \ldots, \xi_k]), \\ X_\sigma &\subseteq \mathbb{C}^{k+1}_{(x_1,\ldots,x_{k+1})} = \mathrm{Specm}(\mathbb{C}[\xi_1, \ldots, \xi_{k+1}]). \end{aligned} \tag{2.65}$$

We also know that at the monoidal level, the relations in S_τ are those in S_σ with the additional one given by $s_k + s_{k+1} = 0$. Therefore R_τ is given by R_σ

with the extra binomial relation $\xi_k \xi_{k+1} = 1$. Now if we project $\mathbb{C}^{k+1}$ onto $\mathbb{C}^k$ by omitting the last coordinate, we will naturally identify X_τ with $X_\sigma \setminus \mathsf{V}(\xi_k)$. The algebraic set $\mathsf{V}(\xi_k)$ is uniquely determined by τ. $\square$

In general, for an affine variety $X = \mathrm{Specm}(R)$, the open sets of the form $X \setminus \mathsf{V}(f)$ for $f \in R$ are called **principal open sets**. The above lemma shows that the variety associated with a face of σ is a principal open set of X_σ.

2.4.2 Toric varieties: definition and examples

The lemma above is a great step toward the proper formalisation of glueing. Suppose now we have two cones σ, σ' in a fan Δ, and let $\tau := \sigma \cap \sigma'$. Lemma 2.51 allows us to write:

$$X_\sigma \setminus \mathsf{V}(f) \cong X_\tau \cong X_{\sigma'} \setminus \mathsf{V}(f') \tag{2.66}$$

Let us call this composition $\psi_{\sigma,\sigma'} \colon X_\sigma \setminus \mathsf{V}(f) \to X_{\sigma'} \setminus \mathsf{V}(f')$ and refer to it as the **glueing map**. We are now ready to state the main theorem of this section.

Theorem 2.52 (Toric varieties from fans) *Let Δ be a fan in $N_\mathbb{R}$. Consider the disjoint union $\bigsqcup_{\sigma \in \Delta} X_\sigma$ and the equivalence relation on it given for $x \in X_\sigma$ and $x' \in X_{\sigma'}$ by $x \sim x'$ if and only if $\psi_{\sigma,\sigma'}(x) = x'$. Quotienting by this relation yields the space*

$$X_\Delta := \bigsqcup_{\sigma \in \Delta} X_\sigma \Big/ \sim \tag{2.67}$$

*which is called the **toric variety associated with the fan** Δ. It is a topological space admitting an open cover by affine toric varieties X_σ for $\sigma \in \Delta$. It is an algebraic variety whose charts are given by binomial relations. It is toric, with embedded torus identified with $X_{\{0\}} = \mathrm{Specm}(\mathbb{C}[M])$, the variety associated with the trivial cone $\{0\} = \bigcap_{\sigma \in \Delta} \sigma$.*

The proof, which we omit, simply checks the compatibility of the torus action between the different affine charts. As in the affine case, we postpone the discussion on normality to Section 3.2.

Remark 2.53 *We can visualise the above construction as a series of inclusions,*

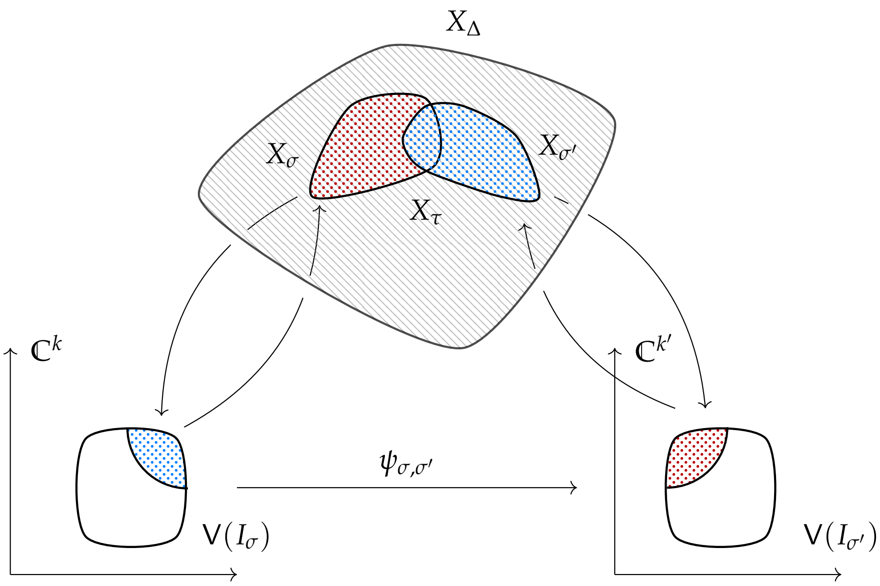


Figure 2.2: A pictorial representation of the glueing of X_σ and $X_{\sigma'}$ along their common intersection X_τ via the glueing map $\psi_{\sigma,\sigma'}$.

giving the commuting diagram

$$
\begin{tikzcd}
 & & & & X_\Delta & & & & \\
 & & & & \bigsqcup_{\sigma\in\Delta} X_\sigma \arrow[u, two heads] & & & & \\
X_\sigma \arrow[urrrr, hook] & & \cdots \arrow[urr, hook] & & X_{\sigma'} \arrow[u, hook] & & \cdots \arrow[ull, hook'] & & X_{\sigma''} \arrow[ullll, hook'] \\
 & & X_\tau \arrow[ull, hook'] \arrow[urr, hook] & & \cdots \arrow[u, hook] & & X_{\tau'} \arrow[ull, hook'] \arrow[urr, hook] & & \\
 & & & & X_{\{0\}} \arrow[ull, hook'] \arrow[urr, hook] & & & &
\end{tikzcd}
\tag{2.68}
$$

where $X_{\{0\}} = \mathrm{Specm}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^n$ *is the embedded torus. Thanks to the commutativity of the above diagram, the final result* X_Δ *does not depend on the order of glueing. We may consider partial glueing of some cones that will then iteratively be glued together without altering the final result.*

Let us now look at some examples.

Example 2.54 *In* $\mathbb{R}^n$ *with standard lattice, set* $e_0 := -(e_1 + \cdots + e_n)$*. Consider* *the fan generated by the cones*

$$\sigma_i := \mathbb{Z}_{\geq 0}\langle e_0, \ldots, \widehat{e_i}, \ldots, e_n \rangle, \tag{2.69}$$

where $\widehat{e_i}$ means that the vector is skipped. The dual of these cones are

$$\check{\sigma}_i = \mathbb{Z}_{\geq 0}\langle e_0^* - e_i^*, \ldots, \widehat{e_0^* - e_i^*}, \ldots, e_n^* - e_i^* \rangle \tag{2.70}$$

where by convention $e_0^* := 0$. The associated algebras are

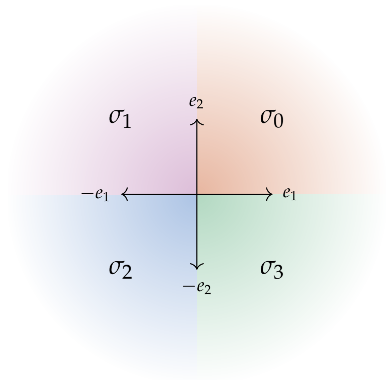
$$R_{\sigma_i} = \mathbb{C}\left[z_0 z_i^{-1}, \ldots, \widehat{z_i z_i^{-1}}, \ldots, z_n z_i^{-1}\right], \tag{2.71}$$

where by convention $z_0 := 1$. The associated varieties are all copies of $\mathbb{C}^n$. A similar analysis for the faces $\sigma_i \cap \sigma_j$ yields the varieties $\mathbb{C}^* \times \mathbb{C}^{n-1}$. For the codimension 2 faces, we find $(\mathbb{C}^*)^2 \times \mathbb{C}^{n-2}$, and so on. In summary, we get the diagram

$$\begin{tikzcd}
 & & \mathbb{P}^n & & \\
 & & \bigsqcup \mathbb{C}^n \sqcup \bigsqcup (\mathbb{C}^* \times \mathbb{C}^{n-1}) \sqcup \cdots \arrow[u, two heads] & & \\
\mathbb{C}^n \arrow[urr, hook] & \cdots \arrow[ur, hook] & \mathbb{C}^n \arrow[u, hook] & \cdots \arrow[ul, hook'] & \mathbb{C}^n \arrow[ull, hook'] \\
\mathbb{C}^* \times \mathbb{C}^{n-1} \arrow[u, hook] \arrow[r, bend left, "\cong"] & \cdots \arrow[r, bend left, "\cong"] & \mathbb{C}^* \times \mathbb{C}^{n-1} \arrow[u, hook] \arrow[r, bend left, "\cong"] & \cdots \arrow[r, bend left, "\cong"] & \mathbb{C}^* \times \mathbb{C}^{n-1} \arrow[u, hook] \\
 & & \vdots & &
\end{tikzcd} \tag{2.72}$$

yielding the projective n-space.

Example 2.55 *In $\mathbb{R}^2$ with the standard lattice, consider the cones and the fan they form.*



The duals of the top-dimensional cones are precisely the same cones in $(\mathbb{R}^2)^*$. *The duals of the rays are the four half-spaces. The dual of the trivial cone is, as usual, the whole* $(\mathbb{R}^2)^*$. *Thus, the monoids corresponding to the top-dimensional cones are*

$$\begin{aligned} S_{\sigma_0} &= \mathbb{Z}_{\geq 0}\langle e_1^*, e_2^* \rangle, & S_{\sigma_1} &= \mathbb{Z}_{\geq 0}\langle -e_1^*, e_2^* \rangle, \\ S_{\sigma_2} &= \mathbb{Z}_{\geq 0}\langle -e_1^*, -e_2^* \rangle, & S_{\sigma_3} &= \mathbb{Z}_{\geq 0}\langle e_1^*, -e_2^* \rangle, \end{aligned} \tag{2.73}$$

with corresponding algebras

$$\begin{aligned} R_{\sigma_0} &= \mathbb{C}[z_1, z_2], & R_{\sigma_1} &= \mathbb{C}[z_1^{-1}, z_2], \\ R_{\sigma_2} &= \mathbb{C}[z_1^{-1}, z_2^{-1}], & R_{\sigma_3} &= \mathbb{C}[z_1, z_2^{-1}], \end{aligned} \tag{2.74}$$

and affine varieties all isomorphic to $\mathbb{C}^2$. *Similarly for the lower dimensional cones. Then the glueings:*

- *of* X_{σ_0} *and* X_{σ_1} *is* $\mathbb{P}^1 \times \mathbb{C}$ *with coordinates* $([t_0, t_1], z_2)$, *where* $z_1 = t_0/t_1$,
- *of* X_{σ_2} *and* X_{σ_3} *is* $\mathbb{P}^1 \times \mathbb{C}$ *with coordinates* $([t_0, t_1], z_2^{-1})$,
- *of these two is* $X_\Delta = \mathbb{P}^1 \times \mathbb{P}^1$ *with coordinates* $([t_0, t_1], [s_0, s_1])$, *where* $z_2 = s_0/s_1$.

The result does not depend on the ordering, as explained in Remark 2.53. We thus obtain the diagram

$$\begin{tikzcd}
X_{\sigma_1} \arrow[drr] \arrow[ddr] & & X_{\langle -e_1 \rangle} \arrow[ll] \arrow[rr] & & X_{\sigma_2} \arrow[dll] \arrow[ddl] \\
& & \mathbb{P}^1 \times \mathbb{C} \arrow[d] & & \\
X_{\langle e_2 \rangle} \arrow[uu] \arrow[dd] & \mathbb{P}^1 \times \mathbb{C} \arrow[r] & \mathbb{P}^1 \times \mathbb{P}^1 & \mathbb{P}^1 \times \mathbb{C} \arrow[l] & X_{\langle -e_2 \rangle} \arrow[uu] \arrow[dd] \\
& & \mathbb{P}^1 \times \mathbb{C} \arrow[u] & & \\
X_{\sigma_0} \arrow[uur] \arrow[urr] & & X_{\langle e_1 \rangle} \arrow[ll] \arrow[rr] & & X_{\sigma_3} \arrow[uul] \arrow[ull]
\end{tikzcd} \tag{2.75}$$

where we also have $X_{\{0\}} = (\mathbb{C}^*)^2$, *which injects into all the varieties associated with the rays (i.e. the one-dimensional cones).*

We conclude the chapter with one last example: the weighted projective space. This example once again highlight the role of the lattice in the construction of toric varieties.

Example 2.56 *Consider, as in Example 2.54, the fan in $\mathbb{R}^n$ generated by the standard basis vectors $e_1, \ldots, e_n$ together with $e_0 := -(e_1 + \cdots + e_n)$. However, we now consider them with respect to the lattice generated by $\frac{1}{d_i} e_i$, where $d_0, \ldots, d_n$ are fixed positive integers. The resulting variety is called the **weighted projective space**, denoted $\mathbb{P}(d_0, \ldots, d_n)$. It can alternatively be defined as*

$$\mathbb{P}(d_0, \ldots, d_n) = \frac{\mathbb{C}^{n+1} \setminus \{ 0 \}}{\mathbb{C}^*}, \tag{2.76}$$

where $\mathbb{C}^*$ acts by a weighted *rescaling:*

$$\lambda \cdot (x_0, \ldots, x_n) := (\lambda^{d_0} x_0, \ldots, \lambda^{d_n} x_n). \tag{2.77}$$

The standard projective space is recovered by setting $d_0 = \cdots = d_n = 1$.

Chapter 3

Geometric properties of toric varieties

Written by Elisa L.
Talks by Noah and Alex

We have seen so far how to construct a toric variety from a fan. Two natural questions now arise: can we describe geometric properties of a toric variety in terms of the combinatorics of the fan? And does every toric variety arise from the fan construction?

The goal of this chapter is to answer these questions. In particular, we will provide a full description of the orbits and their closures and characterize basic geometric properties—smoothness, completeness, and projectivity—in terms of the fan's combinatorics. We will conclude with a sketch of the proof that every toric variety arises as the toric variety of a fan.

3.1 Orbits and their closure

Arguably, the most natural geometric object we can associate with a group action is the collection of orbits.

Definition 3.1 *Let G be a group acting on a set X. For $x \in X$, we define the **orbit** of x under the action of the group G as the set*

$$\mathcal{O}_x := \{ g \cdot x \mid g \in G \} \subseteq X. \tag{3.1}$$

If, in addition, X is a topological space, it is natural to ask about the closures of its orbits. Now, if G and X are the torus and the toric variety associated with a fan, can we describe its orbits and their closures via the combinatorics of the fan? The main idea is to associate an orbit with each cone in the fan. To this end, we assign a distinguished point to each cone, and the orbit of that point corresponds to the orbit of the cone.

3.1.1 Distinguished points

To define the distinguished point, let us recall the torus action on the toric variety and some basic facts about the algebro-geometric description of affine varieties. Consider the torus action on every affine toric variety X_σ, where σ is a strongly convex cone in $N_\mathbb{R}$ with N of rank n. It is given by

$$\begin{aligned} (\mathbb{C}^*)^n \times X_\sigma &\longrightarrow X_\sigma \\ (t, x) &\longmapsto t \cdot x := (t^{s_1}x_1, \ldots, t^{s_k}x_k)\,, \end{aligned} \tag{3.2}$$

where $(s_1, \ldots, s_k)$ are the generators of the monoid S_σ, $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$, and $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$. Here, we denote by t^{s_i} the product $t_1^{s_{i,1}} \cdots t_n^{s_{i,n}} \in \mathbb{C}^*$ under the identification $M \cong \mathbb{Z}^n$, giving $s_i \cong (s_{i,1}, \ldots, s_{i,n})$.

Example 3.2 *Consider the cone $\sigma = \mathrm{Cone}(e_2, 2e_1 - e_2)$ (see Example 2.22). The monoid S_σ is generated by the elements $s_1 \cong (1,0)$, $s_2 \cong (1,1)$, and $s_3 \cong (1,2)$, yielding the double cone*

$$X_\sigma \cong \{\, x_2^2 = x_1x_3 \,\} \subset \mathbb{C}^3. \tag{3.3}$$

For $t \in (\mathbb{C}^*)^2$, we obtain

$$t^{s_1} = t_1, \qquad t^{s_2} = t_1t_2, \qquad t^{s_3} = t_1t_2^2. \tag{3.4}$$

Hence, we identify the torus T with $\{\, (t_1, t_1t_2, t_1, t_2^2) \mid (t_1, t_2) \in (\mathbb{C}^*)^2 \,\}$. For $x = (x_1, x_2, x_3) \in X_\sigma$, the point

$$t \cdot x = (t_1x_1, t_1t_2x_2, t_1t_2^2x_3) \tag{3.5}$$

is also in X_σ.

For a fan Δ in $N_\mathbb{R}$, recall that the action defined on each affine patch extends to the whole X_Δ. Indeed, if τ is a face of σ, we have an identification $X_\tau \cong X_\sigma \setminus \mathrm{V}(f)$ for some $f \in R_\sigma$. We have seen that this identification makes the torus action compatible with the glueing maps $\psi_{\sigma,\sigma'}$. The embedded torus can also be identified with $X_{\{0\}}$, the affine variety associated with the trivial cone. In particular, for any $t \in X_{\{0\}}$, the orbit is the whole $X_{\{0\}}$. This follows from the fact that the action of the torus on itself gives a single orbit, coinciding with the entire torus. This provides the assignment

$$\{\, 0 \,\} \in \Delta \longleftrightarrow \mathcal{O}_{\{0\}} = T \subseteq X_\Delta. \tag{3.6}$$

What about the other orbits? To describe them and associate an orbit with each cone of the fan, we recall and introduce some useful correspondences.

By the Nullstellensatz (Theorem 2.29), there is a one-to-one correspondence between points in affine space $\mathbb{C}^k$ and maximal ideals in the polynomial algebra $\mathbb{C}[\xi_1, \ldots, \xi_k]$:

$$\mathbb{C}^k \overset{1:1}{\longleftrightarrow} \operatorname{Specm}(\mathbb{C}[\xi_1, \ldots, \xi_k]) = \{ \mathfrak{m} \trianglelefteq \mathbb{C}[\xi_1, \ldots, \xi_k] \text{ maximal} \}. \tag{3.7}$$

More explicitly, every $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$ corresponds to the maximal ideal $\mathfrak{m}_x = \langle \xi_1 - x_1, \ldots, \xi_k - x_k \rangle \trianglelefteq \mathbb{C}[\xi_1, \ldots, \xi_k]$. Another useful characterisation is in terms of $\mathbb{C}$-algebra morphisms:

$$\mathbb{C}^k \overset{1:1}{\longleftrightarrow} \operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[\xi_1, \ldots, \xi_k], \mathbb{C}). \tag{3.8}$$

Explicitly, every $x = (x_1, \ldots, x_k) \in \mathbb{C}^k$ corresponds to the homomorphism $\varphi_x \colon \mathbb{C}[\xi_1, \ldots, \xi_k] \to \mathbb{C}$ defined as $\varphi_x(f) := f(x)$. The morphism φ_x is sometimes referred to as the **evaluation morphism**, as it evaluates polynomials at the corresponding point. Notice that the maximal ideal is recovered as the kernel of such map: $\ker(\varphi_x) = \mathfrak{m}_x$.

The above correspondences generalise to affine algebraic sets as follows. Fix an ideal I of $\mathbb{C}[\xi_1, \ldots, \xi_k]$, and take $V = \mathsf{V}(I) = \{ x \in \mathbb{C}^k \mid I \subseteq \mathfrak{m}_x \}$. The coordinate ring of the affine algebraic set V is $R_V = \mathbb{C}[\xi]/I_V$, where $I_V = \mathsf{I}(\mathsf{V}(I))$ is the radical as in Definition 2.31. Then, generalising Theorem 2.33, we obtain the following one-to-one correspondences:

$$V \overset{1:1}{\longleftrightarrow} \operatorname{Specm}(R_V) = \{ \mathfrak{m} \trianglelefteq R_V \text{ maximal} \} \overset{1:1}{\longleftrightarrow} \operatorname{Hom}_{\mathbb{C}\text{-alg}}(R_V, \mathbb{C}). \tag{3.9}$$

Conceptually, the rightmost set allows us to characterise points of affine varieties via the finitely generated $\mathbb{C}$-algebra structure of their coordinate rings.

Up until now, in our description of toric varieties, we have encountered two specific types of $\mathbb{C}$-algebras. The first is the algebra $\mathbb{C}[M]$ of Laurent monomials, generated by $z^{\pm m_1}, \ldots, z^{\pm m_n}$ for $m_1, \ldots, m_n$ forming a $\mathbb{Z}$-basis of M. The second is the algebra $\mathbb{C}[S_\sigma]$ associated with a finitely generated monoid $S_\sigma \subset M$, generated by $z^{s_1}, \ldots, z^{s_k}$ for $s_1, \ldots, s_k$ forming a generating set of S_σ. A natural question arises: Can we describe the points in the associated affine varieties in terms of the group structure of M and the monoidal structure of S_σ? The answer is yes, given by the correspondences

$$\operatorname{Specm}(\mathbb{C}[M]) \overset{1:1}{\longleftrightarrow} \operatorname{Hom}_{\mathrm{gp}}(M, \mathbb{C}^*), \tag{3.10}$$

$$\operatorname{Specm}(\mathbb{C}[S_\sigma]) \overset{1:1}{\longleftrightarrow} \operatorname{Hom}_{\mathrm{sgp}}(S_\sigma, \mathbb{C}). \tag{3.11}$$

Here, 'gp' and 'sgp' stand for 'group' and 'semigroup', respectively, with $\mathbb{C}^*$ and $\mathbb{C}$ considered as a group or semigroup under multiplication.

Both bijections are given by evaluations. Explicitly, the first one is defined as follows. Recall that the lattice M sits inside the coordinate ring of the torus, $\mathbb{C}[M]$, as $m \mapsto z^m$. In particular, z^m defines a nowhere-vanishing function on the torus. Thus, for each point $t \in \operatorname{Specm}(\mathbb{C}[M])$, we can evaluate z^m at t to obtain the map

$$\varphi_t \colon M \longrightarrow \mathbb{C}^*, \qquad m \longmapsto z^m(t). \tag{3.12}$$

This is precisely the group morphism associated to $t \in \operatorname{Specm}(\mathbb{C}[M])$.

For the second bijection, a similar argument applies: for each point $x \in \operatorname{Specm}(\mathbb{C}[S_\sigma])$, we consider the map

$$\varphi_x \colon S_\sigma \longrightarrow \mathbb{C}, \qquad s \longmapsto z^s(x). \tag{3.13}$$

This is precisely the semigroup morphism associated to $x \in \operatorname{Specm}(\mathbb{C}[S_\sigma])$.

The above description is also compatible with the torus action: if $t \in \operatorname{Specm}(\mathbb{C}[M])$ is identified with the group morphism $\varphi_t \colon M \to \mathbb{C}^*$, and $x \in X_\sigma$ with a semigroup morphism $\varphi_x \colon S_\sigma \to \mathbb{C}$, then $t \cdot x$ is identified with the semigroup morphism

$$\varphi_{t\cdot x}(s) = \varphi_t(s) \cdot \varphi_x(s). \tag{3.14}$$

Here, the product on the left represents the torus action, while the product on the right is the usual multiplication in $\mathbb{C}$ between an element of $\mathbb{C}^*$ and an element of $\mathbb{C}$.

We can now define the concept of distinguished points associated with cones in a fan Δ. The orbit of a cone will then be defined as the orbit of its distinguished point.

Definition 3.3 *Let Δ be a fan in $N_\mathbb{R}$ and fix a cone $\sigma \in \Delta$. We associate to each face τ of σ a **distinguished point** x_τ, defined as the point in X_σ corresponding to the semigroup morphism $\varphi_\tau \colon S_\sigma \to \mathbb{C}$ given by*

$$\varphi_\tau(s) := \begin{cases} 1 & \text{if } s \in \tau^\perp, \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

for generators s of S_σ.

Let us continue with our previous example 3.2, the double cone.

Example 3.4 *Recall the monoid generators* $s_1 \cong (1,0)$*,* $s_2 \cong (1,1)$*, and* $s_3 \cong (1,2)$*. Define* τ_1 *as the face generated by* $2e_1 - e_2$*. Then* s_3 *is the only generator in* $\tau_1^\perp$*. Therefore,*

$$\varphi_{\tau_1}(s_1) = \varphi_{\tau_1}(s_2) = 0, \qquad \varphi_{\tau_1}(s_3) = 1, \tag{3.16}$$

so x_{τ_1} *has coordinates* $(0,0,1)$ *in* X_σ*. Similarly, let* τ_2 *be the face generated by* e_2*. Then, since* s_1 *generates* $\tau_2^\perp$*, we have* $x_{\tau_2} = (1,0,0)$ *in* X_σ*. Finally, considering the faces* σ *and* $\{ 0 \}$ *of* σ*, we have* $\sigma^\perp = \{ 0 \}$ *and* $\{ 0 \}^\perp = \mathbb{R}^2$*. Therefore,* $x_\sigma = (0,0,0)$ *and* $x_{\{ 0 \}} = (1,1,1)$*.*

Before proceeding, let us answer an important question. Since in a fan Δ, a cone τ can be a face of different cones, does the definition depend on the choice of the cone? The answer is no. Indeed, suppose that a cone τ is a face of two different cones in the fan, say σ and σ', leading to two different points x_τ and x'_τ. Once embedded in the toric variety X_Δ, the two points get identified via the glueing map $\psi_{\sigma,\sigma'}$. Abusing notation, we denote the point x_τ in X_Δ with the same symbol. To sum up:

The distinguished point x_τ *in* X_Δ
does not depend on the choice of cone $\sigma \succeq \tau$*.*

Now, we are ready to define the orbit associated with a cone.

Definition 3.5 *Let* Δ *be a fan in* $N_\mathbb{R}$ *and fix a cone* $\sigma \in \Delta$*. The* ***orbit associated with the cone*** σ *is defined as the orbit of its distinguished point* x_σ *in* X_Δ*:*

$$\mathcal{O}_\sigma := \mathcal{O}_{x_\sigma} = \{ t \cdot x_\sigma \mid t \in T \}. \tag{3.17}$$

Let us continue again with the double cone example.

Example 3.6 *We compute the orbits of the distinguished points of the double cone (recall the torus action from Example 3.2 and the distinguished points from Example 3.4):*

- *The orbit of* $x_\sigma = (0,0,0)$ *is* $\mathcal{O}_\sigma = \{ (0,0,0) \}$*. In other words,* x_σ *is a fixed point for the torus action.*
- *The orbit of* $x_{\tau_1} = (0,0,1)$ *is* $\mathcal{O}_{\tau_1} = \{ 0 \} \times \{ 0 \} \times \mathbb{C}^*$*.*
- *The orbit of* $x_{\tau_2} = (1,0,0)$ *is* $\mathcal{O}_{\tau_2} = \mathbb{C}^* \times \{ 0 \} \times \{ 0 \}$*.*
- *The orbit of* $x_{\{ 0 \}} = (1,1,1)$ *is* $\mathcal{O}_{\{ 0 \}} = T \cong (\mathbb{C}^*)^2$*, as expected from Equation* (3.6)*.*

3.1.2 A description of orbits

We now describe orbits (and their closures) in terms of the lattice and the fan. Before doing so, we introduce three key concepts: the quotient lattice, the quotient cone, and the star of a cone.

Let Δ be a fan in $N_{\mathbb{R}}$, and let $\tau \in \Delta$ be a fixed cone. Define the sublattice N_τ of N as

$$N_\tau := (\tau \cap N) + (-\tau \cap N). \tag{3.18}$$

That is, N_τ is the smallest sublattice of N containing τ. Thanks to the strong convexity of τ, the quotient

$$N(\tau) := \frac{N}{N_\tau} \tag{3.19}$$

is also a lattice, called the **quotient lattice**. Its dual is naturally identified with $M(\tau) = \tau^\perp \cap M$.

For every cone σ with τ as a face, define the **quotient cone** as

$$\bar{\sigma} := \frac{\sigma + (N_\tau)_{\mathbb{R}}}{(N_\tau)_{\mathbb{R}}} \subseteq \frac{N_{\mathbb{R}}}{(N_\tau)_{\mathbb{R}}} = N(\tau)_{\mathbb{R}}. \tag{3.20}$$

It can be shown that $\bar{\sigma}$ is still a strongly convex lattice cone in $N(\tau)_{\mathbb{R}}$.

Definition 3.7 *For a fan Δ in $N_{\mathbb{R}}$ and a cone $\tau \in \Delta$, define the **star** of τ as*

$$\mathrm{Star}(\tau) := \{\, \bar{\sigma} \subseteq N(\tau)_{\mathbb{R}} \mid \tau \text{ is a face of } \sigma \,\}. \tag{3.21}$$

It can be shown that $\mathrm{Star}(\tau)$ is a fan in the quotient space $N(\tau)_{\mathbb{R}}$.

Example 3.8 *Consider $\mathbb{R}^3$ with its standard lattice, and define the vectors:*

$$v_0 = e_3, \quad v_1 = e_1 + e_3, \quad v_2 = e_2 + e_3, \quad v_3 = -e_2 + e_3, \quad v_4 = -e_1 + e_3. \tag{3.22}$$

Define the following top-dimensional cones:

$$\sigma_0 = \mathrm{Cone}(v_0, v_1, v_2), \qquad \sigma_1 = \mathrm{Cone}(v_0, v_1, v_3), \qquad \sigma_2 = \mathrm{Cone}(v_0, v_2, v_4). \tag{3.23}$$

Let Δ be the fan generated by these cones, along with their faces and intersections. Consider the one-dimensional cone ρ generated by $v_0 = e_3$. Then:

- *The minimal sublattice containing ρ is $N_\rho = \mathbb{Z}e_3$.*

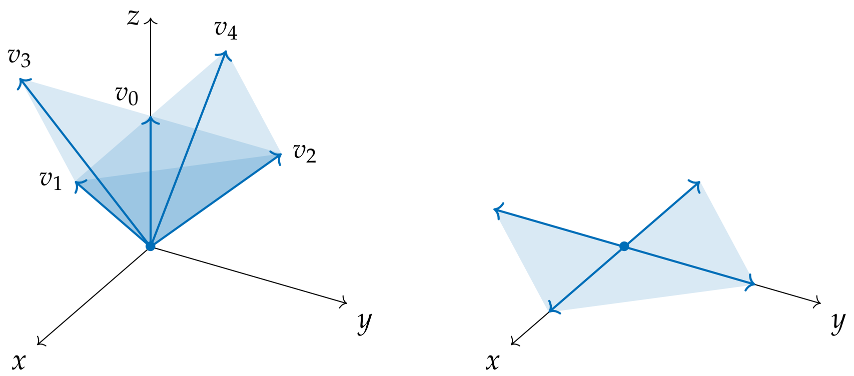


Figure 3.1: A representation of the fan Δ (left) and the fan $\mathrm{Star}(\rho)$ (right).

- *The cones in Δ containing ρ are: the three-dimensional cones σ_0, σ_1, and σ_2; the two-dimensional cones $\sigma_0 \cap \sigma_1$, $\sigma_0 \cap \sigma_2$, $\tau' = \mathrm{Cone}(v_0, v_3) \preceq \sigma_1$, and $\tau'' = \mathrm{Cone}(v_0, v_4) \preceq \sigma_2$; and the one-dimensional cone ρ itself.*

As a consequence, the quotient lattice is identified as $N(\rho) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Moreover, the star of ρ is obtained by projecting the cones listed above onto the (x, y)-plane. To avoid confusion, denote by $\bar{e}_1$ and $\bar{e}_2$ the projections of e_1 and e_2. Then, the quotient cones forming $\mathrm{Star}(\rho)$ *are given by:*

$$\begin{gathered}
\overline{\sigma_0} = \mathrm{Cone}(\bar{e}_1, \bar{e}_2), \qquad \overline{\sigma_1} = \mathrm{Cone}(\bar{e}_1, -\bar{e}_2), \qquad \overline{\sigma_2} = \mathrm{Cone}(-\bar{e}_1, \bar{e}_2), \\
\overline{\sigma_0 \cap \sigma_1} = \mathrm{Cone}(\bar{e}_1), \qquad \overline{\sigma_0 \cap \sigma_1} = \mathrm{Cone}(\bar{e}_2), \\
\overline{\tau'} = \mathrm{Cone}(-\bar{e}_2), \qquad \overline{\tau''} = \mathrm{Cone}(-\bar{e}_1), \\
\bar{\rho} = \{ 0 \} .
\end{gathered} \tag{3.24}$$

See Example 3.8 for a visual representation of $\mathrm{Star}(\rho)$.

The main result of this section, whose proof is omitted, is the following description of the orbits of a cone and their closure in terms of its quotient lattice and star.

Theorem 3.9 *The following holds:*

1. *The orbit $\mathcal{O}_\tau$ is isomorphic to* $\mathrm{Specm}(\mathbb{C}[M(\tau)])$, *which is the torus associated with the dual quotient lattice $M(\tau)$.*

2. *The closure $\overline{\mathcal{O}_\tau}$ is isomorphic to $X_{\mathrm{Star}(\tau)}$, the toric variety associated with the fan* $\mathrm{Star}(\tau)$.

Since the rank of $N(\tau)$ is equal to the codimension of τ, we obtain an immediate corollary.

Corollary 3.10 *The orbit $\mathcal{O}_\tau$ is isomorphic to $(\mathbb{C}^*)^{\mathrm{codim}(\tau)}$. Its closure is a toric variety with $\mathcal{O}_\tau$ as its dense torus.*

Three particularly interesting cases are worth mentioning:

- If σ is a top-dimensional cone, then $\mathcal{O}_\sigma = \overline{\mathcal{O}_\sigma}$ is a fixed point, namely the distinguished point x_σ.
- If ρ is a one-dimensional cone (called a **ray**), then $\overline{\mathcal{O}_\rho}$ is a codimension-one subvariety of X_Δ. Codimension-one subvarieties are particularly important objects, known as divisors, whose role will be discussed in Section 4.1.
- The zero-dimensional cone $\{0\}$ corresponds to the torus T, whose closure is the entire toric variety X_Δ.

From the above theorem, we can deduce an explicit description of the points in the closure of an orbit in fixed affine patch. Let again σ be a cone and τ a face of σ in a fan Δ in $N_\mathbb{R}$. Then $\overline{\mathcal{O}_\tau} \cap X_\sigma$ can be described as follows. Let $s_1, \ldots, s_k$ be a generating set of S_σ and let I_τ be the index set defined as

$$I_\tau := \left\{ i \in \{1, \ldots, k\} \;\middle|\; s_i \notin \tau^\perp \right\}. \tag{3.25}$$

Equivalently,

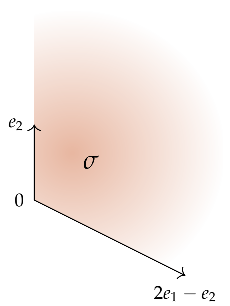
$$i \in I_\tau \qquad \Longleftrightarrow \qquad \langle s_i, v_j \rangle \neq 0 \text{ for some } j \in \{1, \ldots, s\}, \tag{3.26}$$

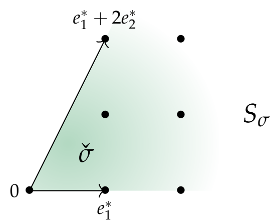
where $v_1, \ldots, v_s$ are a generating set for the face τ. Then,

$$\overline{\mathcal{O}_\tau} \cap X_\sigma = \{ (x_1, \ldots, x_k) \in X_\sigma \mid x_i = 0 \text{ for all } i \in I_\tau \}. \tag{3.27}$$

Let us provide some examples.

Example 3.11 *Consider again the cone σ of the previous examples, providing the double cone.*





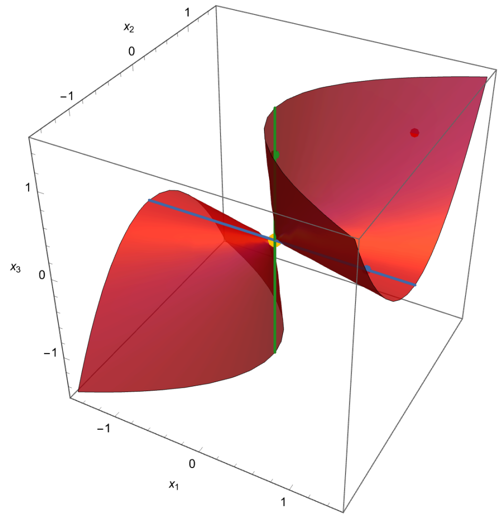


Figure 3.2: The real points of the double cone, together with the orbits and the distinguished points. The colour red correspond to σ, green to τ_1, blue to τ_2, and yellow to $\{0\}$.

Recall the generators $s_1 = e_1$, $s_2 = e_1^* + e_2^*$, and $s_3 = e_1^* + 2e_2^*$. Since the variety X_σ is affine in $\mathbb{C}^3$, the above description directly provides the closure of the orbits.

For $\tau_1 = \mathrm{Cone}(2e_1 - e_2)$ we get

$$i \in I_{\tau_1} \qquad \Longleftrightarrow \qquad \langle s_i, 2e_1 - e_2 \rangle \neq 0. \tag{3.28}$$

We therefore get $I_{\tau_1} = \{1, 2\}$. Thus, the closure of the orbit associated with τ_1 is

$$\overline{\mathcal{O}_{\tau_1}} = \{ (x_1, x_2, x_3) \in X_\sigma \mid x_1 = x_2 = 0 \} = \{0\} \times \{0\} \times \mathbb{C}. \tag{3.29}$$

As for $\tau_2 = \mathrm{Cone}(e_2)$, we have

$$i \in I_{\tau_2} \qquad \Longleftrightarrow \qquad \langle s_i, e_2 \rangle \neq 0. \tag{3.30}$$

which implies $I_{\tau_2} = \{2, 3\}$ and therefore

$$\overline{\mathcal{O}_{\tau_2}} = \{ (x_1, x_2, x_3) \in X_\sigma \mid x_2 = x_3 = 0 \} = \mathbb{C} \times \{0\} \times \{0\}. \tag{3.31}$$

To conclude, consider σ as a face of itself, we get $I_\sigma = \{1, 2, 3\}$ and therefore $\overline{\mathcal{O}_\sigma} = \{(0,0,0)\}$, which is the origin in $\mathbb{C}^3$ and the unique fixed point of the torus action. See Figure 3.2 for an illustration of the (real points) of the orbits.

We conclude with a useful result that describes the affine patches of the closures $\overline{\mathcal{O}_\tau} = X_{\mathrm{Star}(\tau)}$.

Proposition 3.12 *For every quotient cone $\bar{\sigma}$, consider the affine subvariety of $\overline{\mathcal{O}_\tau} = X_{\mathrm{Star}(\tau)}$ defined as*

$$X_\sigma(\tau) := \mathrm{Specm}(\mathbb{C}[(\bar{\sigma})^\vee \cap M(\tau)]), \tag{3.32}$$

It can be alternatively described as

$$X_\sigma(\tau) \cong \mathrm{Specm}(\mathbb{C}[\check{\sigma} \cap \tau^\perp \cap M]). \tag{3.33}$$

Notice that the cone $\check{\sigma} \cap \tau^\perp$ is precisely the cone τ^* from Proposition 2.13, the duality statement between faces of a cone and its dual.

3.1.3 Cone-orbit correspondence

We conclude this section with the cone-orbit correspondence, which completely classifies the orbits of a toric variety in terms of the combinatorics of its fan, providing an explicit description of the orbits and their closures.

Theorem 3.13 (Cone-orbit correspondence) *Let Δ be a fan in $N_\mathbb{R}$. There is a one-to-one correspondence between cones in Δ and orbits:*

$$\{\, \textit{cones in } \Delta \,\} \overset{1:1}{\longleftrightarrow} \{\, \textit{orbits of } X_\Delta \,\}, \tag{3.34}$$

mapping each cone σ to its corresponding orbit $\mathcal{O}_\sigma$. In this correspondence, we have the dimension formula

$$\dim(\sigma) + \dim(\mathcal{O}_\sigma) = n. \tag{3.35}$$

Moreover, the following hold:

1. $X_\sigma = \bigsqcup_{\tau \preceq \sigma} \mathcal{O}_\tau$.
2. $\overline{\mathcal{O}_\tau} = \bigsqcup_{\sigma \succeq \tau} \mathcal{O}_\sigma$.
3. $\mathcal{O}_\tau = \overline{\mathcal{O}_\tau} \setminus \bigcup_{\sigma \succ \tau} \overline{\mathcal{O}_\sigma}$.

Before proving the theorem, let us examine an explicit example: the projective plane.

Example 3.14 *Recall the fan of $\mathbb{P}^2$ from Example 2.50, illustrated again in Figure 3.3. There are seven orbits, corresponding to the seven cones in the fan.*

- *The orbit of the point $[1,1,1]$, which is the open dense torus $T \subset \mathbb{P}^2$ homeomorphic to $(\mathbb{C}^*)^2$. This orbit corresponds to the trivial cone: $T = \mathcal{O}_{\{0\}}$. Its closure is the whole of $\mathbb{P}^2$.*

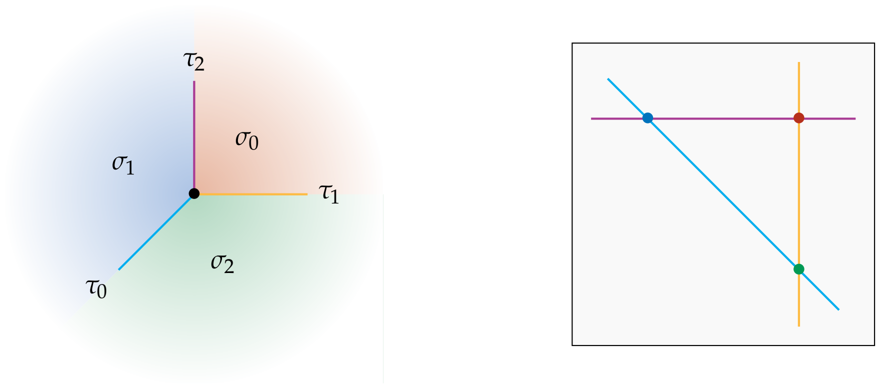


Figure 3.3: The fan of $\mathbb{P}^2$ (left) and a schematic picture of its orbit closures (right).

- *Three orbits corresponding to the points* $[1,1,0]$, $[1,0,1]$, *and* $[0,1,1]$, *each homeomorphic to* $\mathbb{C}^*$. *These correspond to the three rays* τ_0, τ_1 *and* τ_2 *generated by* $-(e_1+e_2)$, e_1, *and* e_2 *respectively. Their closure are homeomorphic to three copies of* $\mathbb{P}^1$ *inside* $\mathbb{P}^2$. *For instance, the closure of the first one is*

$$\overline{\mathcal{O}_{\tau_0}} = \{\, [x_0, x_1, 0] \in \mathbb{P}^2 \,\} \cong \mathbb{P}^1. \tag{3.36}$$

- *Three fixed points:* $[1,0,0]$, $[0,1,0]$, *and* $[0,0,1]$. *These correspond to the distinguished points associated with* σ_i *for* $i = 0,1,2$, *and coincide with their orbits* $\mathcal{O}_{\sigma_i}$ *and their closures.*

See Figure 3.3 for a schematic representation of $\mathbb{P}^2$ *and its orbit closures.*

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Proof (of the cone-orbit correspondence) We will construct an inverse correspondence of $\mathsf{Orb} : \sigma \mapsto \mathcal{O}_\sigma$. Let $\mathcal{O}$ be a T-orbit in X_Δ. Since X_Δ is covered by the T-invariant affine open subsets $X_\sigma \subseteq X_\Delta$ satisfying $X_{\sigma_1} \cap X_{\sigma_2} = X_{\sigma_1 \cap \sigma_2}$, there is a unique minimal cone $\sigma_{\mathcal{O}} \in \Delta$ whose affine variety covers $\mathcal{O}$, which is given by

$$\sigma_{\mathcal{O}} := \bigcap_{\mathcal{O} \subseteq X_\sigma} \sigma. \tag{3.37}$$

Define the correspondence $\mathsf{Con} \colon \mathcal{O} \mapsto \sigma_{\mathcal{O}}$. We then prove $\mathsf{Con} = \mathsf{Orb}^{-1}$.

Firstly, we prove $\mathsf{Orb} \circ \mathsf{Con} = \mathrm{id}$, that is $\mathcal{O} = \mathcal{O}_\sigma$ for $\sigma = \mathsf{Con}(\mathcal{O})$. Let $x \in \mathcal{O}$. From the previous lecture, x corresponds to a semigroup morphism $\varphi_x := S_\sigma \to \mathbb{C}$. Consider the non-zero locus of φ_x defined by

$$\mathrm{NZ}(\varphi_x) := \{\, s \in S_\sigma \mid \varphi_x(s) \neq 0 \,\}. \tag{3.38}$$

It can be shown that, for any semigroup morphism $\varphi\colon S_\sigma \to \mathbb{C}$, the set $\mathrm{NZ}(\varphi)$ can be precisely contained in a face of $\check{\sigma}$. In other words, there exists a unique face $\tau \preceq \sigma$ such that

$$\mathrm{NZ}(\varphi) = \check{\sigma} \cap \tau^\perp \cap M. \tag{3.39}$$

We omit the proof, which is a straightforward computation. As a result, we have $x \in X_\tau$, which implies $\mathcal{O} \subseteq X_\tau$. Hence we have $\sigma \subseteq \tau$ from Equation (3.37), meaning $\tau = \sigma$ and $\mathrm{NZ}(\varphi_x) = \check{\sigma} \cap \sigma^\perp \cap M$. From the definition of distinguished point, we deduce that $x \in \mathcal{O}_\sigma$. Since any two orbits are either equal or disjoint, we conclude that $\mathcal{O} = \mathcal{O}_\sigma$.

Secondly, we prove $\mathsf{Con} \circ \mathsf{Orb} = \mathrm{id}$. Actually, it is equivalent to prove

$$\sigma = \bigcap_{x_\sigma \in X_{\sigma'}} \sigma', \tag{3.40}$$

which is obvious since $x_\sigma \in X_{\sigma'}$ if and only if $\sigma \preceq \sigma'$.

For the remaining part of Theorem 3.13, the dimension formula is exactly Corollary 3.10. The last three items are a direct consequence of Equation (3.27) and the cone-orbit correspondence. This concludes the proof. □

3.2 Smoothness, normality, completeness, projectivity

In this section we will introduce four basic geometric properties of algebraic varieties: smoothness, normality, completeness, and projectivity. The first two are local properties that describe how singular a point can be, while the last two are global properties related to compactness. We will first introduce their general definitions, and then discuss how to understand them in the context of toric varieties from the information of associated fans.

3.2.1 Definitions in algebraic geometry...

Smoothness and normality

We begin with smoothness, whose intuition is similar to that in differential geometry. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and 0 is a regular value of f, which means $d_x f$ is surjective for any $x \in f^{-1}(0)$, then $Z_f := f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^n$ for which we can express the tangent space as

$$T_x Z_f = \ker(d_x f). \tag{3.41}$$

A key insight is that Equation (3.41) provides an alternative way of defining the tangent space even if 0 is not a regular value of f, which may happen for algebraic varieties. As both the tangent space and smoothness are local properties, we can restrict our attention to affine varieties.

Definition 3.15 *Suppose $V \subseteq \mathbb{C}^k$ is an affine variety with vanishing ideal I_V. We define the **tangent space** at a point $x \in V$ as*

$$T_xV := \left\{ v \in \mathbb{C}^k \;\middle|\; d_xf(v) = 0 \text{ for any } f \in I_V \right\}, \tag{3.42}$$

where d_xf is the linear part of f at x, which is also a polynomial in $\mathbb{C}[\xi_1, \ldots, \xi_k]$.

There is another way to define this tangent space under the correspondence $V \cong \mathrm{Specm}(R_V)$, where $x = (x_1, \ldots, x_k)$ corresponds to the maximal ideal $\mathfrak{m}_x = \langle \overline{\xi_1 - x_1}, \ldots, \overline{\xi_1 - x_k} \rangle$.

Lemma 3.16 *Define the restricted differential map*

$$\begin{aligned} \varphi\colon \mathfrak{m}_x &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(T_xV, \mathbb{C}) \\ \bar{f} &\longmapsto d_xf|_{T_xV}. \end{aligned}$$

Then we have $\ker \varphi = \mathfrak{m}_x^2$*. In particular, we have the isomorphism*

$$T_xV \cong \left(\mathfrak{m}_x / \mathfrak{m}_x^2\right)^*. \tag{3.43}$$

We refer to [5, Lemma 4.4.3] for the proof. As mentioned above, the concept of tangent space is local, thus it is sufficient to define it for affine varieties. The above lemma provides a more conceptual definition that does not require the choice of an affine chart: the tangent space is the dual vector space of the quotient of the associated maximal ideal by its square. Intuitively, the square kills all the non-linear components, in the same way the differential does it.

We are now ready to define the concept of smoothness.

Definition 3.17 *Let X be a variety. A point $x \in X$ is called **smooth** if the dimension of its tangent space equals the dimension of the variety:* $\dim(T_xX) = \dim(X)$*. It is called **singular** otherwise. The variety X is called **smooth** if it is smooth at every point.*

Example 3.18 *Consider $X = \mathsf{V}(y^2 - x^3) \subset \mathbb{C}^2_{(x,y)}$, whose real points are depicted in Example 3.18. Take $p = (x_0, y_0) \in X$, then we have*

$$\mathfrak{m}_p / \mathfrak{m}_p^2 = \frac{\langle y^2 - x^3, x - x_0, y - y_0 \rangle}{\langle (y^2 - x^3)^2, (x - x_0)^2, (y - y_0)^2 \rangle}.$$

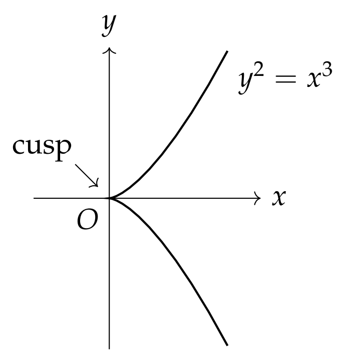


Figure 3.4: The real points of $\mathsf{V}(y^2 - x^3)$. Notice the cusp at the origin.

If p is not the origin, one can easily show that

$$3x_0(x - x_0) = 2y_0(y - y_0) \tag{3.44}$$

in $\mathfrak{m}_p/\mathfrak{m}_p^2$, hence $\dim(\mathfrak{m}_p/\mathfrak{m}_p^2)^* = 1 = \dim(X)$, which implies that p is smooth.

However, this is no longer the case for $p = (0,0)$, where (3.44) vanishes. In contrast $\overline{x}$ and $\overline{y}$ are linearly independent in $\mathfrak{m}_x/\mathfrak{m}_x^2$. As a result, X is not smooth at the origin, which coincides with the intuition that $(0,0)$ is a 'cusp' of X.

Next, we introduce the concept of normality. Normality is a weaker version of smoothness; in other words, it characterises varieties with only mild singularities. The algebraic definition of normality is more straightforward to state than its geometric characterisation, so we focus on the former. Before stating the main definition, we first give some basic concepts in commutative algebra.

Definition 3.19 *Suppose $R \subseteq R'$ is a ring extension. An element $a \in R'$ is called **integral over** R if there exists a monic polynomial $f \in R[x]$ such that $f(a) = 0$. Denote $\overline{R} \subseteq R'$ the set of all integral elements over R. It is apparent that $R \subseteq \overline{R}$. We say that R is **integrally closed in** R' if $\overline{R} = R$.*

Definition 3.20 *An integral domain R is called **normal** if R is integrally closed in its quotient field $\mathrm{Quot}(R)$. In other words, for any $a \in \mathrm{Quot}(R)$, a is integral over R if and only if $a \in R$.*

Intuitively, an normal ring is a ring for which most solutions to polynomial equations are inside the ring itself. Next, we move to the definition of localisation.

Definition 3.21 *Suppose R is a ring and $S \subseteq R$ is a* ***multiplicatively closed subset****, that is $1 \in S$ and $a \cdot b \in S$ for any $a, b \in S$. Then*

$$(a,s) \sim (a',s') \quad \Leftrightarrow \quad \textit{there exists } u \in S \textit{ such that } u(as' - as) = 0 \tag{3.45}$$

is an equivalence relation on $R \times S$, for which we denote

$$[(a,s)] = \frac{a}{s}. \tag{3.46}$$

The set of all equivalent classes

$$S^{-1}R := \left\{ \frac{a}{s} \,\middle|\, a \in R,\, s \in S \right\} \tag{3.47}$$

together with the addition and multiplication

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'} \qquad \textit{and} \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \tag{3.48}$$

is called the ***localisation of** R **at** S. One can easily show that it is a well-defined ring.*

Then we focus on the affine varieties. Let x be a point in an affine variety $X = \mathrm{Specm}(R)$. Denote $\mathfrak{m}_x$ the corresponding maximal ideal.

Definition 3.22 *The* ***local ring at** x is defined as the localization of R at $R \setminus \mathfrak{m}_x$:*

$$R_x := (R \setminus \mathfrak{m}_x)^{-1}R. \tag{3.49}$$

The point x is called ***normal*** *if R_x is a normal ring. The variety X is called* ***normal*** *if it is normal at every point.*

Intuitively speaking, R_x consists of all the rational functions on X which take the value at x, hence it is analogous to the notion of germs of functions at a point in differential geometry. The requirement of the local ring at a point to be normal corresponds to the idea that most solutions to polynomial equations locally around a point are contained in the local ring itself, that it, they are 'known' by the ring of germs of functions at that point.

As expected from the intuition, we state the following proposition. See [6, Example 13.4] for more discussions.

Proposition 3.23 *Let x be a point in a variety X. If x is smooth, then x is normal.*

Completeness and projectivity

We then discuss completeness and projectivity, which are both global properties of algebraic varieties. The notion of completeness is the algebro-geometric analogue of topological compactness.

Definition 3.24 *An algebraic variety X is called **complete** if for any variety Y, the projection*

$$\mathrm{pr}_2 \colon X \times Y \longrightarrow Y \tag{3.50}$$

is a closed map.

Example 3.25 *The affine line $\mathbb{C}$ is not complete. Indeed, one can consider the projection*

$$\begin{aligned} \mathrm{pr}_2 \colon \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto y. \end{aligned}$$

The hyperbola $H := \{ xy = 1 \}$ is closed in $\mathbb{C} \times \mathbb{C}$, while the image $\mathrm{pr}_2(H) = \{ y \neq 0 \}$ is not closed in $\mathbb{C}$. Hence pr_2 is not closed. More generally, one can show that the affine spaces $\mathbb{C}^n$ are not complete.

Example 3.26 *Projective spaces $\mathbb{P}^n$ are complete. See [5, Corollary 3.4.4] for the proof.*

A simple but useful property of complete varieties is that any closed subvariety is also complete.

Proposition 3.27 *Let Z be a closed subvariety of X. If X is complete, then so is Z.*

Proof This is easy to see from the fact that any closed subset of $Z \times Y$ is also closed in $X \times Y$. $\square$

The definition of projectivity is rather concise.

Definition 3.28 *An algebraic variety X is called **projective** if it is a closed subvariety of a projective space.*

Example 3.29 *It is obvious that $\mathbb{P}^n$ itself is projective. In general, a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is called **homogeneous** if there exists an integer $d \geq 0$, such that there is always*

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n), \tag{3.51}$$

*where d is called the **degree** of f. One can define the **projective algebraic sets and varieties** similarly to the affine cases by changing the polynomials to homogenous* *polynomials, and those varieties are all projective (see [5, Chapter 3] for a detailed discussion). Specially, for $X = \mathsf{V}(f)$, we define the **degree** of X as the degree of f.*

One reason to care about projectivity is that it simplifies many situations and provides the right framework for intersection theory. One classical example is **Bézout's theorem**: if C_1 and C_2 are two curves of degrees d_1 and d_2 in $\mathbb{P}^2$, then they intersect in exactly $d_1 \cdot d_2$ points (counted with multiplicities). This statement is definitely false in affine space (e.g., two parallel lines do not intersect).

The proposition below is a direct corollary from Example 3.26 and Proposition 3.27.

Proposition 3.30 *Any projective variety is complete.*

Remark 3.31 *The converse of Proposition 3.30 is not true. A toric counter-example will be given shortly in Example 3.37.*

3.2.2 ... and for toric varieties

Our next goal is to describe the geometric properties of a toric variety associated with a fan in terms of the combinatorics of the fan itself. We start with a series of definitions corresponding to those for algebraic varieties in the previous part of the section. In what follows, we fix a lattice N of rank n.

Definition 3.32 *A strongly convex lattice cone $\sigma \subseteq N_{\mathbb{R}}$ is called **smooth** if there exists a generating set $\{ v_1, \dots, v_r \} \subset N$ of σ such that we can find $v_{r+1}, \dots, v_n \in N$, such that $\{ v_1, \dots, v_n \}$ forms a $\mathbb{Z}$-basis of N. A fan is called smooth if all its cones are smooth.*

Definition 3.33 *A fan Δ in $N_{\mathbb{R}}$ is called **complete** if it covers the entire space:*

$$|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}. \tag{3.52}$$

*The set $|\Delta|$ is called the **support** of Δ.*

Definition 3.34 *A **polytope** P in $N_{\mathbb{R}}$ is defined by the convex hull of finitely many points $v_1, \dots, v_r \in N_{\mathbb{R}}$:*

$$P := \left\{ \sum_{i=1}^{r} a_i v_i \;\middle|\; a_i \geq 0, \sum_{i=1}^{r} a_i = 1 \right\}. \tag{3.53}$$

We can also define a ***face*** *of P by the intersection of P with a supporting affine hyperplane. To be specific, denote*

$$F_u := \{ v \in P \mid \langle u, v \rangle = r \} \tag{3.54}$$

the face generated by u, where $u \in M_{\mathbb{R}}$ is a linear function with $\langle u, v \rangle \geq r$ for any $v \in P$. Also denote by σ_F the polyhedral cone spanned by the face F:

$$\sigma_F := \left\{ \sum_{i=1}^{r} a_i v_i \;\middle|\; v_i \in F, a_i \geq 0 \right\}. \tag{3.55}$$

Definition 3.35 *A fan Δ in $N_{\mathbb{R}}$ is called* ***polytopal*** *if there exists a polytope $P \subseteq N_{\mathbb{R}}$ such that $0 \in \operatorname{int}(P)$ and Δ coincides with the collection of cones spanned by the faces of P.*

The key feature of toric varieties is that the geometric properties of smoothness, completeness and projectivity of X_Δ exactly correspond to the combinatorial properties of smoothness, completeness, and polytopality of Δ.

Theorem 3.36 *Let Δ be a fan in $N_{\mathbb{R}}$ and X_Δ the corresponding toric variety. The following hold.*

1. *X_Δ is always normal;*
2. *X_Δ is smooth if and only if Δ is smooth;*
3. *X_Δ is complete if and only if Δ is complete;*
4. *X_Δ is projective if and only if Δ is polytopal.*

We will now give partial proofs of Theorem 3.36, which includes 1 and the "if" part of 2. The proof of the "only if" part of 2 and the "only if" part of 3 will be postponed till the following sections. The proofs of remaining parts are more delicate, for which we only provide some references. For the "if" part of 3, see e.g. [2, Section 2.4] or [7, § VI, Theorem 9.1]; for 4, see e.g. [7, § VII.3].

Proof (of 1) Since normality is a local property, we can only focus on affine toric varieties. Suppose σ is a strongly convex lattice cone; then it is easy to see that σ can be generated by rays (i.e., one-dimensional cones). That is, we can take $v_1, \ldots v_r \in N$ generating the rays $\rho_1, \ldots \rho_r$, so that

$$\sigma = \sum_{i=1}^{r} \rho_i. \tag{3.56}$$

Thus we have $\check{\sigma} = \bigcap_i \check{\rho}_i$ and $S_\sigma = \bigcap_i S_{\rho_i}$, which shows that

$$R_\sigma = \bigcap_{i=1}^{r} \mathbb{C}[S_{\rho_i}]. \tag{3.57}$$

One can show that R_σ is normal if and only if $\mathbb{C}[S_{\rho_i}]$ is normal for any i. Without loss of generality, we can suppose that ρ is a ray generated by e_1 in $\mathbb{R}^n$. Then we explicitly compute

$$\mathbb{C}[S_\rho] \cong \mathbb{C}\left[z_1, z_2^{\pm 1}, \ldots, z_n^{\pm 1}\right], \tag{3.58}$$

which can be easily checked to be normal (the associated variety is $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$, which is in fact smooth). $\square$

Proof (of the "if" part of 2) We will prove that if Δ is smooth, then X_Δ is smooth. Similar to the proof of 1, we can only focus on affine toric varieties. If σ is a strongly convex lattice cone which is smooth, then we can choose a series of generators $v_1, \ldots, v_r \in N$ which can be completed in a $\mathbb{Z}$-basis of N. It can be seen from Example 2.46 that in this case we have

$$X_\sigma \cong \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}, \tag{3.59}$$

which is obviously smooth. $\square$

We finish this section by using 4 to show the existence of a complete but non-projective toric variety, which provides a counter-example stated in Remark 3.31.

Example 3.37 *Take the cube in $\mathbb{R}^3$ with all coordinates of the vertices ± 1. The 6 faces of this cube provide a polytopal fan containing one 0-dimensional cone (the origin), eight 1-dimensional cones, twelve 2-dimensional cones and six 3-dimensional cones. Now replace the point $(1,1,1)$ by $(1,2,3)$ and consider the corresponding new fan, denoted by Δ. It is clearly complete but not isomorphic to a polytopal one: we cannot construct a polytope whose vertices lie in each of the eight rays from the origin to each new point. We will show this in Chapter 4. Then from 3 and 4 in Theorem 3.36, we conclude that X_Δ is complete but not projective.*

3.3 From toric varieties to fans

We have seen how to construct a toric variety from a fan Δ, and discussed some geometric properties of X_Δ from the combinatorial information of Δ. A natural question is: does every toric variety arise from this construction? In other words:

Given a toric variety X,
can we construct a fan Δ such that $X \cong X_\Delta$?

Is this section we will approach this question, and discuss more about geometric properties of X. We will first answer this question by two steps: recovering the lattice from X and constructing all cones from X.

The lattice. Let us start with the lattice. Recall from Equation (3.10) that if X_Δ is the toric variety associated with a fan Δ in $N_\mathbb{R}$, then we have the natural isomorphism

$$T \cong T_\Delta := \mathrm{Hom}_{\mathrm{gp}}(M, \mathbb{C}^*), \tag{3.60}$$

where M is the dual lattice of N. After fixing a (non-canonical) isomorphism $N \cong \mathbb{Z}^n$, we have the induced isomorphism $T \cong (\mathbb{C}^*)^n$. As a consequence, Equation (3.60) becomes

$$(\mathbb{C}^*)^n \ni (t_1, \ldots, t_n) \longmapsto \left(M \ni \sum_{i=1}^n u_i e_i^* \mapsto \prod_{i=1}^n t_i^{u_i} \in \mathbb{C}^* \right). \tag{3.61}$$

This gives the torus in X_Δ in terms of the lattice and it dual. We can, however, recover the (dual) lattice from the torus itself in a similar fashion.

Recall that a character of a torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ that is a group homomorphism. For instance, $(u_1, \ldots, u_n) \in \mathbb{Z}^n$ gives a character $\chi^u\colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^u(t_1, \ldots, t_n) := t_1^{u_1} \cdots t_n^{u_n}. \tag{3.62}$$

We have shown in Lemma 1.4 that all characters of $(\mathbb{C}^*)^n$ arise this way. Thus the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T. It is customary to say that $u \in M$ gives the character $\chi^u\colon T \to \mathbb{C}^*$.

Dually, consider a map $\lambda\colon \mathbb{C}^* \to T$ that is a group homomorphism, called a cocaracher or one-parameter subgroup. For example, $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda_v\colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda_v(t) := (t^{v_1}, \ldots, t^{v_n}). \tag{3.63}$$

All one-parameter subgroups of $(\mathbb{C}^*)^n$ arise this way. It follows that the group of one-parameter subgroups of $(\mathbb{C}^*)^n$ is naturally isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, the one-parameter subgroups form a free abelian group N of rank equal to the dimension of T. As with the character group, an element $v \in N$ gives the one-parameter subgroup $\lambda_v \colon \mathbb{C}^* \to T$.

There is a natural bilinear pairing $\langle\, , \rangle : M \times N \to \mathbb{Z}$ defined as follows.

- (Intrinsic) Given a character χ^u and a one-parameter subgroup λ_v, the composition $\chi^u \circ \lambda_v \colon \mathbb{C}^* \to \mathbb{C}^*$ is character of $\mathbb{C}^*$, which is given by $t \mapsto t^k$ for some $k \in Z$. Then $\langle u, v\rangle := k$.
- (In coordinates) After fixing an isomorphism $T \cong (\mathbb{C}^*)^n$ which induces isomorphisms $M \cong \mathbb{Z}^n \ni u = (u_1, \ldots, u_n)$, and $N \cong \mathbb{Z}^n \ni v = (v_1, \ldots, v_n)$, then one computes that

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i, \tag{3.64}$$

  i.e., the pairing is the usual Euclidean product.

It follows that the characters and one-parameter subgroups of a torus T form free abelian groups M and N of finite rank with a perfect pairing $\langle\, , \rangle$ that makes them dual to each other.

As seen many times, picking an isomorphism $T \cong (\mathbb{C}^*)^n$ induces dual bases of M and N, i.e., isomorphisms $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$ that turn characters into Laurent monic monomials (cf. Equation (3.62)), one-parameter subgroups into monomial curves (cf. Equation (3.63)), and the pairing into the Euclidean product (cf. Equation (3.64)).

We can summarise the above discussion in the following definition.

Definition 3.38 *Let X be a toric variety of dimension n with torus T. Define the rank-n **character lattice** and **cocharacter lattice** by*

$$M := \mathrm{Hom}_{\mathrm{gp}}(T, \mathbb{C}^*), \qquad N := \mathrm{Hom}_{\mathrm{gp}}(\mathbb{C}^*, T). \tag{3.65}$$

*Elements of M are called characters, elements of N are called **cocharacters** or **one-parameter subgroups**. The character and cocharacter lattices are duals to each other via the perfect pairing*

$$\begin{aligned} M \times N &\longrightarrow \mathrm{Hom}_{\mathrm{gp}}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z} \\ (\chi^u, \lambda_v) &\longmapsto \chi^u \circ \lambda_u. \end{aligned} \tag{3.66}$$

The cones. We then turn to the second step: the construction of cones in $N_{\mathbb{R}}$. The method is to consider the limit point

$$x_v := \lim_{t \to 0} \lambda_v(t) \tag{3.67}$$

for $v \in N$ with corresponding one-parameter subgroup λ_v. The limiting process takes place in the closure of T, i.e. the whole variety X, while the existence of x_v has not been established by now. However, it is noticeable that if $X = X_\Delta$ for a fan Δ in $N_{\mathbb{R}}$, then the following hold.

1. For $v \in |\Delta| \cap N$, let τ be the (unique) cone of Δ containing v in its relative interior. Then for any $u \in \check{\tau} \cap M$, which implies $\langle u, v \rangle \geq 0$, the limit
$$\lim_{t \to 0} (\chi^u \circ \lambda_v)(t) = \lim_{t \to 0} t^{\langle u, v \rangle} \tag{3.68}$$
exists, hence we have a well-defined map
$$\begin{aligned} x_v \colon S_\tau &\longrightarrow \mathbb{C} \\ u &\longmapsto \lim_{t \to 0} (\chi^u \circ \lambda_v)(t). \end{aligned} \tag{3.69}$$
Moreover, one can easily see that x_v is exactly the distinguished point x_τ by directly comparing Equation (3.69) and Definition 3.3.

2. If $v \notin |\Delta| \cap N$, then for any cone $\tau \in \Delta$, there exists $u \in \check{\tau} \cap M$ such that $\langle u, v \rangle < 0$. Hence the definition of x_v through Equation (3.69) fails, which shows that x_v does not exist in X_Δ.

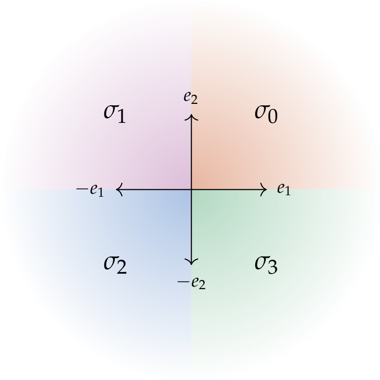
The discussion above not only provides another perspective for the distinguished points as the limit points of one-parameter subgroups, but also reconstruct the cones in Δ through the geometric information of X. The following theorem submits the discussion above.

Proposition 3.39 *Let X_Δ the toric variety associated with the fan Δ in $N_{\mathbb{R}}$. Then there is a well-defined correspondence*

$$\{\, \textit{limit points of one-parameter subgroups in } X_\Delta \,\} \overset{1:1}{\longleftrightarrow} \{\, \textit{cones in } \Delta \,\}, \tag{3.70}$$

which associates to each limit point x the closure σ_x of the positive real span of all cocharacters in $N_{\mathbb{R}}$ limiting to x.

Example 3.40 *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We have seen in Example 2.55 that $X \cong X_\Delta$ where Δ is the following fan.*



Let us check that the cones in Δ are indeed reconstructed from the limiting points. The torus $T \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ can be expressed by

$$(\mathbb{C}^*)^n \ni (t_1, t_2) \longmapsto ([1, t_1], [1, t_2]) \in \mathbb{P}^1 \times \mathbb{P}^1. \tag{3.71}$$

Using the identifications in Equation (3.63)*, we can assign to each $v = (a, b) \in \mathbb{Z}^2$ the cocharacter*

$$\lambda_v(t) = ([1, t^a], [1, t^b]) \in \mathbb{P}^1 \times \mathbb{P}^1. \tag{3.72}$$

One can easily write down all the nine possibilities of x_v corresponding to the nine cones in Δ:

(1) $x_v = ([1, 1], [1, 1])$ *when* $a = 0$, $b = 0$, *corresponding to* $\{0\}$.

(2) $x_v = ([1, 0], [1, 1])$ *when* $a > 0$, $b = 0$, *corresponding to* $\mathrm{Cone}(e_1)$.

(3) $x_v = ([1, 1], [1, 0])$ *when* $a = 0$, $b > 0$, *corresponding to* $\mathrm{Cone}(e_2)$.

(4) $x_v = ([0, 1], [1, 1])$ *when* $a < 0$, $b = 0$, *corresponding to* $\mathrm{Cone}(-e_1)$.

(5) $x_v = ([1, 1], [0, 1])$ *when* $a = 0$, $b < 0$, *corresponding to* $\mathrm{Cone}(-e_2)$.

(6) $x_v = ([1, 0], [1, 0])$ *when* $a > 0$, $b > 0$, *corresponding to* σ_0.

(7) $x_v = ([0, 1], [1, 0])$ *when* $a < 0$, $b > 0$, *corresponding to* σ_1.

(8) $x_v = ([0, 1], [0, 1])$ *when* $a < 0$, $b < 0$, *corresponding to* σ_2.

(9) $x_v = ([1, 0], [0, 1])$ *when* $a > 0$, $b < 0$, *corresponding to* σ_3.

In general, for a toric variety X, we can follow exactly the same steps as above and prove the following theorem, for which we omit the details.

Theorem 3.41 (From toric variety to fan) *Let X be a toric variety of dimension n with torus T. Define M and N as in Definition 3.38. Associate to each limit point*

x the closure σ_x of the positive real span of all cocharacters in $N_{\mathbb{R}}$ limiting to x. Then each σ_x is a cone and they together form a fan in $N_{\mathbb{R}}$ denoted by Δ. Moreover, the following isomorphism of toric varieties holds:

$$X \cong X_\Delta. \tag{3.73}$$

The idea of using one-parameter subgroups to "explore" the geometry of a toric variety is quite powerful and can be used to prove several properties. Let us apply it to prove completeness, as promised.

Proof (of the "only if" part of 3) We will prove that X_Δ complete implies Δ complete. Let $v \in N$ and let $\lambda_v \in \mathrm{Hom}_{\mathrm{gp}}(\mathbb{C}^*, T)$ denote the corresponding one-parameter subgroup. It can be shown that completeness of X_Δ implies the existence of the limit point x_v in X. Denote σ_x the corresponding cone as in Theorem 3.41, which implies that $v \in \sigma_x \cap N$. Hence every point in N is contained in some lattice cone in the fan Δ. This proves that $|\Delta| \cap N = N$, hence Δ is complete. $\square$

3.4 Polytopes

Written by Noah
Talks by Rasmus and Davide

As briefly discussed in the previous section, polytopes play a crucial role in the study of projective toric varieties. In this section, we will discuss the construction of toric varieties from polytopes in more detail.

As for cones, we are interested in duality statements. For reasons that will become clear shortly, we start with the definition of a polytope in the dual vector space V^* of a vector space V.

Definition 3.42 *A **polytope** P in a real vector space V^* is the convex hull of finitely many points: given $u_1, \dots, u_r \in V^*$,*

$$P := \left\{ \sum_{i=1}^{r} a_i u_i \;\middle|\; a_i \geq 0,\ \sum_{i=1}^{r} a_i = 1 \right\}. \tag{3.74}$$

If $V^* = M_{\mathbb{R}}$ for a lattice M, and the vectors u_i can be chosen in the lattice M, then P is called a lattice (or rational) polytope.

As we did for cones, we are interested in dual polytopes. This role is played by the polar polytope.

Definition 3.43 *Let $P \subset V^*$ be a polytope. Its **polar** $P^\circ \subset V$ is defined as*

$$P^\circ := \{\, v \in V \mid \langle u, v \rangle \geq -1 \text{ for all } u \in P \,\}. \tag{3.75}$$

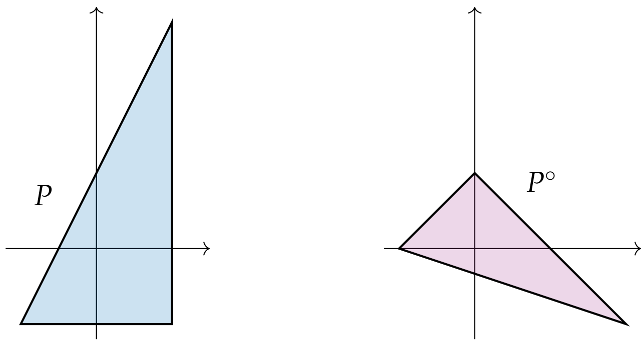
Some natural properties of polytopes follow.

Lemma 3.44 *Let $P \subset V^*$ be a polytope and $P^\circ \subset V$ its polar. Then:*

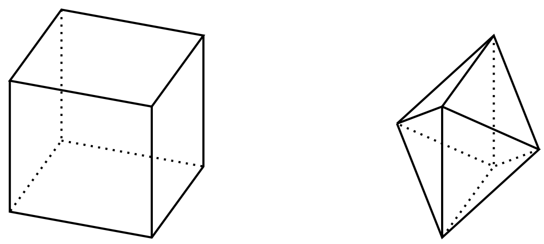
- *P° is a polytope.*
- *If P is rational with respect to a lattice M, then so is P° with respect to the dual lattice $N = \mathrm{Hom}_{\mathrm{gp}}(M, \mathbb{Z})$.*

Proof Consider the cone σ over $P \times \{1\} \subset V^* \times \mathbb{R}$. It is not hard to see that $\check{\sigma}$ is the cone over $P^\circ \times \{1\}$. The result then follows from the analogous results for cones. $\square$

Example 3.45 *Let P be the polytope in $\mathbb{R}^2$ generated by the points $(-1,-1)$, $(1,-1)$, and $(1,3)$. Then its polar is the polytope generated by $(-1,0)$, $(0,1)$, and $(2,-1)$.*



For a second example, consider the unit cube in $\mathbb{R}^3$, which is a polytope whose polar is the octahedron.



Intuitively, a face of a polytope is the intersection of the polytope with an affine hyperplane such that the polytope lies entirely on one side of the affine hyperplane. This naturally leads to the following definition.

Definition 3.46 *Let $P \subset V^*$ be a polytope. A proper* ***face*** *F is a subset of P such that*

$$F = \{ u \in P \mid \langle u, v \rangle = r \}, \tag{3.76}$$

where $v \in V$ is a linear form on V^* such that $\langle u, v \rangle \geq r$ for all $u \in P$. In other words, P lies on one side of the affine hyperplane, and the face is the intersection of the hyperplane with the polytope. The dimension of a face is the dimension of the affine space it spans.

Again, by considering the cone over $P \times \{1\}$, one can translate the one-to-one correspondence between faces of a cone and its dual to the polytopal setting.

Lemma 3.47 *Let $P \subset V^*$ be a polytope such that $0 \in \mathrm{int}(P)$, and let P° be its polar. Given a face F of P, define*

$$F^\circ := \{ v \in P^\circ \mid \langle u, v \rangle = -1 \text{ for all } u \in F \}. \tag{3.77}$$

Then F° is a face of P°. Moreover, the map $F \mapsto F^\circ$ defines a one-to-one, order-reversing correspondence between the faces of P and the faces of P°, satisfying the dimension formula

$$\dim(F) + \dim(F^\circ) = n - 1, \tag{3.78}$$

where $n = \dim(V)$.

As was hinted at above, to associate a toric variety to a polytope, we will first associate to a polytope a fan and proceed as before. To this end, we will associate to each face F of a polytope $P \subset V^*$ a cone $\sigma_F \subseteq V$ by setting

$$\sigma_F := \left\{ v \in V \;\middle|\; \langle u, v \rangle \leq \langle u', v \rangle \text{ for all } u \in F, u' \in P \right\}. \tag{3.79}$$

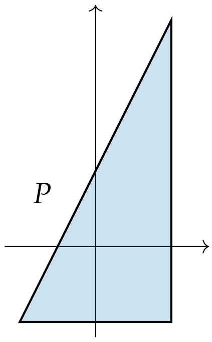
This definition allows for an easy characterisation of the dual cone $\check{\sigma}_F$, namely it is generated by $u' - u$ where u varies among the vertices of F and u' among the vertices of P.

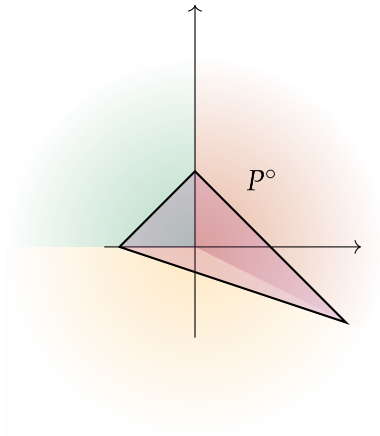
Proposition 3.48 *Let $P \subset M_{\mathbb{R}}$ a rational polytope. Then:*

1. *$\{ \sigma_F \subset N_{\mathbb{R}} \mid F$ is a face of $P \}$ defines a fan in $N_{\mathbb{R}}$, denoted Δ_P.*
2. *If $0 \in \mathrm{int}(P)$, then Δ_P consists of the cones over the faces of the polar P°. Moreover, it is complete.*

In what follows, given a rational polytope P, we denote the associated toric variety X_{Δ_P} simply by X_P.

Example 3.49 *For the polytope P generated by $(-1,-1)$, $(1,-1)$, and $(1,3)$, the fan Δ_P is depicted below. It is generated by faces of the polar.*





Chapter 4

(Co)homology

4.1 Divisors

Divisors provide a powerful way to understand the geometry of algebraic varieties by keeping track of the behaviour of functions and subvarieties. At a basic level, a divisor records the zeros and poles of a rational function. But the concept extends far beyond that, offering a flexible language to describe important geometric objects such as subvarieties of codimension-one and line bundles. In the context of toric varieties, divisors are especially tractable: they often admit explicit combinatorial descriptions in terms of the fan.

4.1.1 On algebraic varieties

There are two possible descriptions of divisors on a general algebraic variety: as formal sums of codimension-one subvarieties, and as local rational functions. Let us start with the former.

Definition 4.1 *Let X be an algebraic variety. A* ***Weil divisor*** *is an element of the form*

$$D = \sum_V a_V V, \tag{4.1}$$

where $a_V \in \mathbb{Z}$ is an integer and the sum is over all closed, irreducible, codimension-one subvarieties of X. The coefficients a_V are zero for all but finitely many V. Weil divisors form a group under addition, denoted $\mathrm{WDiv}(X)$.

Example 4.2 *Consider $X = \mathbb{C}^2$ with coordinates (z, w). Let*

$$A = \left\{ (z,w) \in \mathbb{C}^2 \mid z = 0 \right\}, \qquad B = \left\{ (z,w) \in \mathbb{C}^2 \mid w = 0 \right\}. \tag{4.2}$$

These are closed, irreducible, codimension-one subvarieties. Thus, $D = 2A - B \in \mathrm{WDiv}(X)$ *is an example of a Weil divisor.*

From now on, let X be a normal algebraic variety. The normality assumption is not strictly necessary, but it will simplify several statements.

As mentioned above, a second approach to divisors is via rational functions. Denote by $\mathcal{K}(X)$ the set of **rational functions** on X. We will provide a precise definition in the next section, and for now stick to the intuitive notion. Given a non-zero rational function f on X, we can consider the set of zeros and poles of f. We define the **order** of f along a closed, irreducible, codimension-one subvariety V as

$$\mathrm{ord}_V(f) := \begin{cases} 0 & \text{if } f \text{ is regular and invertible on } V, \\ k & \text{if } f \text{ vanishes along } V \text{ with multiplicity } k, \\ -k & \text{if } f \text{ has a pole along } V \text{ with multiplicity } k. \end{cases} \tag{4.3}$$

In other words, the order is an integer that measures the multiplicity of a zero or pole of the function along that subvariety.

Definition 4.3 *Let* $f \in \mathcal{K}(X)$ *be a non-zero rational function. The* ***principal divisor*** *associated to* f *is given by*

$$\mathrm{div}(f) := \sum_V \mathrm{ord}_V(f)\, V \in \mathrm{WDiv}(X). \tag{4.4}$$

Denote by $\mathrm{PDiv}(X)$ *the set of principal divisors in* X*, which forms a subgroup of* $\mathrm{WDiv}(X)$*, as*

$$\mathrm{div}(f \cdot g) = \mathrm{div}(f) + \mathrm{div}(g). \tag{4.5}$$

Moreover $\mathrm{div}(u \cdot f) = \mathrm{div}(f)$ *if* u *is an invertible regular function.*

Example 4.4 *Consider again* $X = \mathbb{C}^2$ *with coordinates* (z, w)*. Let* $f = z^2/w$*, which is a rational function on* $\mathbb{C}^2$*. Then, with the notation of Example 4.2,*

$$\mathrm{ord}_V(f) = \begin{cases} 2 & \text{if } V = A, \\ -1 & \text{if } V = B, \\ 0 & \text{otherwise}. \end{cases} \tag{4.6}$$

Therefore, the divisor $D = 2A - B = \mathrm{div}(f)$ *is a principal divisor.*

It will prove useful to relax the definition of a principal divisor to allow rational functions that are only defined on subsets $U \subset X$.

Definition 4.5 *A Weil divisor $D = \sum_V a_V V$ is called a **Cartier divisor** if there exists an open cover $(U_i)_{i \in I}$ such that the restrictions $D|_{U_i} = \sum_V a_V (U_i \cap V)$ are principal divisors in U_i for every $i \in I$. Equivalently, a Cartier divisor is the data of:*

- *An open cover (U_i) of X,*
- *Rational functions f_i on U_i,*
- *Such that $\frac{f_i}{f_j}$ is an invertible regular function on the intersection $U_i \cap U_j$.*

Cartier divisors form a group, denoted $\mathrm{CDiv}(X)$*.*

The definitions make it clear that we have the inclusions

$$\mathrm{PDiv}(X) \subseteq \mathrm{CDiv}(X) \subseteq \mathrm{WDiv}(X). \tag{4.7}$$

4.1.2 On toric varieties

Consider now a toric variety $X = X_\Delta$ associated with a fan Δ in $N_\mathbb{R}$. The torus action $T \curvearrowright X$ induces a natural action on Weil divisors:

$$T \times \mathrm{WDiv}(X_\Delta) \longrightarrow \mathrm{WDiv}(X_\Delta), \qquad t \cdot \left(\sum_V a_V \, V \right) := \sum_V a_V \, (t \cdot V), \tag{4.8}$$

where $t \cdot V = \{ \, t \cdot x \mid x \in V \, \}$ is the action of T on V. This action naturally restricts to Cartier and principal divisors. Thus, it makes sense to consider the groups of torus-invariant Weil, Cartier, and principal divisors, denoted $\mathrm{WDiv}^T(X_\Delta)$, $\mathrm{CDiv}^T(X_\Delta)$, and $\mathrm{PDiv}^T(X_\Delta)$ respectively. The main advantage of working with torus-invariant divisors is that the associated groups are much smaller and more tractable. Our next goal is to understand these groups.

The easiest to understand is the group of torus-invariant Weil divisors. Indeed, in the last chapter, we saw that the closures of torus orbits are T-invariant, closed, irreducible subvarieties. Conversely, every torus-invariant, closed, irreducible subvariety is the closure of a torus orbit. By the cone-orbit correspondence, the closures of orbits are of the form $\overline{O_\sigma}$ for cones $\sigma \in \Delta$. Moreover,

$$\mathrm{codim}(\overline{O_\sigma}) = \dim(\sigma), \tag{4.9}$$

so codimension one corresponds to rays (i.e. one-dimensional cones) in Δ. For notational convenience, define

$$\Delta_1 := \{ \, \rho \in \Delta \mid \dim(\rho) = 1 \, \}, \qquad D_\rho := \overline{O_\rho}, \tag{4.10}$$

as the set of rays and their associated divisors. The above discussion implies the following.

Proposition 4.6 (T-invariant Weil divisors) *Torus-invariant Weil divisors are freely generated by rays:*

$$\mathrm{WDiv}^T(X_\Delta) \cong \bigoplus_{\rho\in\Delta_1} \mathbb{Z}\cdot D_\rho. \tag{4.11}$$

Our next goal is to understand the group of torus-invariant Cartier divisors. First, notice that for each $u \in M$, the associated character $\chi^u\colon T \to \mathbb{C}^*$ is a rational function, since T is dense in X_Δ. Moreover, χ^u is T-invariant. Thus, it makes sense to consider $\mathrm{div}(\chi^u)$. The next lemma expresses $\mathrm{div}(\chi^u)$ in terms of the $\mathbb{Z}$-basis of divisors associated with rays.

Lemma 4.7 *For each ray $\rho \in \Delta_1$, let $v_\rho \in N$ be the minimal generator of ρ, i.e. the first lattice point along ρ. Then*

$$\mathrm{div}(\chi^u) = \sum_{\rho\in\Delta_1} \langle u, v_\rho\rangle\, D_\rho. \tag{4.12}$$

In other words, $\mathrm{ord}_{D_\rho}(\chi^u) = \langle u, v_\rho\rangle$.

Since we have not yet formally defined the notion of rational function or that of order, we omit the proof and refer to [2, Section 3.3].

It can be shown that, for X_σ affine, the map $u \mapsto \mathrm{div}(\chi^u)$ is surjective onto $\mathrm{CDiv}^T(X_\sigma)$. That is, every torus-invariant Cartier divisor on an affine toric variety is the divisor of a character. The situation is more subtle for non-affine toric varieties. Indeed, if D and D' are T-invariant Cartier divisors on X_Δ, then for every cone $\sigma \in \Delta$ we have

$$D|_{X_\sigma} = \mathrm{div}(\chi^u), \qquad D'|_{X_\sigma} = \mathrm{div}(\chi^{u'}) \tag{4.13}$$

for some $u, u' \in M$. Furthermore,

$$\mathrm{div}(\chi^u) = \mathrm{div}(\chi^{u'}) \qquad \Longleftrightarrow \qquad u - u' \in \sigma^\perp \cap M = M(\sigma). \tag{4.14}$$

Thus, a torus-invariant Cartier divisor is defined by the local data $u(\sigma) \in M/M(\sigma)$, which determines a local rational function on the affine patch X_σ. We may restrict to the affine patches corresponding to the maximal cones in the fan:

$$\Delta_{\max} := \{\, \sigma \in \Delta \mid \sigma \text{ is maximal} \,\}. \tag{4.15}$$

However, the local data $(X_\sigma, u(\sigma))_{\sigma\in\Delta_{\max}}$ must agree on intersections. This discussion outlines the proof of the following.

Proposition 4.8 (*T*-invariant Cartier divisors) *Torus-invariant Cartier divisors are given by:*

$$\mathrm{CDiv}^T(X_\Delta) \cong \ker \left(\bigoplus_{\sigma \in \Delta_{\max}} M/M(\sigma) \to \bigoplus_{\substack{\sigma', \sigma'' \in \Delta_{\max} \\ \sigma' \neq \sigma''}} M/M(\sigma' \cap \sigma'') \right). \quad (4.16)$$

We conclude this section with a characterisation of torus-invariant Cartier divisors inside the group of Weil divisors.

Proposition 4.9 *A torus-invariant Weil divisor $D = \sum_{\rho \in \Delta_1} a_\rho D_\rho \in \mathrm{WDiv}^T(X_\Delta)$ is Cartier if and only if, for all maximal cones $\sigma \in \Delta_{\max}$, there exists $u(\sigma) \in M/M(\sigma)$ such that $\langle u(\sigma), v_\rho \rangle = -a_\rho$ for all $\rho \in \Delta_1$ that are faces of σ. Here, $v_\rho \in N$ denotes the minimal generator of ρ.*

Remark 4.10 *In the pairing $\langle u(\sigma), v_\rho \rangle$, we may pick any representative of $u(\sigma)$ in $M/M(\sigma)$. Indeed, if u and u' in M are two different representatives, then $\langle u - u', v_\rho \rangle = 0$, since $u - u' \in M(\sigma) = \sigma^\perp \cap M$ and ρ is a face of σ.*

The above results can be used to show that not all Weil divisors are Cartier.

Example 4.11 *Let Δ be the fan of the double cone, see Example 2.40. The rays of Δ are generated by $v_1 = 2e_1 - e_2$ and $v_2 = e_2$. For $(a, b) \in M = \mathbb{Z}^2$, the associated divisor is*

$$\mathrm{div}(\chi^u) = \langle u, v_1 \rangle D_1 + \langle u, v_2 \rangle D_2 = (2a - b)D_1 + bD_2, \quad (4.17)$$

where D_i is the closure of the torus orbit associated with the ray generated by v_i. Since X_Δ is affine, we know that $u \mapsto \mathrm{div}(\chi^u)$ is surjective onto torus-invariant Cartier divisors. We conclude that $D_2 \notin \mathrm{CDiv}(X_\Delta)$ is not a Cartier divisor, while $2D_2 \in \mathrm{CDiv}(X_\Delta)$ is.

4.1.3 Support functions

Proposition 4.8 describes torus-invariant Cartier divisors. However, it can be difficult to work with such characterisation directly. A more convenient characterisation is in terms of support functions.

Definition 4.12 *Let Δ be a fan in $N_\mathbb{R}$. A **support** (or **piecewise linear**) **function** on Δ is a continuous function $\psi\colon |\Delta| \to \mathbb{R}$ satisfying the following conditions.*

- piecewise linearity*: ψ is linear on each cone of Δ, i.e., for each $\sigma \in \Delta$ there exists a $u_\sigma \in M_\mathbb{R}$ such that $\psi(v) = \langle u_\sigma, v \rangle$ for all $v \in \sigma$.*

- Integrality*: ψ takes integral values on the lattice, i.e., $\psi(|\Delta| \cap N) \subseteq \mathbb{Z}$.*

The set of support functions on Δ *is denoted* $\mathrm{SF}(\Delta)$*.*

Before providing a characterisation of torus-invariant Cartier divisors in terms of support functions, let us give an important example of a support function associated with the fan of a polytope (see Section 3.4).

Let $P \subset M_{\mathbb{R}}$ be a rational polytope with $0 \in \mathrm{int}(P)$, and let Δ_P be its associated fan in $N_{\mathbb{R}}$. Define

$$\psi_P \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \qquad \psi_P(v) := \min_{u \in P} \langle u, v \rangle . \tag{4.18}$$

This function is not only a support function, but it also allows for the reconstruction of the polytope P, as well as all its faces.

Lemma 4.13 *The function ψ_P is a support function on Δ_P. Furthermore, the polytope P is given in terms of ψ_P as*

$$P = \{ u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq \psi_P(v) \ \forall v \in N_{\mathbb{R}} \} , \tag{4.19}$$

and the cone σ_F of a face F of P is given by

$$\sigma_F = \{ v \in N_{\mathbb{R}} \mid \langle u, v \rangle = \psi_P(v) \ \forall u \in F^\circ \} . \tag{4.20}$$

Example 4.14 *Consider the polytope P from Example 3.49, namely the polytope generated by $(-1,-1)$, $(1,-1)$, and $(1,3)$, with fan Δ_P generated by the cones*

$$\sigma_1 = \mathrm{Cone}(e_1, e_2), \qquad \sigma_2 = \mathrm{Cone}(-e_1, e_2), \qquad \sigma_3 = \mathrm{Cone}(-e_1, 2e_1 - e_2). \tag{4.21}$$

See the figure below. The support function ψ_P is given by

$$\psi_P(v) = \langle e_1^* - e_2^*, v \rangle \tag{4.22}$$

for all $v \in \mathbb{R}^2$. More generally, every support function on Δ_P is of the form

$$\psi(v) = \begin{cases} a_1 x + b_1 y & \text{if } v = x e_1 + y e_2 \in \sigma_1, \\ a_2 x + b_2 y & \text{if } v = x e_1 + y e_2 \in \sigma_2, \\ a_3 x + b_3 y & \text{if } v = x e_1 + y e_2 \in \sigma_3, \end{cases} \tag{4.23}$$

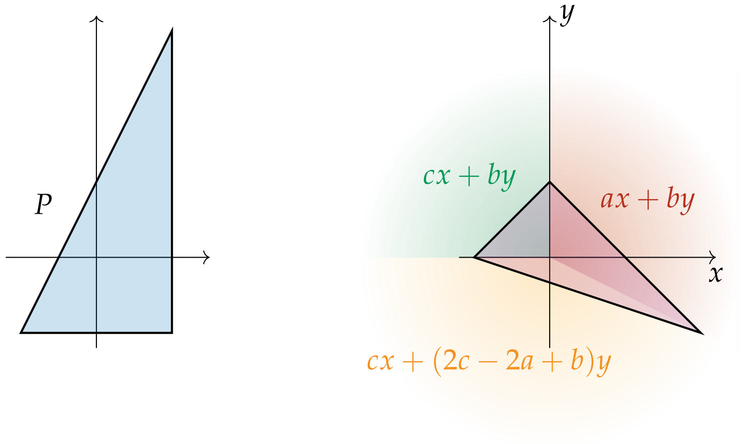
for some $(a_i, b_i) \in \mathbb{Z}^2$. Imposing continuity along the faces, we find the conditions

$$\begin{cases} b_1 = b_2 & \text{along } \sigma_1 \cap \sigma_2, \\ -a_2 = -a_3 & \text{along } \sigma_2 \cap \sigma_3, \\ 2a_3 - b_3 = 2a_1 - b_1 & \text{along } \sigma_3 \cap \sigma_1. \end{cases} \tag{4.24}$$

Choosing $a = a_1$, $b = b_1$, and $c = a_2$ as free parameters, we find that the general form of a support function on Δ_P is

$$\psi(v) = \begin{cases} ax + by & \text{if } v = xe_1 + ye_2 \in \sigma_1, \\ cx + by & \text{if } v = xe_1 + ye_2 \in \sigma_2, \\ cx + (2c - 2a + b)y & \text{if } v = xe_1 + ye_2 \in \sigma_3, \end{cases} \tag{4.25}$$

for any integers a, b, c. It can be visualised as follows.



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Talks by Elisa L. and Janine

Using Proposition 4.9, we can associate every Cartier divisor to a support function. Specifically, for a Cartier divisor D, we have that for each maximal cone $\sigma \in \Delta_{\max}$, there exists an element $u(\sigma) \in M/M(\sigma)$ such that

$$\langle u(\sigma), v_\rho \rangle = -a_\rho \quad \text{for all } \rho \preceq \sigma, \tag{4.26}$$

where $D = \sum_{\rho \in \Delta_1} a_\rho D_\rho$. We can then define a function

$$\psi_D \colon |\Delta| \longrightarrow \mathbb{R}, \qquad v \longmapsto \langle u(\sigma), v \rangle \quad \text{for } v \in \sigma. \tag{4.27}$$

Note that the function ψ_D is well-defined, i.e., the pairing $\langle u(\sigma), v \rangle$ is independent of the choice of representative. Indeed, if $u, u' \in M$ are two different representatives, then $\langle u - u', v \rangle = 0$ for all $v \in \sigma$, since $u - u' \in M(\sigma) = \sigma^\perp \cap M$.

Proposition 4.15 *Let $D \in \mathrm{CDiv}^T(X_\Delta)$ be a Cartier divisor. Then the map $\psi_D \colon |\Delta| \to \mathbb{R}$ defined above is a support function. Furthermore, the map*

$$D \longmapsto \psi_D \tag{4.28}$$

induces a group isomorphism

$$\mathrm{CDiv}^T(X_\Delta) \cong \mathrm{SF}(\Delta). \tag{4.29}$$

Proof The first two parts follow from the construction: ψ_D is continuous because the Cartier data agree on intersections, it is linear on each cone, and integral by definition. The fact that the map $D \mapsto \psi_D$ is a group homomorphism follows from the linearity of the definition. We are then left with proving that it is a bijection.

For the injectivity, suppose $D = \sum_{\rho \in \Delta_1} a_\rho D_\rho$. Then for each ρ, we have

$$a_\rho = -\psi_D(u_\rho), \tag{4.30}$$

which uniquely determines the coefficients a_ρ. Hence, ψ_D determines D, proving injectivity. As for surjectivity, let $\psi \in \mathrm{SF}(\Delta)$. By the integrality condition, ψ restricts to an $\mathbb{N}$-linear map on each cone: $\psi|_{\sigma \cap N} \colon \sigma \cap N \to \mathbb{Z}$. In particular, by extending linearly to the negative of the cone, we obtain an $\mathbb{Z}$-linear map $\Psi_\sigma \colon N_\sigma \to \mathbb{Z}$, where $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$. Since $N_\sigma^* = M(\sigma)$, there exists $u(\sigma) \in M(\sigma)$ such that $\psi|_\sigma = \langle u(\sigma), \cdot \rangle$. Thus,

$$\psi = \psi_D \qquad \text{for } D = -\sum_{\rho \in \Delta_1} \psi(v_\rho) D_\rho, \tag{4.31}$$

which proves surjectivity. □

We will come back to an application of the above characterization after a brief discussion of the class and Picard groups.

4.1.4 Class and Picard groups

Let X be a normal algebraic variety. The groups of Weil and Cartier divisors of X are often too large to control. A more manageable group can be obtained by modding out principal divisors.

Definition 4.16 *Let $D, D' \in \mathrm{WDiv}(X)$ (or $D, D' \in \mathrm{CDiv}(X)$). We say that D and D' are **linearly equivalent** whenever $D - D' \in \mathrm{PDiv}(X)$. This defines an equivalence relation.*

*We then define the **divisor class group** and the **Picard group** as the groups of Weil and Cartier divisors, modulo such equivalence:*

$$\mathrm{Cl}(X) := \frac{\mathrm{WDiv}(X)}{\mathrm{PDiv}(X)}, \qquad \mathrm{Pic}(X) := \frac{\mathrm{CDiv}(X)}{\mathrm{PDiv}(X)}. \tag{4.32}$$

Elements of these groups are denoted by $[D]$.

The class and Picard groups are in general difficult to compute. However, for toric varieties, we can describe them more easily through the group of toric-invariant divisors.

Theorem 4.17 *Let Δ be a fan in $N_{\mathbb{R}}$. Then we have the exact sequence:*

$$M \longrightarrow \mathrm{WDiv}^T(X_\Delta) \longrightarrow \mathrm{Cl}(X_\Delta) \longrightarrow 0, \tag{4.33}$$

where the first map sends $u \in M$ to $\mathrm{div}(\chi^u)$, and the second map sends a toric-invariant Weil divisor to its class in $\mathrm{Cl}(X_\Delta)$. Furthermore, this sequence is short exact, i.e.,

$$0 \longrightarrow M \longrightarrow \mathrm{WDiv}^T(X_\Delta) \longrightarrow \mathrm{Cl}(X_\Delta) \longrightarrow 0 \tag{4.34}$$

if and only if the set of rays Δ_1 spans $N_{\mathbb{R}}$.

Since $\mathrm{div}(\chi^u)$ is a Cartier divisor for all $u \in M$, we can replace $\mathrm{WDiv}^T(X_\Delta)$ with $\mathrm{CDiv}^T(X_\Delta)$ and $\mathrm{Cl}(X_\Delta)$ with $\mathrm{Pic}(X_\Delta)$. We then obtain the following result:

Corollary 4.18 *Let Δ be a fan in $N_{\mathbb{R}}$. Then we have the exact sequence:*

$$M \longrightarrow \mathrm{CDiv}^T(X_\Delta) \longrightarrow \mathrm{Pic}(X_\Delta) \longrightarrow 0, \tag{4.35}$$

where the first map sends $u \in M$ to $\mathrm{div}(\chi^u)$, and the second map sends a toric-invariant Cartier divisor to its class in $\mathrm{Pic}(X_\Delta)$. Again, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{CDiv}^T(X_\Delta) \longrightarrow \mathrm{Pic}(X_\Delta) \longrightarrow 0 \tag{4.36}$$

if and only if the set of rays Δ_1 spans $N_{\mathbb{R}}$.

Combining Proposition 4.15 and Corollary 4.18 above, we obtain a useful way to compute the Picard group of a toric variety:

Corollary 4.19 *Let Δ be a fan in $N_{\mathbb{R}}$. Then we have the exact sequence:*

$$M \longrightarrow \mathrm{SF}(\Delta) \longrightarrow \mathrm{Pic}(X_\Delta) \longrightarrow 0, \tag{4.37}$$

where the first map sends $u \in M$ to the support function $v \mapsto -\langle u, v\rangle$, and the second map sends a support function $\psi \in \mathrm{SF}(\Delta)$ to the class

$$\left[-\sum_{\rho \in \Delta_1} \psi(v_\rho) D_\rho \right] \in \mathrm{Pic}(X_\Delta). \tag{4.38}$$

Again, the sequence is short exact if and only if Δ_1 spans $N_{\mathbb{R}}$.

Notice that the map $v \mapsto -\langle u, v\rangle$ is a *globally* linear, integral function (rather than piecewise linear). Hence, if Δ_1 spans $N_{\mathbb{R}}$, we find that:

The Picard group is isomorphic to
integral piecewise linear functions on Δ
modulo integral linear functions.

We can use the above corollary to prove that the fan from Example 3.37 is non-polytopal. To this end, we need a fact from the general theory of polytopes.

Lemma 4.20 *Let* $P \subset M_{\mathbb{R}}$ *be a polytope with* $0 \in \mathrm{int}(P)$. *Recall that we have a canonical support function on* Δ_P*, the complete fan associated with* P*, defined as*

$$\psi_P(v) := \min_{u \in P} \langle u, v \rangle. \tag{4.39}$$

Then $[\psi_P] \in \mathrm{Pic}(X_P)$ *is non-zero.*

In other words, projective toric varieties have non-trivial Picard group. This gives a criterion to prove that a toric variety is non-projective, or equivalently, that a fan is non-polytopal.

Example 4.21 (Continuation of Example 3.37) *Recall the definition of the fan* Δ *in* $\mathbb{R}^3$ *obtained by replacing* $(1,1,1)$ *with* $(1,2,3)$ *in the standard cube. Its rays are generated by:*

$$\begin{gathered} v_1 = (1,2,3),\ v_2 = (1,-1,1),\ v_3 = (1,1,-1),\ v_4 = (-1,-1,1), \\ v_5 = (1,-1,-1),\ v_6 = (-1,-1,1),\ v_7 = (-1,1,-1),\ v_8 = (-1,-1,-1). \end{gathered} \tag{4.40}$$

See Figure 4.1 for a representation. We have the following maximal cones:

$$\begin{aligned} \sigma_1 &= \mathrm{Cone}(v_1, v_2, v_3, v_5), & \sigma_2 &= \mathrm{Cone}(v_1, v_3, v_4, v_7), \\ \sigma_3 &= \mathrm{Cone}(v_1, v_2, v_4, v_6), & \sigma_4 &= \mathrm{Cone}(v_2, v_5, v_6, v_8), \\ \sigma_5 &= \mathrm{Cone}(v_3, v_5, v_7, v_8), & \sigma_6 &= \mathrm{Cone}(v_4, v_6, v_7, v_8). \end{aligned} \tag{4.41}$$

Our goal is to show that all support functions are linear, so that by Corollary 4.19 we deduce that $\mathrm{Pic}(X_\Delta) = 0$ *and as a consequence* Δ *cannot be polytopal. To this end, fix* $\psi \in \mathrm{SF}(\Delta)$ *and take* $u \in M$ *such that* $\psi|_{\sigma_1} = \langle u, \cdot \rangle$. *Define* $\varphi := \psi - \langle u, \cdot \rangle$, *which vanishes on* v_1, v_2, v_3, v_5. *We want to show that* $\varphi = 0$ *everywhere, so that* $\psi = \langle u, \cdot \rangle$ *everywhere.*

First, notice that each cone σ_i *has four generators. Since we are in* $\mathbb{R}^3$, *the four generators of each cone must be linearly dependent, i.e. they must satisfy a linear relation.*

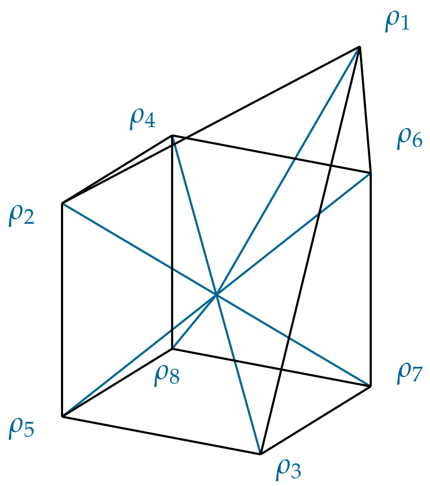

Figure 4.1: Fan associated with the modified cube Δ in $\mathbb{R}^3$, where the vertex $(1,1,1)$ is replaced with $(1,2,3)$. The rays $\rho_1, \ldots, \rho_8$ generate the fan structure.

Cone	*Relation*
σ_1	$2v_1 + 5v_5 = 4v_2 + 3v_3$
σ_2	$2v_1 + 4v_7 = 3v_3 + 5v_4$
σ_3	$2v_1 + 3v_6 = 4v_2 + 5v_4$
σ_4	$v_2 + v_8 = v_5 + v_6$
σ_5	$v_3 + v_8 = v_5 + v_7$
σ_6	$v_4 + v_8 = v_6 + v_7$

Applying φ to the relations associated with $\sigma_2, \ldots, \sigma_8$ and using that φ evaluates to zero on v_1, v_2, v_3, v_5, we obtain:

$$4\varphi(v_7) = 5\varphi(v_4), \quad 3\varphi(v_6) = 5\varphi(v_4), \quad \varphi(v_8) = \varphi(v_6) = \varphi(v_7). \tag{4.42}$$

It follows that φ vanishes on all rays, so $\psi = \langle u, \cdot \rangle$ globally. Hence, $\mathrm{Pic}(X_\Delta) = 0$*, and Δ is non-polytopal.*

4.2 Sheaves and cohomology

Sheaves provide a natural framework to study the geometric objects associated with a variety, such as divisors. In particular, the Picard group, which classifies Cartier divisors up to principal divisors, can be understood through the language of sheaves and their cohomology.

4.2.1 Sheaf cohomology

We define sheaves in a more general context than what we have been working with so far. Rather than restricting ourselves to algebraic varieties, we consider a general topological space X.

Definition 4.22 *A **pre-sheaf** $\mathcal{F}$ on X assigns an abelian group $\mathcal{F}(U)$ to every open set $U \subseteq X$, together with a map*

$$r_{UV} \colon \mathcal{F}(V) \longrightarrow \mathcal{F}(U) \tag{4.43}$$

for each inclusion $U \subseteq V$, such that:

1. *$\mathcal{F}(\emptyset) = 0$;*
2. *$r_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and for $U \subseteq V \subseteq W$, we have $r_{UV} \circ r_{VW} = r_{UW}$, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "r_{VW}"] \arrow[rr, bend right, "r_{UW}"'] & \mathcal{F}(V) \arrow[r, "r_{UV}"] & \mathcal{F}(U) \end{tikzcd} \tag{4.44}$$

*An element $s \in \mathcal{F}(U)$ is called a **local section** of $\mathcal{F}$ over U. An element $s \in \mathcal{F}(X)$ is called a **global section**. The map $r_{UV} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is called the **restriction map**; for a section s of V, we often write $r_{UV}(s)$ as $s|_U$ when the larger open set is clear from context.*

Remark 4.23 *For the categorical enjoyers, a presheaf can be defined as follows. First, given a topological space X, one can define the category* **Open**(X)*, whose objects are the open subsets of X, and with a single morphism from U to V if and only if $U \subseteq V$. The category* **Open**(X) *is called the poset category of open sets on X. Then, a presheaf is a contravariant functor from* **Open**(X) *to the category* **Ab** *of abelian groups.*

Intuitively, a pre-sheaf is an assignment of function-like objects to each open set, which can be consistently restricted when we pass to smaller open sets.

Example 4.24

- $\mathcal{C}^0(U) := \{\, f \colon U \to \mathbb{R} \mid f \text{ is continuous} \,\}$, *with r_{UV} the usual restriction;*
- $\underline{\mathbb{R}}^{\mathrm{psh}}(U) := \{\, f \colon U \to \mathbb{R} \mid f \text{ is constant} \,\} \cong \mathbb{R}$, *with r_{UV} the usual restriction.*

A pre-sheaf is called a sheaf if it satisfies a glueing property.

Definition 4.25 *A pre-sheaf $\mathcal{F}$ is called a* ***sheaf*** *if the following conditions hold for all U open covered by open sets $\{ U_i \}_{i \in I}$.*

- *Locality. If we have sections $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t_{U_i}$ for all $i \in I$, the $s = t$.*
- *Glueing. If we have sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.*

Intuitively, if a pre-sheaf assigns to each open set function-like objects satisfying a certain property $\mathscr{P}$, then it is a sheaf if the property $\mathscr{P}$ is *local*. Let us see this in practice with an example.

Example 4.26 *We now revisit Example 4.24:*

- *The pre-sheaf $C^0(U) = \{ f \colon U \to \mathbb{R} \mid f \text{ is continuous} \}$ is a sheaf. Indeed, given continuous functions f_i on U_i that agree on overlaps $U_i \cap U_j$, we can define a unique continuous function f on U by setting $f(x) = f_i(x)$ if $x \in U_i$.*
- *The pre-sheaf $\underline{\mathbb{R}}^{\mathrm{psh}}(U) = \{ f \colon U \to \mathbb{R} \mid f \text{ is constant} \}$ is not a sheaf. For instance, consider two disjoint open sets U_0 and U_1, and define $f_0 \equiv 0$ on U_0 and $f_1 \equiv 1$ on U_1. These agree trivially on $U_0 \cap U_1 = \emptyset$, but cannot be glued to a constant function on $U_0 \cup U_1$.*

The constant pre-sheaf $\underline{\mathbb{R}}^{\mathrm{psh}}$ can be turned into an actual sheaf, denoted $\underline{\mathbb{R}}$, by turning the condition of being constant into a local property as follows:

$$\underline{\mathbb{R}}(U) := \bigoplus_{U_0 \in \pi_0(U)} \{ f \colon U_0 \to \mathbb{R} \mid f \text{ is constant} \} \cong \mathbb{R}^{\pi_0(U)}, \tag{4.45}$$

where the direct sum is over all connected components of U. A similar definition can be given for any abelian group G: set

$$\underline{G}(U) := G^{\pi_0(U)}, \tag{4.46}$$

called the ***constant sheaf*** *$\underline{G}$.*

Let us now focus on the case $X = \mathrm{Specm}(R)$, an affine variety. We are interested in constructing a sheaf $\mathcal{O}_X$ on X that captures the behaviour of regular algebraic functions on open subsets. These should be thought of as the analogues of continuous functions in topology, differentiable functions in real analysis, or holomorphic functions in complex analysis. As a first attempt, consider the pre-sheaf $\mathcal{O}_X^{\mathrm{psh}}$ defined as follows:

$$\mathcal{O}_X^{\mathrm{psh}}(U) := \left\{ \varphi \colon U \to \mathbb{C} \;\middle|\; \varphi = \frac{f}{g} \text{ for } f, g \in R,\ g \neq 0 \text{ on } U \right\}, \tag{4.47}$$

where $U \subset X$ is an open subset. However, this does not define a sheaf, because being a ratio of elements in the coordinate ring is a global condition.

As an example, consider the affine variety $X = \mathsf{V}(x_1x_4 - x_2x_3) \subset \mathbb{C}^4$, and let $U = X \setminus \mathsf{V}(x_2, x_4) = \{ (x_1, \dots, x_4) \in X \mid x_2 \neq 0 \text{ or } x_4 \neq 0 \}$. Define the function $\varphi\colon U \to \mathbb{C}$ by

$$\varphi(x_1, x_2, x_3, x_4) := \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0, \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0. \end{cases} \tag{4.48}$$

This definition is well-posed, since $x_1x_4 = x_2x_3$ on X. However, neither of the two expressions in Equation (4.48) defines a regular function on all of U: the first fails at $(0, 0, 0, 1)$, the second at $(0, 1, 0, 0)$.

The problem can be resolved by modifying the condition to be local (or, as algebraic geometers like to say, by sheafifying $\mathcal{O}_X^{\text{psh}}$).

Definition 4.27 *Let $X = \operatorname{Specm}(R)$ be an affine variety. The* ***sheaf of regular functions*** *(or* ***structure sheaf****) on X, denoted $\mathcal{O}_X$, is defined by*

$$\mathcal{O}_X(U) := \left\{ \varphi\colon U \to \mathbb{C} \;\middle|\; \begin{array}{c} \forall x_0 \in U,\ \varphi = \frac{f}{g} \text{ on a neighbourhood } U_x \text{ of } x \\ \text{for some } f, g \in R,\ g \neq 0 \text{ on } U_x \end{array} \right\}. \tag{4.49}$$

The restriction maps are the usual restrictions of functions.

We record two useful properties of the sheaf of regular functions on $X = \operatorname{Specm}(R)$:

- The global sections are precisely the elements of the coordinate ring: $\mathcal{O}_X(X) = R$.
- If $U = X \setminus \mathsf{V}(f) \subset X$ is a principal open subset, for $f \in R$, then the corresponding sections are exactly the localisation of R at f: $\mathcal{O}_X(U) \cong R_f$, where R_f is the localisation with respect to the multiplicatively closed subset $\{ f^n \mid n \in \mathbb{N} \}$.

For a general variety X, which is by definition a topological space covered by affine open subsets $\{ U_i \}$ (each with their own sheaf of regular functions $\mathcal{O}_{U_i}$), we define the structure sheaf $\mathcal{O}_X$ by gluing the affine sheaves on overlaps.

Remark 4.28 *One can think of sheaves as a collection of abelian groups varying over the topological space X. As such, all natural operations that can be performed with groups, such as direct sums, kernels, images, cokernels, etc., can be performed with sheaves too. One caveat: for some operations (such as the image), the naive* *definition only gives a presheaf. The issue can be solved by taking the sheafification of such a presheaf. We will not go into the details here and refer to [5] for the interested reader.*

We are now ready to introduce the notion of sheaf cohomology. Let X be a topological space, and let $\mathcal{F}$ be a sheaf on it. Suppose $U_1, \ldots, U_m$ is a finite open cover of X. Set $U_{ij} = U_i \cap U_j$, and write r_{ij} for the restriction maps $r_{ij} \colon \mathcal{F}(U_i) \to \mathcal{F}(U_{ij})$. Then it is easy to see that the group of global sections $\mathcal{F}(X)$ is precisely the kernel of the map

$$\bigoplus_{i=1}^{m} \mathcal{F}(U_i) \xrightarrow{r_{ij}-r_{ji}} \bigoplus_{1\leq i<j\leq m} \mathcal{F}(U_{ij}). \tag{4.50}$$

The reason is intuitive: global sections are local sections that agree on overlaps. A natural question then arises: what happens when we consider triple overlaps? And more generally, what about p-fold intersections? This leads to the notion of sheaf cohomology, which, informally, measures the obstructions to gluing local data into global data. The higher cohomology groups $H^p(X, \mathcal{F})$ for $p > 0$ quantify how local sections—while agreeing on lower-order overlaps—might still fail to glue consistently across more complex intersections. In this way, cohomology captures the "gaps" between local and global information, describing how gluing works (or fails to work) at increasingly subtle levels.

Let us start from some general notions. Let $(C^\bullet, d^\bullet) = (C^p, d^p)_{p\geq 0}$ be a sequence of abelian groups with homomorphisms, called **differentials**,

$$0 = C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} \cdots \tag{4.51}$$

such that $d^{p+1} \circ d^p = 0$ for all $p \geq 0$. That is,

$$\mathrm{im}(d^p) \subseteq \ker(d^{p+1}). \tag{4.52}$$

Such a sequence is called a **cochain complex**.

Definition 4.29 *Given a cochain complex $(C^\bullet, d^\bullet)$, define the p-th* ***cohomology group*** *as the quotient*

$$H^p(C^\bullet, d^\bullet) := \frac{\ker(d^{p+1})}{\mathrm{im}(d^p)}. \tag{4.53}$$

Let X be a topological space and $\mathcal{F}$ a sheaf on X. Let $U_1, \ldots, U_m$ be an open cover of X. For any index set $I = \{ i_1, \ldots, i_p \} \subseteq \{ 1, \ldots, m \}$, define

$$U_I := U_{i_1} \cap \cdots \cap U_{i_p}, \tag{4.54}$$

and set

$$C^p(X, \mathcal{F}) := \bigoplus_{|I|=p} \mathcal{F}(U_I), \tag{4.55}$$

where the sum ranges over all subsets $I \subseteq \{ 1, \ldots, m \}$ of size p. We omit the dependence of $C^\bullet(X, \mathcal{F})$ on the open cover for ease of notation.

We now define the differential $d^p \colon C^p(X, \mathcal{F}) \to C^{p+1}(X, \mathcal{F})$ as follows. For each I of size $p+1$, say $I = \{ i_1, \ldots, i_{p+1} \}$, there are precisely $p+1$ subsets of size p obtained by removing the k-th element of I. We set $I_k := I \setminus \{ i_k \}$. We then define

$$\bigoplus_{k=1}^{p+1} \mathcal{F}(U_{I_k}) \longrightarrow \mathcal{F}(U_I), \qquad (s_k)_{k=1,\ldots,p+1} \longmapsto \sum_{k=1}^{p+1} (-1)^{k-1} s_k|_{U_I}. \tag{4.56}$$

Assembling these maps for each I of size $p+1$, we get a map $d^p \colon C^p(X, \mathcal{F}) \to C^{p+1}(X, \mathcal{F})$.

Theorem 4.30 *The sequence $(C^\bullet(X, \mathcal{F}), d^\bullet)$ forms a cochain complex, called the Čech complex associated with the open cover $U_1, \ldots, U_m$. In particular, it makes sense to consider the cohomology groups*

$$H^p(X, \mathcal{F}) := H^p(C^\bullet(X, \mathcal{F}), d^\bullet), \tag{4.57}$$

called the ***sheaf cohomology groups*** *of the sheaf $\mathcal{F}$. In particular, $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ is the group of global sections. If clear from the context, we will omit the space X from the notation.*

As stated, the definition of sheaf cohomology depends on the choice of open cover $U_1, \ldots, U_m$ of X. To make the definition independent of such a choice (and non-trivial—for instance, if X is taken as a single open set, then $H^p(X, \mathcal{F}) = 0$ for all $p > 0$), one must take a "fine enough" cover. We will not discuss this notion in detail (for the interested reader, this corresponds to the notion of fine enough cover in Čech cohomology). One can simply keep in mind the following choices, which are fine enough for the cases at hand:

- For an n-dimensional real manifold X, the U_i should be isomorphic to balls in $\mathbb{R}^n$.
- For an algebraic variety X, the U_i should consist of affine opens.
- For a toric variety $X = X_\Delta$, we can take X_σ for $\sigma \in \Delta$ as our cover.

Let us see the definition in action for the structure sheaf on the punctured plane $\mathbb{C}^2 \setminus \{ 0 \}$.

Example 4.31 *Let $X = \mathbb{C}^2 \setminus \{ 0 \}$. Global regular functions are nothing but the polynomials in two variables:*

$$H^0(X, \mathcal{O}_X) = \mathbb{C}[x_1, x_2]. \tag{4.58}$$

Global sections do not detect the puncture at the origin: every regular function on $\mathbb{C}^2 \setminus \{ 0 \}$ extends to a global regular function on the whole of $\mathbb{C}^2$. In order to detect the removal of the origin, we need to consider higher cohomology groups. To this end, consider the fine enough open cover

$$U_1 = \left\{ (x_1, x_2) \in \mathbb{C}^2 \;\middle|\; x_1 \neq 0 \right\}, \quad U_2 = \left\{ (x_1, x_2) \in \mathbb{C}^2 \;\middle|\; x_2 \neq 0 \right\}. \tag{4.59}$$

Then $X = U_1 \cup U_2$, and $U_1 \cap U_2 = \left\{ (x_1, x_2) \in \mathbb{C}^2 \;\middle|\; x_1 x_2 \neq 0 \right\}$. The cochain complex associated to this open cover and the structure sheaf is:

$$0 \longrightarrow \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2) \xrightarrow{d^1} \mathcal{O}_X(U_1 \cap U_2) \longrightarrow 0, \tag{4.60}$$

where the differential d^1 is given by:

$$d^1(s, t) = s|_{U_1 \cap U_2} - t|_{U_1 \cap U_2}. \tag{4.61}$$

As U_1, U_2, and $U_1 \cap U_2$ are principal open sets, we have:

$$\begin{aligned} \mathcal{O}_X(U_1) &= \mathbb{C}[x_1, x_2]_{x_1}, \\ \mathcal{O}_X(U_2) &= \mathbb{C}[x_1, x_2]_{x_2}, \\ \mathcal{O}_X(U_1 \cap U_2) &= \mathbb{C}[x_1, x_2]_{x_1 x_2}. \end{aligned} \tag{4.62}$$

Then the first cohomology group is:

$$H^1(X, \mathcal{O}_X) = \frac{\mathcal{O}_X(U_1 \cap U_2)}{\operatorname{im}(d^1)} = \frac{\mathbb{C}[x_1, x_2]_{x_1 x_2}}{\mathbb{C}[x_1, x_2]_{x_1} + \mathbb{C}[x_1, x_2]_{x_2}}. \tag{4.63}$$

This quotient is isomorphic to the (infinite-dimensional) $\mathbb{C}$-vector space of Laurent polynomials with no holomorphic part, i.e., the space spanned by the monomials:

$$\left\{ x_1^{-i} x_2^{-j} \;\middle|\; i, j > 0 \right\}. \tag{4.64}$$

This aligns with the intuition that local regular functions on $\mathbb{C}^2 \setminus \{ 0 \}$ face an obstruction to gluing, due to the presence of a puncture at the origin. This obstruction is not detected by H^0, but it is detected by H^1.

We conclude with a particularly useful computational tool: the long exact sequence in cohomology arising from a short exact sequence of sheaves. More precisely, given

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{4.65}$$

a short exact sequence, one deduces the **long exact sequence in cohomology**:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{F}'') \arrow[dll, out=-10, in=170] & \\ & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}'') \arrow[r] & \cdots \end{tikzcd} \tag{4.66}$$

This provides a second intuition for sheaf cohomology: taking global sections of a short exact sequence does not yield a short exact sequence (it is only left exact), but it can be completed to a long one via the higher cohomology groups. In other words, higher cohomology groups measure how far global sections are from being exact.

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4.2.2 Cartier and Picard revisited

The goal of this section is to reinterpret the group of Cartier divisors and the Picard groups in terms of sheaf cohomology and line bundles. To this end, we need to discuss some variants of the structure sheaf on algebraic varieties.

Definition 4.32 *Let X be an algebraic variety. The following are sheaves on X: for all $U = \operatorname{Specm}(R)$ affine,*

$$\mathcal{O}_X(U) := \left\{ \varphi\colon U \to \mathbb{C} \;\middle|\; \begin{array}{c} \forall x_0 \in U,\ \varphi = \frac{f}{g} \text{ on a neighbourhood } U_x \text{ of } x \\ \text{for some } f,g \in R,\ g \neq 0 \text{ on } U_x \end{array} \right\}, \tag{4.67}$$

$$\mathcal{O}_X^*(U) := \{ \varphi \in \mathcal{O}_X(U) \mid \varphi \text{ is invertible} \}, \tag{4.68}$$

$$\mathcal{K}_X(U) := \left\{ \psi \;\middle|\; \begin{array}{c} \forall x_0 \in U,\ \psi = \frac{f}{g} \text{ on a neighbourhood } U_x \text{ of } x \\ \text{for some } f,g \in R,\ g \not\equiv 0 \text{ on } U_x \end{array} \right\}, \tag{4.69}$$

$$\mathcal{K}_X^*(U) := \{ \psi \in \mathcal{K}_X(U) \mid \psi \text{ is invertible} \}. \tag{4.70}$$

Notice that $\mathcal{K}_X^*(U) = \mathcal{K}_X(U) \setminus \{0\}$, since the only non-invertible rational function is the constant function zero.

The sheaf $\mathcal{O}_X^*$ is called the ***sheaf of units*** *(or invertible regular functions), the sheaf $\mathcal{K}_X$ is called the* ***sheaf of rational functions****, and $\mathcal{K}_X^*$ is called the* ***sheaf of invertible*** *(or non-zero)* ***rational functions****.*

Before proceeding, two important remarks are in order.

First, notice that both $\mathcal{O}_X$ and $\mathcal{K}_X$ are sheaves of rings (not just of abelian groups), since regular and rational functions can be added and multiplied.

However, the sheaves $\mathcal{O}_X^*$ and $\mathcal{K}_X^*$ are only sheaves of abelian groups with respect to multiplication. In particular, the notation for sheaf cohomology for such sheaves must be adapted accordingly. For instance, the map d^1 becomes

$$d^1 \colon \bigoplus_{i=1}^{m} \mathcal{F}(U_i) \xrightarrow{r_{ij} \cdot r_{ji}^{-1}} \bigoplus_{1 \leq i < j \leq m} \mathcal{F}(U_{ij}). \tag{4.71}$$

and so on.

Secondly, notice that $\mathcal{K}_X$ and $\mathcal{K}_X^*$ are constant sheaves, since every rational function can be uniquely extended to the whole of X.

Recall from Definition 4.5 that a Cartier divisor on X is the data of

- an open cover (U_i) of X by affine sets,
- non-zero rational functions $f_i \in \mathcal{K}_X^*(U_i)$,
- such that $\frac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})$ is an invertible regular function on the intersection.

As the sheaf of invertible regular functions is a subsheaf of the sheaf of non-zero rational functions, we can take elements in the quotient and observe that a Cartier divisor on X is equivalently the data of:

- an open cover (U_i) of X by affine sets,
- elements $[f_i] \in (\mathcal{K}_X^*/\mathcal{O}_X^*)(U_i)$,
- such that $[\frac{f_i}{f_j}] = 0$ in $(\mathcal{K}_X^*/\mathcal{O}_X^*)(U_{ij})$.

In other words, the data of a Cartier divisor $D \in \mathrm{CDiv}(X)$ is equivalent to the data of $([f_i])_i \in C^1(\mathcal{K}_X^*/\mathcal{O}_X^*)$ such that $d^1([f_i])_i = 0$ (cf. Equation (4.71)). The kernel of d^1 is precisely the definition of the 0-th cohomology group, i.e., the global sections. Moreover, since the Picard group is the group of Cartier divisors modulo global non-zero rational functions, we obtain the following restatements of the definitions of the group of Cartier divisors and the Picard group:

Definition 4.33 *On an algebraic variety X, Cartier divisors are defined as the global sections of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ of non-zero rational functions modulo units:*

$$\mathrm{CDiv}(X) := H^0(\mathcal{K}_X^*/\mathcal{O}_X^*). \tag{4.72}$$

Moreover, the Picard group is given by

$$\mathrm{Pic}(X) := \frac{H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)}{H^0(\mathcal{K}_X^*)}. \tag{4.73}$$

This sheaf perspective allows us to give yet another interpretation of the Picard group as the group of line bundles modulo isomorphism. Let us start with the definition of line bundles. Intuitively, a line bundle is a sheaf that locally looks like the structure sheaf.

Definition 4.34 (Cartier and Picard, revisited) *Let X be a variety. A **line bundle** $\mathcal{L}$ is a sheaf on X such that there exists an open cover $U_1, \dots, U_m$ of X for which $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. We denote by*

$$\phi_i \colon \mathcal{O}_X(U_i) \xrightarrow{\cong} \mathcal{L}(U_i) \tag{4.74}$$

the local isomorphisms with the structure sheaf.

*A sheaf satisfying the above condition is also called **locally free** (locally, it looks like a number of copies of the structure sheaf) of **rank** 1 (only one copy).*

Despite being locally isomorphic, a line bundle can still differ from $\mathcal{O}_X$, since the above isomorphisms for U_i, U_j in the covering do not necessarily agree on the intersection $U_{ij} = U_i \cap U_j$. One can thus measure the obstruction for the isomorphisms to agree on intersections. Indeed, the composition

$$\phi_j^{-1}|_{U_{ij}} \circ \phi_i|_{U_{ij}} \colon \mathcal{O}_X(U_{ij}) \xrightarrow{\cong} \mathcal{O}_X(U_{ij}) \tag{4.75}$$

is an isomorphism. It can be shown that, for all $U \subseteq X$, we have a natural identification $\mathrm{Aut}(\mathcal{O}_X(U)) \cong \mathcal{O}_X^*(U)$. Thus, the above isomorphism corresponds to a unique $u_{ij} \in \mathcal{O}_X^*(U_{ij})$. Furthermore,

$$u_{ij}(u_{ik})^{-1}u_{jk} = (\phi_j^{-1} \circ \phi_i) \circ (\phi_i^{-1} \circ \phi_k) \circ (\phi_k^{-1} \circ \phi_j) = 1. \tag{4.76}$$

In the above equation, all sections are restricted to U_{ijk}. As a result, a line bundle is determined by an element $(u_{ij})_{i<j}$ in the kernel of the map:

$$d^2 \colon \bigoplus_{i<j} \mathcal{O}_X^*(U_{ij}) \longrightarrow \bigoplus_{i<j<k} \mathcal{O}_X^*(U_{ijk}). \tag{4.77}$$

This perspective offers two insights.

First, there is a natural operation between line bundles, called the tensor product, that corresponds to the operation on $C^2(\mathcal{O}_X^*)$. Explicitly, given the data (u_{ij}) and (u'_{ij}) defining line bundles $\mathcal{L}$ and $\mathcal{L}'$, define the new line bundle $\mathcal{L} \otimes \mathcal{L}'$ by the functions $(u_{ij} \cdot u'_{ij})$. Notice that each line bundle $\mathcal{L}$ has an inverse with respect to the tensor product, called the dual line bundle $\mathcal{L}^\vee$, defined by the multiplicative inverse of the local data.

Secondly, it is then natural to ask ourselves if there is also a geometric interpretation of the image $\mathrm{im}(d^1)$. This is linked to the notion of isomorphic line bundles.

Definition 4.35 *Two* ***line bundles*** *$\mathcal{L}$ and $\mathcal{L}'$, given by the local isomorphisms ϕ_i and ϕ'_i on U_i respectively, are called* ***isomorphic*** *when there exist isomorphisms*

$$\psi_i \colon \mathcal{L}(U_i) \longrightarrow \mathcal{L}'(U_i) \qquad \textit{such that} \qquad \psi_i|_{U_{ij}} = \psi_j|_{U_{ij}}. \tag{4.78}$$

Given two isomorphic line bundles, we can now combine the local isomorphisms ϕ_i for $\mathcal{L}$ and ϕ'_i for $\mathcal{L}'$ with ψ_i and obtain that

$$\phi_i'^{-1} \circ \psi_i \circ \phi_i \in \mathrm{Aut}(\mathcal{O}_X(U_i)), \tag{4.79}$$

corresponding to a unique element $u_i \in \mathcal{O}_X^*(U_i)$. Moreover, it is not hard to see that

$$\frac{u_{ij}}{u'_{ij}} = \frac{u_i}{u_j}. \tag{4.80}$$

In other words, isomorphic line bundles are those for which the data u_{ij}, u'_{ij} differ exactly by the image of $\bigoplus_i \mathcal{O}_X^*(U_i) \to \bigoplus_{i<j} \mathcal{O}_X^*(U_{ij})$. Putting these two results together and considering the sequence

$$\bigoplus_i \mathcal{O}_X^*(U_i) \xrightarrow{d^1} \bigoplus_{i<j} \mathcal{O}_X^*(U_{ij}) \xrightarrow{d^2} \bigoplus_{i<j<k} \mathcal{O}_X^*(U_{ijk}), \tag{4.81}$$

we obtain a cohomological reinterpretation of line bundles up to isomorphism.

Lemma 4.36 *Isomorphism classes of line bundles are precisely the* 1*-st cohomology group of the sheaf of units:*

$$\frac{\{\textit{ line bundles on } X \,\}}{\textit{isomorphism}} \cong H^1(\mathcal{O}_X^*). \tag{4.82}$$

To conclude, let us show that $H^1(\mathcal{O}_X^*) \cong \mathrm{Pic}(X)$. To this end, recall the definition of $\mathrm{Pic}(X)$ from the beginning of this subsection and consider the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0 \tag{4.83}$$

which gives the following long exact sequence in cohomology:

$$0 \to H^0(\mathcal{O}_X^*) \to H^0(\mathcal{K}_X^*) \to H^0(\mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(\mathcal{O}_X^*) \to H^1(\mathcal{K}_X^*) \to \cdots \tag{4.84}$$

There is the following result concerning constant sheaves and cohomology:

Lemma 4.37 *Let X be an algebraic variety, and $\underline{G}$ a constant sheaf on X with values in the abelian group G. Then*

$$H^0(X, \underline{G}) = \begin{cases} G & \text{if } p = 0, \\ 0 & \text{else.} \end{cases} \tag{4.85}$$

From the above lemma, we deduce that

$$H^0(\mathcal{K}_X^*) \longrightarrow H^0(\mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow H^1(\mathcal{O}_X^*) \longrightarrow 0, \tag{4.86}$$

which implies that $H^1(X, \mathcal{O}_X^*) \cong H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)/H^0(\mathcal{K}_X^*) = \mathrm{Pic}(X)$. Thus, we obtain another interpretation of the Picard group as the group parametrising line bundles up to isomorphism.

Theorem 4.38 *Let X be an algebraic variety. Then*

$$\mathrm{Pic}(X) = \frac{H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)}{H^0(\mathcal{K}_X^*)} = H^1(\mathcal{O}_X^*) = \frac{\{\text{line bundles on } X\}}{\text{isomorphism}}. \tag{4.87}$$

In plain English:

The Picard group is equivalently given by:
Cartier divisors modulo principal divisors,
the first cohomology group of the sheaf of units,
line bundles modulo isomorphism.

4.2.3 Line bundles on toric varieties

We conclude this chapter by studying line bundle and their cohomology on toric varieties. First, recall that any Cartier divisor is a global section of D of $\mathcal{K}_X^*/\mathcal{O}_X^*$. By unpacking the isomorphisms discussed in the previous section, it is easy to see that to such a section we can associate a line bundle, denoted $\mathcal{O}_X(D)$, defined as

$$\mathcal{O}_X(D)(U) := \left\{ f \in \mathcal{K}_X(U) \;\middle|\; \frac{f}{D|_U} \in \mathcal{O}_X(U) \right\}. \tag{4.88}$$

The Picard class of D corresponds to the isomorphism class of $\mathcal{O}_X(D)$, realising the isomorphism in Theorem 4.38. The above definition can be written in a more suggestive way in the language of divisors (rather than rational functions) as follows. First, for a Weil divisor $D = \sum_V a_V V$, we say that $D \geq 0$ if $a_V \geq 0$ for all V. Then, by looking at a Cartier divisors as special Weil divisor, we can recast the above definition as

$$\mathcal{O}_X(D)(U) := \{ f \in \mathcal{K}_X(U) \mid \operatorname{div}(f) - D|_U \geq 0 \} . \tag{4.89}$$

Intuitively, $\mathcal{O}_X(D)$ parametrises rational functions whose zeros an poles are bounded by D.

Our goal in this section is to better understand and compute the cohomology of $\mathcal{O}_X(D)$ for X toric. Let $X = X_\Delta$ be a toric variety such that Δ_1 spans $N_\mathbb{R}$, and let D be a torus-invariant Cartier divisor, defined by the local data $(u(\sigma))_{\sigma\in\Delta_{\max}}$.

Definition 4.39 *Define the polytope $P_D \subseteq M_\mathbb{R}$ as*

$$P_D := \{ u \in M_\mathbb{R} \mid \langle u, v\rangle \geq \psi_D(v) \ \forall v \in |\Delta| \} \tag{4.90}$$

where $\psi_D(v) := \langle u(\sigma), v\rangle$ for $v \in \sigma$ is the support function defining D.

We start with the 0-th cohomology. The characterisation follows from the fact that, if $X = X_\sigma$ is affine, then the map $M \to \operatorname{CDiv}^T(X)$ sending $u \mapsto \operatorname{div}(\chi^u)$ is surjective.

Proposition 4.40 *The following holds:*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u\in P_D\cap M} \mathbb{C}\cdot\chi^u. \tag{4.91}$$

In particular, $\dim H^0(X, \mathcal{O}_X(D)) = \#(P_D \cap M)$.

Let us generalise this result to higher cohomology. First notice that $H^0(|\Delta|) \cong \mathbb{C}\cdot\chi^u$, where the cohomology is the sheaf cohomology for the constant sheaf $\underline{\mathbb{C}}$. From now on, we will omit $\underline{\mathbb{C}}$ for such cohomology groups. This hits at looking at the cohomology of the support in order to understand the cohomology of $\mathcal{O}_X(D)$. This motivates the introduction of certain subsets of the support based on the divisor D.

Definition 4.41 *For $u \in M$ define the* ***conical set***

$$Z(u) := \{ v \in |\Delta| \mid \langle u, v\rangle \geq \psi_D(v) \} . \tag{4.92}$$

The conical set $Z(u)$ depends on D, but we omit the dependence for ease of notation.

By definition of the conical set we have the following:

$$u \in P_D \iff Z(u) = |\Delta| \iff |\Delta| \setminus Z(u) = \emptyset \iff H^0(|\Delta| \setminus Z(u)) = 0. \tag{4.93}$$

The cohomology of the difference is very much related to the concept of relative cohomology, that we briefly recall here.

Let Y be a topological space with a chain complex $(C_\bullet(Y), d_\bullet)$. Let $A \subseteq Y$ be an open subset, and define $C_p(Y, A) := C_p(Y)/C_p(A)$. This defined a short exact sequence

$$0 \to C_p(A) \to C_p(Y) \to C_p(Y, A) \to 0, \tag{4.94}$$

which we can dualise to a short exact sequence of cochain complexes by reversing all the above arrows. By taking cohomology, we obtain a long exact sequence in cohomology:

$$0 \to H^0(Y, A) \to H^0(Y) \to H^0(A) \to H^1(Y, A) \to H^1(Y) \to H^1(A) \to \cdots \tag{4.95}$$

In particular, this gives $H^0(Y, A) = \ker(H^0(Y) \to H^0(A))$. Roughly speaking, the relative cohomology $H^p(Y, A)$ measures the cohomology classes on X that "vanish on A".

Going back to our toric variety, we use the above results with $Y = |\Delta|$, $A = |\Delta| \setminus Z(u)$. For ease of notation, we denote

$$H^p_{Z(u)}(|\Delta|) := H^p(|\Delta|, |\Delta| \setminus Z(u)). \tag{4.96}$$

We therefore get $H^0_{Z(u)}(|\Delta|) = \ker(H^0(|\Delta|) \to H^0(|\Delta| \setminus Z(u)))$, and

$$H^0_{Z(u)}(|\Delta|) \cong \begin{cases} \mathbb{C} \cdot \chi^u & \text{if } u \in P_D, \\ 0 & \text{else,} \end{cases} \tag{4.97}$$

and hence $H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^0_{Z(u)}(|\Delta|)$.

This can be generalised as follows.

Proposition 4.42 *For $p \geq 0$, we have*

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p_{Z(u)}(|\Delta|). \tag{4.98}$$

We will not prove this result (which requires further cohomological tools), but instead examine some applications.

Before proceeding, observe that since $|\Delta|$ is always contractible, we have $H^p(|\Delta|) = 0$ for all $p > 0$. Hence, the long exact sequence in cohomology implies that

$$H^p_{Z(u)}(|\Delta|) = H^{p-1}(|\Delta| \setminus Z(u)). \tag{4.99}$$

This gives a concrete algorithm for computing cohomology groups of line bundles over toric varieties: given D, compute the associated support function ψ_D which in turn gives

- the polytope P_D and, as a consequence, $H^0(X, \mathcal{O}_X(D))$ via Proposition 4.40,
- the conical sets $Z(u)$ and, as a consequence, $H^p(X, \mathcal{O}_X(D))$ for $p > 0$ via Proposition 4.42 and Equation (4.99).

Before looking at examples, we remark that the computation of higher cohomology groups is often simplified by vanishing theorems in the convex case.

Proposition 4.43 *If $|\Delta|$ is convex and ψ_D is convex, then*

$$H^p(X, \mathcal{O}_X(D)) = 0 \quad \textit{for all } p > 0. \tag{4.100}$$

Proof Since for all $u \in M$, both $|\Delta|$ and $|\Delta| \setminus Z(u)$ are convex (and hence contractible), the relative cohomology groups vanish, as they are sandwiched between zeros in the associated long exact sequence. □

It follows that if $|\Delta|$ is convex (e.g. $|\Delta|$ is complete) and ψ_D is convex, then the Euler characteristic of $\mathcal{O}_X(D)$ is given by:

$$\chi(X, \mathcal{O}_X(D)) := \sum_{p \geq 0} (-1)^p \dim H^p(X, \mathcal{O}_X(D)) = \#(P_D \cap M), \tag{4.101}$$

since the only non-zero term in the sum is the one corresponding to $p = 0$, given by Proposition 4.40.

In Section 5.2, we will see how to compute the Euler characteristic above via the celebrated Riemann–Roch theorem, providing a beautiful application of toric geometry to combinatorics: the computation of lattice points in polytopes.

Examples on projective spaces

Before we begin, a few words about $\mathbb{P}^n$. The fan Δ of $\mathbb{P}^n$ can be realised by taking all cones generated by the basis vectors of $\mathbb{R}^n$ and the negative of their sum:

$$e_0 := -\sum_{i=1}^{n} e_i, e_1, \ldots, e_n. \tag{4.102}$$

This yields $n+1$ rays, $\rho_0, \ldots, \rho_n$, where any selection of n elements spans all of $\mathbb{R}^n$. A support function on Δ is uniquely determined by its values on $e_0, \ldots, e_n$. In other words, we have an isomorphism:

$$\mathrm{SF}(\Delta) \xrightarrow{\cong} \mathbb{Z}^{n+1}, \qquad \psi \longmapsto (\psi(e_0), \psi(e_1), \ldots, \psi(e_n)). \tag{4.103}$$

From the description of the Picard group in terms of support functions, we deduce that

$$\mathrm{Pic}(\mathbb{P}^n) \cong \frac{\mathrm{SF}(\Delta)}{M} \cong \mathbb{Z}, \tag{4.104}$$

where we identify $\mathrm{SF}(\Delta) \cong \mathbb{Z}^{n+1}$ as above and $M = \mathbb{Z}^n \subset \mathbb{Z}^{n+1}$ via the map (cf. Corollary 4.19):

$$M = \mathbb{Z}^n \longrightarrow \mathrm{SF}(\Delta) = \mathbb{Z}^{n+1}, \qquad u \mapsto -\big(\langle u, e_0 \rangle, \langle u, e_1 \rangle, \ldots, \langle u, e_n \rangle \big). \tag{4.105}$$

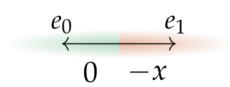
This proves the claimed isomorphism.

The isomorphism class corresponding to the integer d can be represented geometrically as follows. Let $D_0 \subset \mathbb{P}^n$ be the closure of the orbit associated with ρ_0. Then the line bundle $\mathcal{O}_{\mathbb{P}^n}(dD_0)$ is the sheaf of rational functions on $\mathbb{P}^n$ having poles of order at most d along D_0, and its isomorphism class is precisely the integer d. For this reason, it is customary to simply denote it by $\mathcal{O}_{\mathbb{P}^n}(d)$. We omit the projective space from the notation if it is clear from the context. A representative of the Picard class of the support function $\psi_{\mathcal{O}(d)}$ is the support function taking value $-d$ on one of the generators e_i and 0 on the others.

On $\mathbb{P}^1$, consider $\mathcal{O}(1)$. The support function $\psi_{\mathcal{O}(1)}$ described above is

$$\psi_{\mathcal{O}(1)} \colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad \psi_{\mathcal{O}(1)}(x) = \begin{cases} -x & \text{if } x \geq 0, \\ 0 & \text{else.} \end{cases} \tag{4.106}$$

Here we made the canonical identification $x \cong xe_1$. The above support function is achieved by choosing the value -1 on e_1 and 0 on $e_0 = -e_1$. It can be visualised as follows.



As a consequence $P_{\mathcal{O}(1)} = [-1,0] \subset \mathbb{R}^*$, so that

$$P_{\mathcal{O}(1)} \cap \mathbb{Z} = \{-1,0\} \qquad \Rightarrow \qquad H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C} \cdot \chi^{-1} \oplus \mathbb{C} \cdot \chi^0. \tag{4.107}$$

On the other hand, the conical sets $Z(a)$ for $a \cong ae_1^*$, $u \in \mathbb{Z}$ are given by

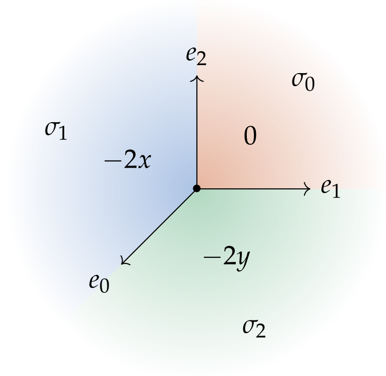
$$Z(a) = \begin{cases} \{ x \geq 0 \} & \text{if } a < 0, \\ \mathbb{R} & \text{if } a = 0, \\ \{ x \leq 0 \} & \text{if } a > 0. \end{cases} \tag{4.108}$$

In particular, $H^{p-1}(\mathbb{R} \setminus Z(a)) = 0$ for all $p > 0$.

Consider now $\mathbb{P}^2$ and the line bundle $\mathcal{O}(-2)$. Consider the support function determined by the value 2 on $e_0 = -2_1 - e_2$ and zero otherwise:

$$\psi_{\mathcal{O}(-2)} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \psi_{\mathcal{O}(-2)}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \sigma_0, \text{ i.e. } x, y \geq 0, \\ -2x & \text{if } (x,y) \in \sigma_1, \text{ i.e. } x \leq 0, x \geq y, \\ -2y & \text{if } (x,y) \in \sigma_2, \text{ i.e. } y \leq 0, x \leq y. \end{cases} \tag{4.109}$$

Here we identified $(x,y) \cong xe_1 + ye_2$. It can be visualised as follows.



A direct computation shows that $P_{\mathcal{O}(-2)} = \emptyset$. Thus, $H^0(\mathbb{P}^2, \mathcal{O}(-2)) = 0$. For higher cohomology groups, we consider the identification $(a,b) \cong ae_1^* + be_2^*$ and compute

$$Z(a,b) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{ll} ax + by \geq 0 & \text{on } \sigma_0 \\ ax + by \geq -2x & \text{on } \sigma_1 \\ ax + by \geq -2y & \text{on } \sigma_2 \end{array} \right\}. \tag{4.110}$$

By checking all cases, one finds that $Z(u)$ is always simply connected and never isolates zero. Thus, the higher cohomology groups of the complement are zero. We conclude that

$$H^p(\mathbb{P}^2, \mathcal{O}(-2)) = 0 \quad \text{for all } p. \tag{4.111}$$

The general analysis of cohomology groups of line bundle on projective spaces is left as Exercise 6.20.

Chapter 5

Topics in toric geometry

5.1 Resolution of singularities

Written by Rasmus
Talks by Jonas and Yu-Yuan

In general, resolving the singularities of an algebraic variety can be a very complicated task. H. Hironaka proved the existence of resolutions of singularities over an algebraically closed field of characteristic zero, a result for which he was awarded the Fields Medal [8]. The goal of this section is to explore the resolution of singularities in the context of toric varieties, where the problem becomes significantly more tractable. We begin by recalling the definition of a resolution of singularities:

Definition 5.1 *Given an algebraic variety X, a* ***resolution of singularities*** *of X is a morphism $f\colon X' \to X$, where X' is an algebraic variety, such that:*

1. *X' is smooth;*
2. *f is proper;*
3. *f induces an isomorphism $f^{-1}(X \setminus X_{\mathrm{sing}}) \cong X \setminus X_{\mathrm{sing}}$,*

where $X_{\mathrm{sing}} := \{\, x \in X \mid x \text{ is singular} \,\}$ is the ***singular locus****.*

The main idea is the following: given a fan Δ whose associated toric variety has singularities, we seek a refinement Δ' of Δ such that the variety $X_{\Delta'}$ is smooth. The morphism $f\colon X_{\Delta'} \to X_\Delta$ induced by the refinement is then a resolution of singularities, and it respects the torus action on both varieties.

5.1.1 Singular locus and star subdivision

The first goal in resolving singularities is to determine the singular locus of a toric variety X_Δ associated with a fan Δ. This can be described explicitly as follows.

Proposition 5.2 *Let $X = X_\Delta$ be a toric variety. Then:*

$$X_{\text{sing}} = \bigcup_{\sigma \text{ singular}} \overline{O_\sigma}, \tag{5.1}$$

$$X \setminus X_{\text{sing}} = \bigcup_{\sigma \text{ smooth}} O_\sigma. \tag{5.2}$$

Proof Recall that a cone σ is smooth if and only if its minimal lattice generators can be extended to a basis of the lattice, and that σ is smooth if and only if the corresponding orbit O_σ is smooth. Also observe that:

1. if σ is smooth, then every face of σ is also smooth;
2. if σ is singular, then every cone of Δ containing σ is also singular.

Condition (1) implies that the smooth cones of Δ form a subfan whose toric variety is $\bigcup_{\sigma \text{ smooth}} O_\sigma$. This open subset of X is clearly smooth, and by the cone-orbit correspondence, its complement in X is $\bigcup_{\sigma \text{ singular}} O_\sigma$. From (2) we obtain:

$$\bigcup_{\sigma \text{ singular}} \overline{O_\sigma} = \bigcup_{\sigma \text{ singular}} O_\sigma. \tag{5.3}$$

Hence, we are done once we show that $\bigcup_{\sigma \text{ singular}} O_\sigma \subseteq X_{\text{sing}}$. We omit this here and refer to [3, Proposition 11.1.2]. $\square$

From this we see that any desired refinement of a fan Δ must leave the smooth cones of Δ unchanged, which motivates the following definition.

Definition 5.3 *Given a fan Δ in $N_\mathbb{R}$ and a primitive element $v \in |\Delta| \cap N \setminus \{0\}$ generating a cone ρ, let $\Delta^\star(\rho)$ be the set consisting of the following cones:*

1. *σ if $v \notin \sigma \in \Delta$;*
2. *$\operatorname{Cone}(\rho, \tau)$ if $v \notin \tau \in \Delta$ and $\rho \cup \tau \subseteq \sigma \in \Delta$.*

*We call $\Delta^\star(\rho)$ the **star subdivision** of Δ at ρ.*

The star subdivision defines a fan. Moreover, the name is justified by the following result.

Lemma 5.4 *The star subdivision $\Delta^\star(\rho)$ is a fan. Moreover, it refines Δ, meaning that every cone of $\Delta^\star(\rho)$ is contained in a cone of Δ, and the supports coincide.*

The key properties of the star subdivision, whose simple proof is omitted, are as follows.

Proposition 5.5 *The star subdivision $\Delta^\star(\rho)$ has the following properties:*

- *The* 1*-dimensional cones of $\Delta^\star(\rho)$ consist of the* 1*-dimensional cones of Δ plus the ray ρ:*

$$\Delta^\star(\rho)_1 = \Delta_1 \cup \{\, \rho \,\}\,. \tag{5.4}$$

- *The refinement induces a proper map $f\colon X_{\Delta^\star(\rho)} \to X_\Delta$. Moreover, the map is a toric morphism, meaning $f(T^\star) \subseteq T$ and its restriction is a group homomorphism. Here, $T^\star$ and T denote the tori in $X_{\Delta^\star(\rho)}$ and X_Δ, respectively.*

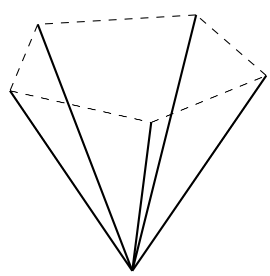
We can now move towards the main objective of this section: given a fan Δ, construct a new fan Δ' by a sequence of star subdivisions such that Δ' is smooth. There are two possible reasons why a cone may be singular: first, its generators may not be linearly independent; second, even if they are linearly independent, they may not extend to a basis of the lattice. Our next goal is to address these two issues separately: we will first perform a **simplicialisation** procedure to resolve the first issue, and then a **regularisation** procedure to handle the second.

5.1.2 Simplicialisation

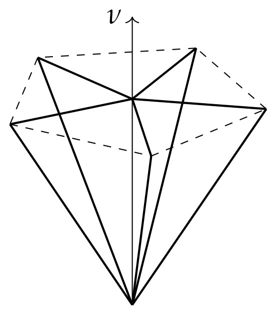
The next step in the process of resolving singularities is to show that every fan admits a simplicial refinement, which we will obtain by successively subdividing using star subdivisions. Let us start with the definition of simplicial cones and fans.

Definition 5.6 *A strongly convex lattice cone $\sigma \subseteq N_{\mathbb{R}}$ is called **simplicial** if there exists a generating set $\{\, v_1, \dots, v_r \,\} \subset N$ of linearly independent vectors. A fan is called simplicial if all its cones are simplicial.*

Before stating the main result, let us analyse some examples. First, observe that every two-dimensional cone is simplicial. Let us consider the following 3-dimensional example:



Clearly, the cone is not simplicial (it has five generators), but we can take the star subdivision defined by ν as the midpoint of the pentagon generating the cone:



It is easy to see that each 3-dimensional cone created by the star subdivision defined by ν is simplicial, hence we have found a simplicial refinement of our original cone (viewed as a fan containing one cone and its faces). One can continue this procedure for cones of higher dimensions, though we omit it here. Given a fan, one simply repeats the process for each cone in the fan. The following proposition formalises the results on simplicialisation.

Proposition 5.7 (Simplicialisation) *Every fan Δ has a refinement Δ' with the following properties:*

1. *Δ' is simplicial and is obtained from Δ by a sequence of star subdivisions.*
2. *Δ' contains every simplicial cone of Δ.*
3. *$\Delta'_1 = \Delta_1$.*

The key properties are conditions (1) and (2), which assert that every fan can be transformed into a simplicial one via a sequence of star subdivisions and this procedure does not change the simplicial cones (in particular, it does not affect the smooth locus of X_Δ). The third property is a nice bonus: it ensures that that no new rays are introduced in the process.

Proof (Sketch) We call a ray $\rho \in \Delta$ in a fan Δ free if, in every cone $\sigma \in \Delta$ containing ρ as an edge, all the other rays lie in a single "complementary" face $\tau \preceq \sigma$. That is, we have $\sigma = \mathrm{Cone}(\tau, \rho)$.

Clearly, a ray $\rho \in \Delta$ is free if and only if $\Delta^\star(\rho) = \Delta$. In particular, ρ becomes a free ray in $\Delta^\star(\rho)$. If another ray $\rho' \in \Delta$ is already free, then it remains free when regarded as a ray of $\Delta^\star(\rho)$. Furthermore, Δ is simplicial if and only if all its rays are free.

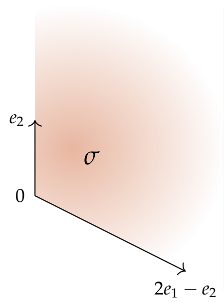
Now, successively applying stellar subdivisions with respect to the rays $\rho_1, \ldots, \rho_k$ of the fan Δ yields a sequence of fans:

$$\Delta = \Delta^{(0)}, \Delta^{(1)}, \ldots, \Delta^{(k)}, \tag{5.5}$$

where $\Delta^{(i+1)}$ is obtained from $\Delta^{(i)}$ by stellar subdivision with respect to ρ_i. Using the properties stated above, it is clear that in the final fan $\Delta^{(k)}$, all rays are free. Hence, the final fan is a simplicial refinement of Δ. □

5.1.3 Regularisation

As mentioned above, for a cone to be simplicial is not enough to guarantee smoothness. A simple example is our beloved double cone in $\mathbb{R}^2 = (\mathbb{Z}^2)_{\mathbb{R}}$:



Although e_2 and $2e_1 - e_2$ are linearly independent, they do not form a basis of $\mathbb{Z}^2$. For instance, it is impossible to write e_1 as a $\mathbb{Z}$-linear combination of them. This can be clearly seen by looking at the lattice generated by these vectors: it is a sublattice of $\mathbb{Z}^2$ that misses some points.

The main idea underlying the regularisation process is to measure how far each cone is from being smooth, i.e., how many lattice points are missing, and then find refinements that decrease this measure. More formally, given a k-dimensional simplicial cone σ with generators $v_1, \ldots, v_k$, and letting $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$, we define the **multiplicity** of σ as the index of the group generated by the generators in N_σ:

$$\operatorname{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k]. \tag{5.6}$$

Intuitively, this measures how many points of N_σ are "captured by" $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$. Note that σ is smooth if and only if $\operatorname{mult}(\sigma) = 1$. Thus, given a simplicial but singular cone, we seek a subdivision such that each resulting cone has smaller multiplicity, and we repeat the process until all cones have multiplicity 1, i.e., are smooth.

Proposition 5.8 (Regularisation) *Every simplicial fan Δ has a refinement Δ' with the following properties:*

1. *Δ' is smooth and is obtained from Δ by a sequence of star subdivisions.*
2. *Δ' contains every smooth cone of Δ.*

Proof (Sketch) Suppose we have a simplicial cone σ generated by $v_1, \ldots, v_k$ and with multiplicity $\operatorname{mult}(\sigma) > 1$. Then there exists a point in $N_\sigma \setminus (\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k)$, hence a ray through that point. Let $\nu = \sum_{i=1}^k t_i v_i$ be the point on that ray in N_σ which is closest to the origin. Then $t_i \in \mathbb{Q}$ for all i, and we can choose ν so that $0 \leq t_i < 1$ for every i. Write $t_i = \frac{c_i}{d_i}$ and let $d = \operatorname{lcd}(d_1, \ldots, d_k)$. Then:

$$d \cdot \nu = d \cdot \sum_{i=1}^{k} \frac{c_i}{d_i} v_i \in \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k, \tag{5.7}$$

so d divides the order of $\overline{\nu}$, where $\overline{\nu} \in N_\sigma / (\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k)$ is the image of ν. Using the fact that the group $N_\sigma / (\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k)$ has order equal to $\operatorname{mult}(\sigma)$, we deduce for each i:

$$d_i \mid d, \qquad d \mid \operatorname{ord}(\overline{\nu}), \quad \text{and} \quad \operatorname{ord}(\overline{\nu}) \mid \operatorname{mult}(\sigma), \tag{5.8}$$

which implies $d_i \mid \operatorname{mult}(\sigma)$, so that $\frac{c_i}{d_i} \cdot \operatorname{mult}(\sigma) = t_i \cdot \operatorname{mult}(\sigma) \in \mathbb{N}$.

We now claim: the cone

$$\sigma_j = \operatorname{Cone}\left(v_1, \ldots, v_{j-1}, \sum_{i=1}^{k} t_i v_i, v_{j+1}, \ldots, v_k \right) \tag{5.9}$$

with $t_j > 0$, has multiplicity $\operatorname{mult}(\sigma_j) = t_j \cdot \operatorname{mult}(\sigma)$.

This can be shown using the following fact: if σ is a simplicial cone with generators $u_1, \ldots, u_k$, and if $\{ e_j \}$ is a basis of N_σ such that $u_i = \sum_j a_{ij} e_j$, then

$\mathrm{mult}(\sigma) = |\det(a_{ij})|$. We omit the proof of this fact, but with it we compute:

$$\begin{aligned}
\det(v_1, \ldots, \sum t_i v_i, \ldots, v_k) &= \det\left([v_1, \ldots, v_j, \ldots, v_k] \cdot \begin{bmatrix} 1 & \cdots & t_1 & \cdots & 0 \\ 0 & \cdots & t_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & t_k & \cdots & 1 \end{bmatrix} \right) \\
&= \det(v_1, \ldots, v_k) \cdot \det\left(\begin{bmatrix} 1 & \cdots & t_1 & \cdots & 0 \\ 0 & \cdots & t_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & t_k & \cdots & 1 \end{bmatrix} \right) \\
&= \det(v_1, \ldots, v_k) \cdot t_j \\
&= t_j \cdot \mathrm{mult}(\sigma).
\end{aligned} \tag{5.10}$$

Since $t_j < 1$, we get $\mathrm{mult}(\sigma_j) < \mathrm{mult}(\sigma)$, and both are positive integers. This proves that $\Delta^\star(\rho)$, where ρ is the ray generated by ν, is a refinement where the cone σ has been substituted by cones of strictly smaller multiplicity. Hence, after finitely many iterations of this procedure, we arrive at a cone with multiplicity 1, i.e., a smooth cone.

Performing this for every cone in a given fan yields a smooth subdivision (which is a sequence of star subdivisions), thereby constructing a resolution of singularities. □

Combining the results on simplicialisation and regularisation, we conclude that every toric variety X_Δ admits a resolution of singularities $f\colon X_{\Delta'} \to X_\Delta$, where $X_{\Delta'}$ is a smooth toric variety obtained by a finite sequence of star subdivisions. Moreover, the morphism f is toric.

5.2 Chow groups, Riemann–Roch, and Pick's formula

5.2.1 Chow groups

Written by Davide
Talks by Elisa M. and Zheming

In this section, we define Chow groups and Chow rings; the former generalises the class group. We first study them for a general variety X, then we present some results in the toric setting. In fact, we will see that both Chow groups and Chow rings admit much simpler descriptions for toric varieties.

Definition 5.9 *Let X be a variety. We define $Z_k(X)$ as the free abelian group generated by the k-dimensional irreducible closed subvarieties of X, that is:*

$$Z_k(X) := \mathbb{Z}\left\{ V \subseteq X \,\middle|\, \substack{\text{closed irreducible subvariety} \\ \text{with } \dim(V) = k} \right\}. \tag{5.11}$$

To define Chow groups, we introduce an equivalence relation on $Z_k(X)$, known as rational equivalence. For $U, V \in Z_k(X)$, we write $U \sim_k V$ if and only if there exists a $(k+1)$-dimensional closed irreducible subvariety $W \subseteq X$ and a non-zero rational function $f \in \mathcal{K}^*(W)$ such that

$$U - V = \operatorname{div}(f). \tag{5.12}$$

In other words, U and V differ by the principal divisor associated to a rational function on a $(k+1)$-dimensional subvariety.

Definition 5.10 *The k-th **Chow group** is defined as the quotient:*

$$A_k(X) := Z_k(X)/\sim_k . \tag{5.13}$$

The equivalence class of $V \in Z_k(X)$ is denoted by $[V]$.

By definition, we have $A_k(X) = 0$ for all $k > n = \dim(X)$. The next lemma describes the n-th and $(n-1)$-th Chow groups.

Lemma 5.11 *Let X be an n-dimensional variety. Then there are natural isomorphisms:*

$$A_n(X) \cong \mathbb{Z}, \qquad A_{n-1}(X) \cong \operatorname{Cl}(X). \tag{5.14}$$

Proof To prove $A_n(X) \cong \mathbb{Z}$, observe that there are no $(n+1)$-dimensional subvarieties of X, so $U \sim_n V$ if and only if $U = V$, and hence $A_n(X) = Z_n(X)$. The only n-dimensional closed irreducible subvariety of X is X itself, so $A_n(X)$ is the free abelian group generated by X, and the result follows.

For $A_{n-1}(X)$, note that $Z_{n-1}(X) = \operatorname{WDiv}(X)$ is the group of Weil divisors. As above, since X is the only n-dimensional irreducible closed subvariety, the equivalence $\sim_{n-1}$ coincides with the relation of linear equivalence of Weil divisors. Hence $A_{n-1}(X) \cong \operatorname{Cl}(X)$. □

Before turning to the toric case, we describe two important features of Chow groups: the pull-back and push-forward maps and the ring structure.

Definition 5.12 *Let $f\colon X \to Y$ be a morphism. The induced pull-back is a map*

$$f^*\colon A_k(Y) \longrightarrow A_k(X). \tag{5.15}$$

For $[V] \in A_k(Y)$ in good position with respect to f, it is simply $[f^{-1}(V)] \in A_k(X)$.

Being 'in good position with respect to f' simply means that $f^{-1}(V)$ has the same codimension in X as V had in X. An important tool in the theory of Chow groups, known as Chow moving lemma, states that it is always possible to choose a rational representative that is in good position.

The pull-back allows to take Chow elements in the codomain and pull them back to the domain. The converse is only possible when the map f is proper.

Definition 5.13 *Let $f\colon X \to Y$ be a proper morphism. The induced push-forward is defined as:*

$$\begin{aligned} f_*\colon A_k(X) &\longrightarrow A_k(Y) \\ [V] &\longmapsto \begin{cases} \deg(V/f(V))[f(V)] & \textit{if } \dim(f(V)) = \dim(V), \\ 0 & \textit{otherwise.} \end{cases} \end{aligned} \tag{5.16}$$

We will not define the degree $\deg(V/f(V))$ formally. Intuitively, it is an integer measuring how V maps onto $f(V)$ under f.

An important special case is the **integration map**. Suppose that X is complete. Then the map $X \to \mathrm{Specm}(\mathbb{C})$ is proper, hence induces

$$\int_X\colon A_k(X) \longrightarrow \mathbb{Z}, \tag{5.17}$$

which is zero for $k > 0$, since $A_k(\mathrm{Specm}(\mathbb{C})) = 0$, and for $k = 0$ it reduces to

$$\sum_{x\in X} a_x[x] \longmapsto \sum_{x\in X} a_x \tag{5.18}$$

also called the **degree map**.

A useful tool for computing Chow groups is the excision theorem, which mirrors the excision property in singular homology.

Lemma 5.14 (Excision) *Let $U \subseteq X$ be open and set $Z := X \setminus U$. Then there is a right exact sequence of Chow groups:*

$$A_k(Z) \xrightarrow{i*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0, \tag{5.19}$$

where $i\colon Z \to X$ and $j\colon U \to X$ denote the inclusions.

Another key structure of Chow groups is their ring structure. We treat the smooth case, though the construction can be extended to singular varieties using rational coefficients.

Definition 5.15 *Let X be a smooth n-dimensional variety. Define $A^k(X) := A_{n-k}(X)$. There exists a product*

$$A^p(X) \times A^q(X) \longrightarrow A^{p+q}(X), \tag{5.20}$$

called the ***intersection product****, which turns $A^\bullet(X) := \bigoplus_{k \geq 0} A^k(X)$ into a unital, commutative graded ring called the* ***Chow ring*** *of X.*

We do not define the intersection product in full generality, as it is rather technical. However, the main geometric idea is simple: suppose $[U] \in A^p(X)$ and $[V] \in A^q(X)$ intersect transversely. Then $U \cap V$ decomposes into irreducible components $W_1, \dots, W_m$ of codimension $p + q$, and we define

$$[U] \cdot [V] = \sum_{i=1}^{m} [W_i]. \tag{5.21}$$

The general definition makes this procedure rigorous. Notice that, from this idea, it is clear that the unit is given by $1 = [X]$, also called the **fundamental class** of X.

In general, Chow groups and Chow rings are hard to compute. However, the toric setting is a notable exception. In what follows, we fix a lattice N and denote its dual by M.

Proposition 5.16 *Let $X = X_\Delta$ be an n-dimensional toric variety. Then the Chow group $A_k(X)$ is generated by the classes $[\overline{O_\sigma}]$, where $\sigma \in \Delta$ has dimension $n - k$.*

Proof Define $X_i := \bigcup_{\dim(\sigma) \geq n-i} \overline{O_\sigma}$. Then we obtain a stratification:

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_{-1} = \emptyset, \tag{5.22}$$

and, by the orbit-cone correspondence theorem,

$$X_i \setminus X_{i-1} = \bigsqcup_{\dim(\sigma) = n-i} O_\sigma. \tag{5.23}$$

Using excision, we obtain:

$$A_k(X_{i-1}) \longrightarrow A_k(X_i) \longrightarrow \bigoplus_{\dim(\sigma)=n-i} A_k(O_\sigma) \longrightarrow 0. \tag{5.24}$$

Each O_σ is a torus, and one can show that the only non-trivial Chow group of a torus is the top-dimensional one, isomorphic to $\mathbb{Z}$. Since the map sends $[\overline{O_\sigma}]$ to $[O_\sigma]$, induction on i gives the claim. □

In the toric setting, we can also describe the intersection product explicitly. If X_Δ is smooth, and $\sigma_1, \sigma_2 \in \Delta$, then

$$[\overline{O_{\sigma_1}}] \cdot [\overline{O_{\sigma_2}}] = \begin{cases} [\overline{O_\tau}] & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ span } \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{5.25}$$

Finally, the following gives a complete description of the Chow ring of a smooth projective toric variety in terms of its torus-invariant divisors. See [2, Section 5.2] for a proof.

Theorem 5.17 *Let $X = X_\Delta$ be a smooth projective toric variety, and let $D_1, \ldots, D_d$ be the divisors corresponding to the closures of the orbits of rays in Δ. Then there is an isomorphism:*

$$A^\bullet(X) \cong \mathbb{Z}[D_1, \ldots, D_d]/I(X), \tag{5.26}$$

where $I(X)$ is the ideal generated by:

- *monomials $D_{i_1} \cdots D_{i_k}$ such that the corresponding rays $v_{i_1}, \ldots, v_{i_k}$ do not span a cone in Δ,*
- *linear relations $\sum_{i=1}^d \langle u, v_i \rangle D_i$ for all $u \in M$.*

5.2.2 Characteristic classes

In this section, we define characteristic classes. These are elements of the Chow group associated with vector bundles, and they measure how 'twisted' a vector bundle is. We begin with the first Chern class of a line bundle, which is simply the divisor of a generic rational section of the bundle.

Unless stated otherwise, we assume that X is a smooth projective variety.

Definition 5.18 *Let $\mathcal{L}$ be a line bundle on X. A rational section of $\mathcal{L}$ is a global section of $\mathcal{L} \otimes \mathcal{K}_X$.*

Rational sections generalise rational functions, which are precisely the rational sections of the structure sheaf $\mathcal{O}_X$. As for rational functions, we can define the order of vanishing of a rational section along a codimension-one irreducible closed subvariety, and the associated divisor.

Definition 5.19 *Let $\mathcal{L}$ be a line bundle on X, s a non-zero rational section of $\mathcal{L}$, and V a codimension-one irreducible closed subvariety of X. Define the order of vanishing of s along V by:*

$$\mathrm{ord}_V(s) := \mathrm{ord}_V(s/t), \tag{5.27}$$

where t is a non-zero section of $\mathcal{L}$ defined on some open set meeting V. The divisor of s is defined as:

$$\operatorname{div}(s) := \sum_V \operatorname{ord}_V(s) \cdot V, \tag{5.28}$$

where the sum runs over all codimension-one irreducible closed subvariety of X.

Definition 5.20 *Let $\mathcal{L}$ be a line bundle and s a non-zero rational section of $\mathcal{L}$. The **first Chern class** of $\mathcal{L}$ is defined as:*

$$\mathrm{c}_1(\mathcal{L}) := [\operatorname{div}(s)] \in \mathrm{Cl}(X) = A^1(X). \tag{5.29}$$

Note that the first Chern class $\mathrm{c}_1(\mathcal{L})$ does not depend on the choice of rational section s. Indeed, if s_1 and s_2 are two non-zero rational sections of $\mathcal{L}$, then:

$$\operatorname{div}(s_2) = \operatorname{div}(s_1) + \operatorname{div}(s_2/s_1), \tag{5.30}$$

and $\operatorname{div}(s_2/s_1)$ is a principal divisor. Hence $[\operatorname{div}(s_1)] = [\operatorname{div}(s_2)]$ in the class group.

This definition generalises to vector bundles, which are sheaves locally isomorphic to a fixed number of copies of the structure sheaf.

Definition 5.21 *A rank-r (algebraic) **vector bundle** $\mathcal{E}$ on X is a locally free sheaf of rank r; that is, there exists an open cover on which $\mathcal{E}$ is isomorphic to r copies of the structure sheaf $\mathcal{O}_X$.*

Definition 5.22 *Let $\mathcal{E}$ be a rank-r vector bundle on X. The **total Chern class** of $\mathcal{E}$ is defined as:*

$$\mathrm{c}(\mathcal{E}) := 1 + \mathrm{c}_1(\mathcal{E}) + \cdots + \mathrm{c}_r(\mathcal{E}) \in A^\bullet(X), \tag{5.31}$$

where $\mathrm{c}_p(\mathcal{E}) \in A^p(X)$ for all $1 \le p \le r$, and such that the following properties hold:

1. *For any line bundle $\mathcal{L}$ on X, $\mathrm{c}_1(\mathcal{L})$ is defined as in Definition 5.20.*
2. *For any short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of bundles, we have:*

   $$\mathrm{c}(\mathcal{E}) = \mathrm{c}(\mathcal{E}') \cdot \mathrm{c}(\mathcal{E}'') \tag{5.32}$$

   where the product is the intersection product on $A^\bullet(X)$.
3. *If $f\colon X \to Y$ is a morphism and $\mathcal{E}$ is a vector bundle on Y, then:*

   $$f^*\mathrm{c}(\mathcal{E}) = \mathrm{c}(f^*\mathcal{E}), \tag{5.33}$$

   where f^* denotes the pull-back on Chow rings on the left, and the pull-back of vector bundles on the right.

4. *If $\mathcal{E}$ is a vector bundle on X and $\mathcal{E}^\vee$ its dual, then:*

$$\mathrm{c}_p(\mathcal{E}^\vee) = (-1)^p \mathrm{c}_p(\mathcal{E}). \tag{5.34}$$

We will not define the pull-back of vector bundles or the dual bundle in detail here. However, it is important to know that the above properties uniquely determine an element $\mathrm{c}(\mathcal{E}) \in A^\bullet(X)$, depending only on the isomorphism class of the vector bundle $\mathcal{E}$ on X.

5.2.3 Riemann–Roch theorem

The goal of this section is to state and understand Hirzebruch's version of the celebrated Riemann–Roch theorem, which relates the Euler characteristic of a vector bundle—a purely topological quantity—to its characteristic classes.

We begin with a brief motivation. If we consider a regular function $f\colon X \to \mathbb{C}$ from a complete variety X, then it follows that f must be constant. This is a generalisation of Liouville's theorem from complex analysis, which states that every entire function on $\mathbb{C}$ (i.e. a regular function on $\mathbb{P}^1$) is constant. In algebraic terms:

$$H^0(X, \mathcal{O}_X) = \mathbb{C} \tag{5.35}$$

for complete X.

To avoid the lack of interesting global functions, we adopt two strategies:

1. Consider local functions;
2. Allow 'mild' singularities.

Regarding the second point, let D be a divisor on X and consider meromorphic functions with zeros and poles no worse than D:

$$\mathscr{L}(D) := \{\, f \in \mathcal{K}_X^* \mid \operatorname{div}(f) + D \geq 0 \,\} \cup \{\, 0 \,\}. \tag{5.36}$$

Then $\mathscr{L}(D)$ is a vector space, which can be identified with the space of global sections of the line bundle $\mathcal{O}_X(D)$. In particular, $\dim \mathscr{L}(D) = h^0(X, \mathcal{O}_X(D))$.

Example 5.23 *Consider $X = \mathbb{P}^1$. Then:*

- *for $D = [0]$, $\mathscr{L}(D) = \mathbb{C}\langle 1, z^{-1}\rangle$;*
- *for $D = [0] + [\infty]$, $\mathscr{L}(D) = \mathbb{C}\langle 1, z, z^{-1}\rangle$.*

One of the main reasons for being interested in the space $\mathscr{L}(D)$ is that, under suitable conditions, it gives rise to a map into projective space.

Theorem 5.24 *Let X be a smooth variety and let $\mathcal{L}$ be a line bundle on X. If $H^0(X,\mathcal{L}) \neq 0$, then for any basis $\{ s_0, \dots, s_n \}$ of $H^0(X,\mathcal{L})$, there is a rational map:*

$$\varphi_{\mathcal{L}} \colon X \dashrightarrow \mathbb{P}^n, \qquad x \mapsto [s_0(x), \dots, s_n(x)], \tag{5.37}$$

well-defined on the open set where not all s_i vanish simultaneously. Moreover:

- *If $\mathcal{L}$ is globally generated, then $\varphi_{\mathcal{L}}$ is a morphism;*
- *If $\mathcal{L}$ is very ample, then $\varphi_{\mathcal{L}}$ is a closed embedding, and thus realises X as a subvariety of projective space.*

This explains why the dimension $h^0(X,\mathcal{L})$ is a crucial invariant: it not only encodes the number of linearly independent global sections, but also determines whether $\mathcal{L}$ can be used to map X into projective space.

In particular, computing the dimension of $\mathscr{L}(D)$, or more generally $H^0(X,\mathcal{L})$, is a central problem in Algebraic Geometry. The Riemann–Roch theorem gives a way to compute this dimension from the geometric and topological data of X and $\mathcal{L}$ when X is complete and one-dimensional, i.e. a Riemann surface. More precisely, the Euler characteristic of $\mathcal{O}_X(D)$ is given by:

$$\begin{aligned} \chi(X,\mathcal{O}_X(D)) &:= h^0(X,\mathcal{O}_X(D)) - h^1(X,\mathcal{O}_X(D)) \\ &= \deg(D) + 1 - g_X, \end{aligned} \tag{5.38}$$

where $\deg(D)$ is the degree map and g_X is the genus of X.

The Hirzebruch–Riemann–Roch formula is a far-reaching generalisation of this result, which computes the Euler characteristic of vector bundles over smooth projective varieties.

Before stating the theorem, we need to introduce certain invariants of vector bundles, built from Chern classes: the Chern character and the Todd class. As in the case of Chern classes, we begin with line bundles.

Definition 5.25 *Let $\mathcal{L}$ be a line bundle on a smooth variety X. The* ***Chern character*** *of $\mathcal{L}$ is defined as:*

$$\mathrm{ch}(\mathcal{L}) := \exp(\mathrm{c}_1(\mathcal{L})). \tag{5.39}$$

The ***Todd class*** *of $\mathcal{L}$ is defined as:*

$$\mathrm{td}(\mathcal{L}) := \frac{\mathrm{c}_1(\mathcal{L})}{1 - \exp(-\mathrm{c}_1(\mathcal{L}))}. \tag{5.40}$$

We define the Chern character and Todd class of a general vector bundle as follows:

Definition 5.26 *Let $\mathcal{E}$ be a vector bundle on X. We define* $\mathrm{ch}(\mathcal{E})$ *and* $\mathrm{td}(\mathcal{E})$ *by requiring that, for any short exact sequence* $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, *we have:*

- $\mathrm{ch}(\mathcal{E}) = \mathrm{ch}(\mathcal{E}') + \mathrm{ch}(\mathcal{E}'')$;
- $\mathrm{td}(\mathcal{E}) = \mathrm{td}(\mathcal{E}') \cdot \mathrm{td}(\mathcal{E}'')$.

Finally, the Hirzebruch–Riemann–Roch formula expresses the Euler characteristic of a vector bundle—that is, the alternating sum of the dimensions of its sheaf cohomology groups—in terms of the Chern character of the bundle and the Todd class of the tangent bundle of the variety.

Theorem 5.27 (Hirzebruch–Riemann–Roch) *Let X be a smooth projective variety and $\mathcal{E}$ a vector bundle on X. Then:*

$$\chi(X, \mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X), \tag{5.41}$$

where $\chi(X, \mathcal{E})$ is the Euler characteristic of $\mathcal{E}$ and $\mathrm{td}(X) = \mathrm{td}(T_X)$ *is the Todd class of the tangent bundle of X.*

Our next goal is to dissect the Riemann–Roch formula in the case of toric varieties. The following theorem provides an explicit expression for the Todd class of a smooth projective toric variety.

Theorem 5.28 *Let X be a smooth projective toric variety, and let $D_1, \ldots, D_d$ be the divisors associated with the rays of X. Then the Todd class of X is given by*

$$\mathrm{td}(X) = \prod_{i=1}^{d} \frac{D_i}{1 - \exp(-D_i)}. \tag{5.42}$$

Proof (Sketch) To prove the theorem, we need the following ingredients:

- Let $\Omega_X(\log D)$ be the sheaf of differentials with at worst simple poles along $D = \sum_i D_i$. Then the sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(\log D) \longrightarrow \bigoplus_{i=1}^{d} \mathcal{O}_{D_i} \longrightarrow 0 \tag{5.43}$$

  is exact.

- The map

$$\begin{aligned} M \otimes_{\mathbb{Z}} \mathcal{O}_X &\longrightarrow \Omega_X(\log D) \\ u &\longmapsto d\log(\chi^u) \end{aligned} \tag{5.44}$$

  is an isomorphism.

The first point is equivalent to saying that the map $\sum f_i \, d\log(x_i) \mapsto (f_i|_{D_i})_{i=1}^d$ is surjective with kernel Ω_X. This holds because the residue of a form vanishes if and only if it lies in Ω_X. The second point follows from a direct computation.

Applying the multiplicativity of Chern classes, we find:

$$\mathrm{c}(T_X^\vee) \cdot \prod_{i=1}^{d} \mathrm{c}(\mathcal{O}_{D_i}) = 1. \tag{5.45}$$

Using the fact that $T_X = \Omega_X^\vee$ and applying the same reasoning to the Todd class yields:

$$\mathrm{td}(X) = \mathrm{td}(T_X) = \prod_{i=1}^{d} \mathrm{td}(\mathcal{O}_{D_i}) = \prod_{i=1}^{d} \frac{D_i}{1 - \exp(-D_i)}. \tag{5.46}$$

□

We can illustrate this formula explicitly in the case of toric surfaces.

Example 5.29 *Let X be a smooth projective toric surface, and let $D_1, \ldots, D_d$ be the divisors associated with the rays of X. Then:*

$$\mathrm{td}(X) = 1 + \frac{1}{2} \sum_{i=1}^{d} [D_i] + [x], \tag{5.47}$$

where the first two summands follow immediately from the formula for the Todd class by expanding the exponential in a Taylor series. The final term $[x]$ lies in $A^2(X)$ and depends on the intersection theory of the surface. Its precise form requires further geometric analysis.

5.2.4 Volume of polytopes and Pick's formula

In this section, we provide a formula for computing the number of lattice points and the volume of polytopes; it is a direct consequence of the Hirzebruch–Riemann–Roch theorem. In the two-dimensional case, a more explicit result is known as Pick's formula.

Corollary 5.30 *Let $P \subset N_{\mathbb{R}}$ be a convex rational polytope, and denote by $\#P := |P \cap N|$ the number of lattice points. Suppose that the associated toric variety $X = X_P$ is smooth, and denote by $D = D_P$ the associated divisor. Then*

$$\#P = \int_X \mathrm{ch}(\mathcal{O}(D)) \cdot \mathrm{td}(X). \tag{5.48}$$

Proof This follows from the fact that $h^0(X, \mathcal{O}(D)) = \#P$ and there is no higher cohomology, cf. Section 4.2.3. $\square$

A direct, perhaps unexpected, consequence of the above formula is that the volume of a polytope in Euclidean n-space is always a non-negative integral multiple of $\frac{1}{n!}$.

Corollary 5.31 *In the above setting,*

$$\mathrm{Vol}(P) \in \frac{1}{n!} \cdot \mathbb{N}, \tag{5.49}$$

where $n = \dim(N_{\mathbb{R}})$.

Proof Observe that

$$\mathrm{ch}(\mathcal{O}(D)) = e^D = \sum_{k=0}^{n} \frac{1}{k!} D^k, \tag{5.50}$$

and write $\mathrm{td}(X) = \sum_{k=0}^{n} \mathrm{td}_k(X)$ with $\mathrm{td}_k(X) \in A_k(X)$. Moreover, by definition of the Euclidean volume,

$$\mathrm{Vol}(P) = \lim_{\nu \to \infty} \frac{\#(\nu P)}{\nu^n}, \qquad \nu \in \mathbb{N}. \tag{5.51}$$

Consequently, we obtain:

$$\begin{aligned} \mathrm{Vol}(P) &= \lim_{\nu \to \infty} \frac{\#(\nu P)}{\nu^n} \\ &= \lim_{\nu \to \infty} \sum_{k=0}^{n} \frac{\nu^k}{k!\, \nu^n} \deg(D^k \cdot \mathrm{td}_k(X)) \\ &= \frac{1}{n!} \deg(D^n \cdot \mathrm{td}_n(X)) \\ &= \frac{\deg(D^n)}{n!}. \end{aligned} \tag{5.52}$$

Here, deg denotes the degree map on $A_0(X)$, see Equation (5.18). Since $\deg(D^n) \in \mathbb{Z}$ and volume is non-negative, the claim follows. $\square$

A more explicit formula, known as Pick's formula, can be derived in the two-dimensional case. To prepare for it, consider a convex rational polygon $P \subseteq N_{\mathbb{R}} = \mathbb{R}^2$. Define its perimeter as

$$\mathrm{Perim}(P) := \sum_{e} \ell(e), \tag{5.53}$$

where the sum runs over all edges of P, and $\ell(e)$, the length of an edge, is defined as $\ell(e) := |N \cap e| - 1$. In this two-dimensional setting, we also refer to the volume of P as its area.

Theorem 5.32 (Pick's formula) *Let $P \subseteq N_{\mathbb{R}} = \mathbb{R}^2$ be a convex rational polygon. Then:*

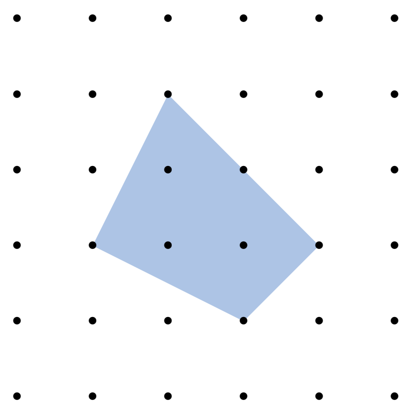
$$\#P = \mathrm{Area}(P) + \frac{1}{2}\mathrm{Perim}(P) + 1. \tag{5.54}$$

Proof The number of lattice points in P is given by:

$$\begin{aligned} \#P &= \deg \mathrm{td}_0(X) + \deg(D \cdot \mathrm{td}_1(X)) + \frac{1}{2}\deg(D^2 \cdot \mathrm{td}_2(X)) \\ &= 1 + \frac{1}{2}\sum \deg(D \cdot D_i) + \mathrm{Area}(P) \\ &= 1 + \frac{1}{2}\mathrm{Perim}(P) + \mathrm{Area}(P). \end{aligned} \tag{5.55}$$

The first equality follows from Hirzebruch–Riemann–Roch; the second from the explicit form of the Todd class for toric surfaces; and the third from applying the same reasoning to the boundary divisors corresponding to the edges of P. □

Example 5.33 *Consider the polygon P below.*



The total number of lattice points is $\#P = 8$. The area is $9/2$, and the perimeter is 5. Hence, we verify Pick's formula: $8 = 9/2 + 5/2 + 1$.

Also, observe that the area is a half-integer, in accordance with our general observation. This also follows from Pick's formula, as the area can be written as the number of lattice points minus one (an integer), minus half of the perimeter (a half-integer).

5.3 Euler's formula and Stanley's theorem

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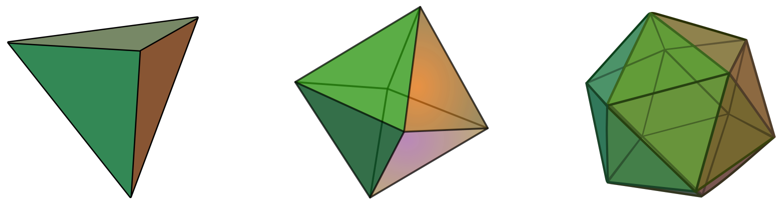


Figure 5.1: From left to right: the tetrahedron, the octahedron, and the icosahedron.

5.3.1 Euler's formula

Let us begin with a classical combinatorial invariant of (convex) simplicial polyhedra (i.e. polytopes in 3-space whose faces are triangles): the **Euler characteristic**. Let f_0, f_1, and f_2 respectively denote the number of vertices, edges, and triangles in such a polyhedron. Then the Euler characteristic is defined as

$$\chi = f_0 - f_1 + f_2. \tag{5.56}$$

The simplest possible example is a tetrahedron, for which $(f_0, f_1, f_2) = (4, 6, 4)$, so that

$$\chi_{\text{tetrahedron}} = 4 - 6 + 4 = 2. \tag{5.57}$$

Similarly, the a octahedron has $(f_0, f_1, f_2) = (6, 12, 8)$, so that

$$\chi_{\text{octahedron}} = 6 - 12 + 8 = 2. \tag{5.58}$$

Euler observed that this alternating sum is invariant under the choice of polyhedron, as it coincides with a topological invariant of the underlying surface: the Euler characteristic of the 2-sphere:

$$f_0 - f_1 + f_2 = 2. \tag{5.59}$$

You can check this with another polyhedron, such as the icosahedron.

One interesting aspect of Euler's formula is that it generalises to arbitrary dimensions. Consider, for instance, a polygon (a polytope in 2-space) composed of k vertices and k edges. Then,

$$(f_0, f_1) = (k, k), \qquad \chi_{k\text{-polygon}} = k - k = 0. \tag{5.60}$$

Once again, the Euler characteristic of the 1-sphere, the circle, coincides with the alternating sum of the number of k-faces: it is always zero. In dimension zero, consider the 0-sphere, which consists of two points. In this case, we have only two vertices, so the Euler characteristic is $\chi = f_0 = 2$.

At this point, the pattern should be clear: for an convex simplicial polyhedron in $\mathbb{R}^n$ (simplicial means that facets are all simplices), the Euler characteristic is either 0 or 2, depending on the parity of the dimension:

$$\chi := \sum_{d=0}^{n-1} (-1)^d f_d = 1 + (-1)^n, \tag{5.61}$$

where f_d is the number of d-dimensional faces. As a sanity check, consider the higher-dimensional generalisation of the triangle (in dimension 2) and the tetrahedron (in dimension 3). Take $n+1$ points in general position in $\mathbb{R}^n$ and form their convex hull. Since all points are in general position, there are

$$f_d = \binom{n+1}{d+1} \tag{5.62}$$

faces of dimension d. This is because every choice of $d+1$ points out of the $n+1$ determines a d-dimensional simplex. Thus, the Euler characteristic reads

$$\chi = \sum_{d=0}^{n-1} (-1)^d \binom{n+1}{d+1} = 1 + (-1)^n. \tag{5.63}$$

This can be seen from the binomial theorem:

$$(x+y)^{n+1} = \sum_{m=0}^{n+1} \binom{n+1}{m} x^m y^{n+1-m}, \tag{5.64}$$

so by setting $x = -1$ and $y = 1$, we find

$$\begin{aligned} 0 = \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} &= \sum_{d=-1}^{n} (-1)^{d+1} \binom{n+1}{d+1} = \\ &= 1 - \sum_{d=0}^{n-1} (-1)^d \binom{n+1}{d+1} - (-1)^n, \end{aligned} \tag{5.65}$$

hence the thesis.

This distinction between even and odd dimensions can be a bit unsettling. There is, however, a way to uniformise Euler's formula. Given a convex polyhedron in $\mathbb{R}^n$, define its f**-vector** to be

$$f := (f_0, f_1, \ldots, f_{n-1}). \tag{5.66}$$

The idea is to build a Pascal-like triangle using dif and only iferences instead of sums. More precisely, write the integers f_i down the right side and 1 down the left, then fill in entries from the top down so that each interior value is the dif and only iference between the one above it to the right and the one above it to the left. For instance, in dimension two, the triangle would be

$$\begin{array}{ccccccc} & & & 1 & & f_0 & & \\ & & 1 & & f_0-1 & & f_1 & \\ & 1 & & f_0-2 & & f_1-f_0+1 & & f_2 \\ 1 & & f_0-3 & & f_1-2f_0+3 & & f_2-f_1+f_0-1 & \end{array} \tag{5.67}$$

We define the h-vector as the last line of this Pascal-like triangle, read from right to left.

Definition 5.34 *Given a simplicial polytope in $\mathbb{R}^n$ with f-vector $f=(f_0,\dots,f_{n-1})$, define the **h-vector** as*

$$h=(h_0,h_1,\dots,h_n), \qquad h_p := \sum_{k=p}^{n}(-1)^{k-p}\binom{k}{p} f_{n-1-k}, \tag{5.68}$$

where by convention $f_{-1}:=1$ and $\binom{x}{p}:=\frac{x(x-1)\cdots(x-p+1)}{p!}$.

With this convention, Euler's formula simply becomes

$$h_0=h_n, \tag{5.69}$$

as $h_n=1$ (by design) and $h_0=\sum_{k=0}^{n}(-1)^k f_{n-1-k}$, which is independent of the dimension.

Interestingly, Euler's formula is just the tip of the iceberg of many more symmetries satisfied by the h-vector. To unravel such symmetries, let us compute the h-vector for a couple of examples more.

- Icosahedron: $(f_0,f_1,f_2)=(12,30,20)$:

$$\begin{array}{ccccccc} & & & 1 & & 12 & & \\ & & 1 & & 11 & & 30 & \\ & 1 & & 10 & & 19 & & 20 \\ 1 & & 9 & & 9 & & 1 & \end{array} \tag{5.70}$$

Figure 5.2: The projection from $\mathbb{R}^4$ to $\mathbb{R}^3$ of the pentachoron (left) and the hexadecachoron (right). They are the 3d analogue of the tetrahedron and the octahedron, respectively.

- Hexadecachoron: $(f_0, f_1, f_2, f_3) = (8, 24, 32, 16)$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 8 & & \\
 & & & 1 & & 7 & & 24 & \\
 & & 1 & & 6 & & 17 & & 32 \\
 & 1 & & 5 & & 11 & & 15 & & 16 \\
1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{5.71}$$

From the above examples, one notices another pattern in the h-vector: not only are the first and last entries equal, but the entire h-vector appears to be palindromic:

$$h_p = h_{n-p}, \qquad p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{5.72}$$

These relations are known as the **Dehn–Sommerville equations**, and generalise Euler's formula for $p = 0$. They were expressed in this form by Sommerville in the 1920s. Moreover, the h-vector seems to be weakly increasing up to the midpoint and weakly decreasing thereafter. This property is called the unimodality of the h-vector. One way to express this feature is via a third vector, known as the g-vector.

Definition 5.35 *Given a simplicial polytope in $\mathbb{R}^n$ with f-vector $f = (f_0, \ldots, f_{d-1})$, define the g-vector as*

$$g = (g_1, \ldots, g_{\lfloor \frac{n}{2} \rfloor}), \qquad g_p := h_p - h_{p-1}, \tag{5.73}$$

where $h = (h_1, \ldots, h_n)$ is the associated h-vector.

Then the unimodality of the h-vector can be restated as

$$g_p \geq 0, \qquad p = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{5.74}$$

For example, in dimension 2, the above inequality reads $f_0 \geq 4$, expressing the fact that a simplicial polyhedron must have at least four vertices. The inequality $g_p \geq 0$ can thus be seen as a higher-dimensional generalisation of this basic condition.

In summary, given a simplicial polytope with f-vector f, we have defined the h-vector and the g-vector as in Equations (5.68) and (5.73). Our experiments suggest that such vectors always satisfy two properties:

- Palindromicity of the h-vector (the Dehn–Sommerville equations); and
- Non-negativity of the g-vector.

A natural question arises: are these two conditions necessary and sufficient for a positive integer vector to be the f-vector of a simplicial polytope? The answer is 'almost': there is a third condition that the g-vector must satisfy. For example, since any two vertices can be joined by at most one edge, we must have

$$f_1 \leq \binom{f_0}{2}. \tag{5.75}$$

The complete characterisation is given in the following theorem, conjectured by P. McMullen [9].

Theorem 5.36 *A vector of positive integers $f = (f_0, f_1, \ldots, f_{n-1})$ is the f-vector of a simplicial polytope if and only if the following conditions hold:*

1. *The h-vector is palindromic (Dehn–Sommerville):*
$$h_p = h_{n-p}, \qquad p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{5.76}$$
2. *The g-vector is non-negative:*
$$g_p \geq 0, \qquad p = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{5.77}$$
3. *The g-vector is a Macaulay vector: let $n_p > n_{p-1} > \cdots > n_r \geq r \geq 1$ be the unique integers such that*
$$g_p = \binom{n_p}{p} + \cdots + \binom{n_r}{r}. \tag{5.78}$$
Then
$$g_{p+1} \leq \binom{n_p+1}{p+1} + \cdots + \binom{n_r+1}{r+1}, \qquad p = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{5.79}$$

The existence of a convex polytope for each f-vector satisfying the above conditions was established by Billera and Lee [10] via a direct and ingenious construction. The necessity of the conditions was proved by Stanley using methods from toric geometry [11]. We now present Stanley's argument.

5.3.2 Stanley's theorem

The goal of this section is to prove that the above conditions are necessary: given a simplicial polytope in $\mathbb{R}^n$, then conditions 1 to 3 hold. The main idea is to construct a toric variety $X = X_\Delta$ for some complete simplicial fan Δ, in such a way that the number d_k of k-dimensional cones in Δ coincides with the number f_{k-1} of $(k-1)$-dimensional faces of the given simplicial polytope.

To achieve this, observe that since the polytope in $\mathbb{R}^n$ is simplicial, a slight perturbation of all its vertices yields a new polytope with the same number of faces in each dimension. By such a perturbation, we can assume that the vertices lie in the set $\mathbb{Q}^n$ of rational points. By choosing an appropriate lattice N, we may further assume that all vertices lie in N. We can also translate the polytope so that the origin lies in its interior.

Define Δ to be the fan consisting of the cones over the faces of the polytope (with vertex at the origin), together with the trivial cone $\{0\}$. Then it is immediate that

$$d_k = f_{k-1}, \qquad d_0 = f_{-1} = 1. \tag{5.80}$$

The resulting toric variety $X = X_\Delta$ is projective. In fact, if $P \subset M_\mathbb{R}$ denotes the polar dual of the original polytope, then $\Delta = \Delta_P$ is the fan of P. The key result of Stanley is the relation between the h-vector of the original polytope and the Chow groups of the toric variety X.

Lemma 5.37 *The h-vector associated with the starting polytope is equal to the vector of dimensions of the rational Chow groups of the toric variety X:*

$$h_p = \dim_\mathbb{Q} A^p(X)_\mathbb{Q}, \tag{5.81}$$

where $A^p(X)_\mathbb{Q} := A^p(X) \otimes \mathbb{Q}$.

If we assume for a moment the above result, then it is clear that the conditions 1 to 3 can be restated as conditions on the dimensions of the Chow groups of X. From now on, we assume X to be smooth (although all of the results below can be generalised to the case of orbifolds, which is the geometric property corresponding to the simplicial condition). In this case, there is an isomorphism

$$A^p(X)_{\mathbb{Q}} \cong H^{2p}(X, \mathbb{Q}), \tag{5.82}$$

where the left-hand side is the rational Chow group, and the right-hand side denotes the singular cohomology of X with $\mathbb{Q}$-coefficients (or, equivalently, the cohomology of the constant sheaf $\underline{\mathbb{Q}}$). Moreover, the odd-degree cohomology groups vanish. The intersection product in the Chow ring corresponds to the cup product in cohomology. Hence, we can equivalently work with even rational cohomology groups.

With this in mind, the Dehn–Sommerville equations (condition 1) reduce to Poincaré duality.

Theorem 5.38 (Poincaré duality) *Let X be a smooth, proper, n-dimensional variety. Then the pairing defined by the composition of the intersection product and the integral,*

$$H^d(X)_{\mathbb{Q}} \otimes H^{2n-d}(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}, \qquad \alpha \otimes \beta \longmapsto \int_X \alpha \cdot \beta, \tag{5.83}$$

is perfect. In particular, we have an isomorphism

$$A^d(X)_{\mathbb{Q}} \cong (A^{2n-d}(X)_{\mathbb{Q}})^* \tag{5.84}$$

as $\mathbb{Q}$-vector spaces.

It is then clear that the Dehn–Sommerville equations follow:

$$\begin{aligned} h_p = \dim_{\mathbb{Q}} A^p(X)_{\mathbb{Q}} = \dim_{\mathbb{Q}} H^{2p}(X, \mathbb{Q}) = \\ = \dim_{\mathbb{Q}} H^{2n-2p}(X, \mathbb{Q}) = \dim_{\mathbb{Q}} A^{n-p}(X)_{\mathbb{Q}} = h_{n-p}. \end{aligned} \tag{5.85}$$

The unimodality condition, i.e., condition 2, is a consequence of another deep theorem in Algebraic Geometry known as the Hard Lefschetz theorem. It can be viewed as a refinement of Poincaré duality for projective (not just proper) varieties.

Theorem 5.39 (Hard Lefschetz) *Let X be a smooth, projective, n-dimensional variety. Then there exists an element $\omega \in H^2(X, \mathbb{Q})$ such that, for all $d = 0, \dots, n$, the map*

$$H^d(X, \mathbb{Q}) \longrightarrow H^{2n-d}(X, \mathbb{Q}), \qquad \alpha \longmapsto \omega^{n-d} \cdot \alpha \tag{5.86}$$

is an isomorphism of $\mathbb{Q}$-vector spaces.

Back to the toric case, where Chow groups and even cohomology groups are identified. An immediate consequence of the Hard Lefschetz theorem is that the map

$$L_p \colon A^p(X)_{\mathbb{Q}} \longrightarrow A^{p+1}(X)_{\mathbb{Q}}, \qquad \alpha \longmapsto \omega \cdot \alpha \tag{5.87}$$

is injective for $p \leq \lfloor \frac{n}{2} \rfloor$. In particular, we find that $h_p \leq h_{p+1}$, i.e., $g_p \geq 0$.

The Macaulay condition, i.e., condition 3, is more delicate, and follows from the description of the Chow ring of toric varieties discussed in the previous lecture: it is generated in degree 1. More precisely, we found that

$$A^\bullet(X)_{\mathbb{Q}} = \frac{\mathbb{Q}[D_1, \dots, D_d]}{I(X)_{\mathbb{Q}}}, \tag{5.88}$$

where $I(X)_{\mathbb{Q}}$ is an explicit ideal, and, most importantly, $D_1, \dots, D_d$ are the closures of the orbits associated to rays, which generate $A^1(X)_{\mathbb{Q}} = \mathrm{Cl}(X)_{\mathbb{Q}}$.

To deduce the Macaulay condition, we introduce an auxiliary algebra. For ease of notation, set $A^p := A^p(X)_{\mathbb{Q}}$. Let $J \subseteq A^\bullet$ be the ideal generated by ω and $A^{\lfloor n/2 \rfloor + 1}$, and set

$$R^p := \frac{A^p}{J \cap A^p}, \qquad p \geq 0. \tag{5.89}$$

Since $A^\bullet$ is generated by A^1, for $p > \lfloor n/2 \rfloor$ we have $A^p \subseteq J$, and thus $R^p = 0$. Therefore,

$$R^\bullet = \bigoplus_{p=0}^{\lfloor n/2 \rfloor} R^p, \tag{5.90}$$

which is a commutative graded $\mathbb{Q}$-algebra generated by R^1. Moreover,

$$R^0 = \frac{A^0}{J \cap A^0} = \frac{\mathbb{Q}}{(0)} = \mathbb{Q}. \tag{5.91}$$

Macaulay characterised the vectors of integers that can arise as the vector of dimensions of such an algebra.

Lemma 5.40 (Macaulay) *Let $R^\bullet$ be a commutative graded $\mathbb{Q}$-algebra, with $R^0 = \mathbb{Q}$ and generated by R^1. Then the vector of dimensions $(\dim_{\mathbb{Q}} R^0, \dim_{\mathbb{Q}} R^1, \dots)$ is a Macaulay vector.*

The Macaulay condition then follows. Indeed, notice that for $p = 1, \dots, \lfloor n/2 \rfloor$ we have

$$h_{p-1} = \dim_{\mathbb{Q}} A^{p-1} = \dim_{\mathbb{Q}} \mathrm{im}(L_{p-1}) + \dim_{\mathbb{Q}} \ker(L_{p-1}) = \dim_{\mathbb{Q}} (J \cap A^{p-1}), \tag{5.92}$$

where the second equality is the rank–nullity theorem, and the third uses the fact that L_{p-1} is injective. Therefore,

$$\dim_{\mathbb{Q}} R^p = \dim_{\mathbb{Q}} A^p - \dim_{\mathbb{Q}}(J \cap A^p) = h_p - h_{p-1} = g_p. \tag{5.93}$$

To conclude, we provide a proof of Lemma 5.37, which states that the h-vector equals the vector of dimensions of the Chow groups. By construction, the number d_k of k-dimensional cones in Δ coincides with the number f_{k-1} of $(k-1)$-dimensional faces. Using this and the definition of the h-vector, we deduce that the claim is equivalent to the identity

$$\dim_{\mathbb{Q}} A^p = \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} d_{n-k}. \tag{5.94}$$

This is an interesting result in its own right: it provides an explicit formula for the dimensions of the Chow groups—also called Betti numbers—in terms of the much simpler combinatorial data of the fan, namely the number of cones.

To prove the identity above, it helps to introduce the **Poincaré polynomial**, which packages the Betti numbers into a single generating function. It is defined for any *smooth, complete* variety X as

$$P_X(t) := \sum_{d \geq 0} \dim_{\mathbb{Q}} H^d(X, \mathbb{Q}) t^d, \tag{5.95}$$

and can be extended to singular and non-proper spaces using three fundamental properties:

- **Excision:** If $Z \subset X$ is closed and $U = X \setminus Z$, then

$$P_X(t) = P_Z(t) + P_U(t). \tag{5.96}$$

- **Additivity:** If $X = \bigsqcup_i X_i$ is a disjoint union of finitely many locally closed subsets, then

$$P_X(t) = \sum_i P_{X_i}(t). \tag{5.97}$$

- **Multiplicativity:** If $X = Y \times Z$, then

$$P_X(t) = P_Y(t) \cdot P_Z(t). \tag{5.98}$$

These properties make the Poincaré polynomial a practical tool for computations.

Example 5.41 *For $\mathbb{C}^*$, apply excision to $X = \P^1$, with $Z = \{0\} \sqcup \{\infty\}$ and $U = \mathbb{C}^*$. Since $P_{\P^1}(t) = 1 + t^2$ and $P_Z(t) = 2$, we find:*

$$P_{\mathbb{C}^*}(t) = t^2 - 1. \tag{5.99}$$

By multiplicativity, we obtain $P_{(\mathbb{C}^*)^k}(t) = (t^2 - 1)^k$.

These properties, together with the cone–orbit correspondence, yield more information about the Poincaré polynomial of a toric variety. Indeed, the cone–orbit correspondence implies that $X = \bigsqcup_{\sigma \in \Delta} O_\sigma$, and each $O_\sigma \cong (\mathbb{C}^*)^{\dim(\sigma)}$. Hence, by additivity,

$$P_X(t) = \sum_{\sigma \in \Delta} P_{O_\sigma}(t) = \sum_{k=0}^{n} d_{n-k}(t^2 - 1)^k. \tag{5.100}$$

Expanding the binomial power, we find

$$P_X(t) = \sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{p} d_{n-k} t^{2p} = \sum_{p=0}^{n} \underbrace{\sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} d_{n-k}}_{=\dim_{\mathbb{Q}} H^{2p}(X,\mathbb{Q}) = \dim_{\mathbb{Q}} A^p(X)_{\mathbb{Q}}} t^{2p}. \tag{5.101}$$

For simplicial (rather than smooth) fans, the same conclusion holds, although one must use more sophisticated cohomological tools: namely, intersection cohomology and its mixed Hodge structure.

We conclude with a nice application of the above formula.

Corollary 5.42 *The Euler characteristic of a smooth projective toric variety X of dimension n equals the number of top-dimensional cones:*

$$\chi(X) = d_n. \tag{5.102}$$

Proof Notice that $\chi(X) = P_X(-1)$. Thus,

$$\chi(X) = \sum_{k=0}^{n} d_{n-k}(t^2 - 1)^k \big|_{t=-1} = d_n. \tag{5.103}$$

□

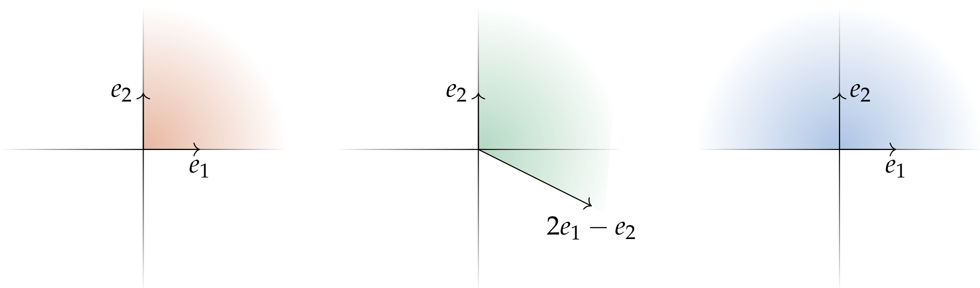
Chapter 6

Exercises

Sheet 1

Exercise 6.1 For each of the following cones:

1. Write the generators and compute the dimension.
2. Compute the dual cone.
3. List all the faces of σ, expressed as $\sigma \cap u^{\perp}$ for some u in the dual ambient vector space. Do the same for $\check{\sigma}$.
4. Verify the orientation-reversing bijection $\tau \leftrightarrow \tau^*$ and the dimension formula $\dim(\tau) + \dim(\tau^*) = n$, where n is the dimension of the ambient space.



Bonus: Try the following case in $\mathbb{R}^3$, where the cone is generated by e_1, $-e_1$, $e_2 + e_3$, and $-e_2 + e_3$. □

Exercise 6.2 Compute the generators of the monoid $\sigma \cap \mathbb{Z}^2$, where σ is the green cone above. □

Exercise 6.3 Let $\sigma \subset \mathbb{R}^n$ be a cone, and let $M \subset \mathbb{R}^n$ be a lattice. Gordon's lemma states that if σ is a rational cone, then the monoid $\sigma \cap M$ is finitely generated. Can you provide an example of a non-rational cone for which $\sigma \cap M$ is not finitely generated? $\square$

Exercise 6.4 Let $\sigma = \{0\}$ be the trivial cone in $\mathbb{Z}^n \subset \mathbb{R}^n$. Compute the associated monoid S_σ, the associated algebra R_σ, and determine the corresponding variety. $\square$

Sheet 2

Exercise 6.5 Consider the cone σ in $\mathbb{R}^2$ generated by e_1 and $3e_1 - 2e_2$. Describe $\check{\sigma}$, find generators of $S_\sigma = \check{\sigma} \cap \mathbb{Z}^2$, compute the toric ideal of the affine variety X_σ, and describe the torus in X_σ. $\square$

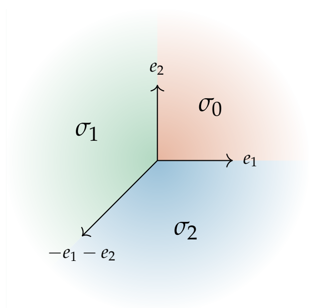
Exercise 6.6 Consider the cone σ in $\mathbb{R}^3$ generated by e_1, e_2 and $e_1 + e_2 + 2e_3$. Describe $\check{\sigma}$, find generators of $S_\sigma = \check{\sigma} \cap \mathbb{Z}^3$, compute the toric ideal of the affine variety X_σ, and describe the torus in X_σ. $\square$

Exercise 6.7 Let $\sigma \subset \mathbb{R}^n$ be a cone. Prove that the following are equivalent.

- σ is strongly convex.
- $\{ 0 \}$ is a face of σ.
- σ contains no positive-dimensional subspace of $\mathbb{R}^n$.
- $\sigma \cap (-\sigma) = \{ 0 \}$
- $\dim \check{\sigma} = n$. $\square$

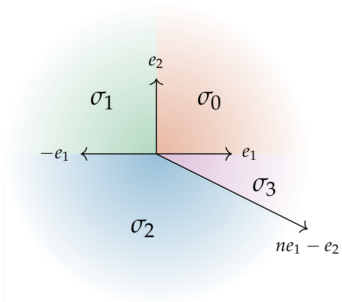
Sheet 3

Exercise 6.8 Work out the details of the construction of $\mathbb{P}^2$ from the following fan in $\mathbb{R}^2 \supset \mathbb{Z}^2$.



More precisely, the fan consists of the two-dimensional cones $\sigma_0, \sigma_1, \sigma_2$, the one-dimensional cones given by their faces, namely $\sigma_0 \cap \sigma_1$, $\sigma_0 \cap \sigma_2$, $\sigma_1 \cap \sigma_2$, and the zero-dimensional cone, the origin, which is the triple intersection $\sigma_0 \cap \sigma_1 \cap \sigma_2 = \{0\}$. $\square$

Exercise 6.9 For a fixed non-negative integer $n \geq 0$, consider the fan Δ_n in $\mathbb{R}^2 \supset \mathbb{Z}^2$ represented as follows.



As above, the fan consists of the two-dimensional cones $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ and all their intersections. Compute the associated variety X_{Δ_n}, known as the n-th Hirzebruch surface and denoted Σ_n. Convince yourself that the resulting variety is given by

$$\Sigma_n \cong \frac{(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})}{\mathbb{C}^* \times \mathbb{C}^*}, \tag{6.1}$$

where the action of $\mathbb{C}^* \times \mathbb{C}^*$ is defined as[1]

$$(\lambda, \mu) \cdot (z_1, z_2, w_1, w_2) = (\lambda z_1, \lambda z_2, \mu w_1, \lambda^n \mu w_2). \tag{6.2}$$

$\square$

Exercise 6.10 Let Δ be a fan in $\mathbb{R}^n$ with lattice N, and Δ' be a fan in $\mathbb{R}^{n'}$ with lattice N'. Define

$$\Delta \times \Delta' = \left\{ \sigma \times \sigma' \;\middle|\; \sigma \in \Delta, \sigma' \in \Delta' \right\}. \tag{6.3}$$

Convince yourself that $\Delta \times \Delta'$ is a fan in $\mathbb{R}^{n+n'}$ with lattice $N \oplus N'$, and that the associated variety is $X_{\Delta \times \Delta'} \cong X_\Delta \times X_{\Delta'}$. Use this to give a short proof that the fan Δ_0 from Exercise 6.9 gives $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

[1]From the above representation, one can deduce that Σ_n is the total space of the projectivisation of the vector bundle $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathbb{P}^1$, where the action of $\mathbb{C}^*_{(\lambda)}$ gives the total space of $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathbb{P}^1$ and the action of $\mathbb{C}^*_{(\mu)}$ projectivise the fibres.

Sheet 4

Exercise 6.11 In the definition of the distinguished point x_τ associated with a cone τ in a fan, we required τ to be the face of a cone σ. Does the definition depend on the choice of σ? $\square$

Exercise 6.12 Compute the distinguished points, the orbits, and their closures for every cone in the following toric varieties:

- The double cone.
- The projective line $\mathbb{P}^1$.
- The surface $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Sheet 5

Exercise 6.13 Verify the orbit–cone correspondence for $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Exercise 6.14 The goal of this exercise is to recover the fan of $\mathbb{P}^2$ from its toric structure. More precisely, write down:

- The embedded torus $T \subset \mathbb{P}^2$ and its action on $\mathbb{P}^2$.
- The isomorphism between $\mathbb{Z}^2$ and the character and cocharacter lattices.
- The one-to-one correspondence between the limit points of one-parameter subgroups and the cones of the fan associated with $\mathbb{P}^2$. $\square$

Exercise 6.15 The goal of this exercise is to understand rational functions on the projective line $\mathbb{P}^1$.

- Convince yourself that all rational functions on $\mathbb{P}^1$ are of the form

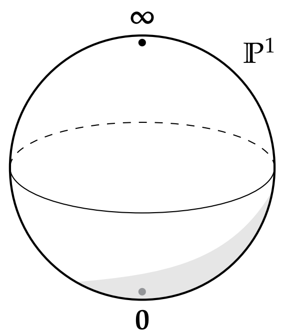
$$f([z,w]) = \frac{p(z,w)}{q(z,w)}, \qquad [z,w] \in \mathbb{P}^1, \tag{6.4}$$

  where $p, q \in \mathbb{C}[z,w]$ are *homogeneous* polynomials *of the same degree* and *with no common factors*.

- To the rational function f as above, associate the function

$$F\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad F([z,w]) = [p(z,w), q(z,w)], \tag{6.5}$$

  with p and q as above. Convince yourself that this defines a one-to-one correspondence. In particular, zeros of f correspond to points in $F^{-1}([0:1]) = p^{-1}(0,0)$, and poles of f correspond to points in $F^{-1}([1:0]) = q^{-1}(0,0)$. For this reason, it is customary to denote $\mathbf{0} = [0:1]$ and $\boldsymbol{\infty} = [1:0]$ in $\mathbb{P}^1$, see figure below. $\square$



Recall that the Weil group of X is defined as the free abelian group generated by codimension-1 irreducible subvarieties of X. If X is one-dimensional, then the Weil group is simply the free abelian group generated by the points of X. Explicitly, for X one-dimensional:

$$\mathrm{WDiv}(X) := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{Z}, \text{ finitely many } a_x \text{ are non-zero} \right\}. \tag{6.6}$$

Recall that $\mathrm{PDiv}(X)$ is the subgroup of divisors associated with global rational functions on X, called principal divisors. The divisor of a rational function f, denoted $\mathrm{div}(f)$, is the formal sum of zeros minus poles, counted with multiplicity. The quotient $\mathrm{WDiv}(X)/\mathrm{PDiv}(X)$ is called the class group, denoted $\mathrm{Cl}(X)$.

Exercise 6.16 The goal of this exercise is to compute the class group of $\mathbb{P}^1$. We proceed in three steps.

- For f as in Equation (6.4), write down $\mathrm{div}(f)$. (Hint: use the fundamental theorem of algebra.)
- Consider the map

$$\deg\colon \mathrm{WDiv}(\mathbb{P}^1) \longrightarrow \mathbb{Z}, \qquad \deg\left(\sum_{p \in \mathbb{P}^1} a_p p\right) = \sum_{p \in \mathbb{P}^1} a_p, \tag{6.7}$$

  called the *degree map*. It is clearly surjective. Its kernel is called the group of degree-zero Weil divisors. Show that $\mathrm{PDiv}(\mathbb{P}^1) \subseteq \ker(\deg)$, i.e., every principal divisor has degree zero.

- Show that the converse is also true on $\mathbb{P}^1$: every degree-zero Weil divisor is principal. Conclude that

$$\mathrm{Cl}(\mathbb{P}^1) = \frac{\mathrm{WDiv}(\mathbb{P}^1)}{\mathrm{PDiv}(\mathbb{P}^1)} \cong \mathbb{Z}. \tag{6.8}$$

□

Sheet 6

Exercise 6.17 Prove that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. □

Exercise 6.18 Let P be a polytope with $0 \in \mathrm{int}(P)$ with associated fan Δ_P. Show that

$$\psi_P \colon |\Delta_P| \longrightarrow \mathbb{R}, \qquad \psi_P(v) := \min_{u \in P} \langle u, v \rangle, \tag{6.9}$$

is a support function and that $[\psi_P] \neq 0$ in $\mathrm{Pic}(X_P)$. Use this to conclude that the fan obtained from the standard cube in $\mathbb{R}^3$ by replacing $(1,1,1)$ with $(1,2,3)$ is non-polytopal. □

Exercise 6.19 Let $X = \mathbb{P}^2$. Compute $H^p(X, \mathcal{O}_X)$ from the definition of sheaf cohomology, taking the affine cover of defined by the fan of $\mathbb{P}^2$ as an open cover. □

Exercise 6.20 Let $X = \mathbb{P}^n$. For any $d \in \mathbb{Z}$, consider $D := dD_0$, where D_0 is the closure of the orbit associated with the ray generated by $e_0 = -(e_1 + \cdots + e_n)$. Set $\mathcal{O}(d) := \mathcal{O}_{\mathbb{P}^n}(D)$

- Prove that the support function ψ_D evaluates to zero on $e_1, \ldots, e_n$ and to m on e_0. Deduce that ψ_D is zero on the cone generated by $e_1, \ldots, e_n$, and is $m \langle e_i^*, \cdot \rangle$ on the cone generated by $e_0, \cdots, \widehat{e_i}, \cdots, e_n$.

- For $(u_1, \ldots, u_n) \in \mathbb{Z}^n$, prove that χ^u is the rational function

$$\chi^u \colon [z_0, z_1, \cdots, z_n] \longmapsto \frac{z_1^{u_1} \cdots z_n^{u_n}}{z_0^{u_1 + \cdots + u_n}}. \tag{6.10}$$

After setting $z_0 = 1$, we obtain a monic Laurent monomial in $z_1, \ldots, z_n$.

- For $d \geq 0$, verify that ψ_D is convex. Verify that

$$P_D = \left\{ (u_1, \ldots, u_n) \in \mathbb{Z}^n \;\middle|\; u_i \geq 0 \text{ and } \sum_i u_i \leq d \right\}. \tag{6.11}$$

Deduce that

$$H^p(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \mathbb{C}[z_1, \ldots, z_n]_d & \text{if } p = 0, \\ 0 & \text{else,} \end{cases} \tag{6.12}$$

where $\mathbb{C}[z_1, \cdots, z_n]_d$ is the space of polynomials of degree d.

- For $d < 0$ verify that ψ_D is concave, so that $H^p_{Z(u)}(|\Delta|) = 0$ unless $Z(u) = 0$. Deduce that

$$H^p(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} (z_1^{-1} \cdots z_n^{-1}\mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}])_d & \text{if } p = n, \\ 0 & \text{else,} \end{cases} \tag{6.13}$$

where by convention z_i^{-1} is in degree -1. $\square$

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