Introduction to Gromov-Witten theory

Alessandro Giacchetto & Campbell Wheeler

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This is a collection of notes based on [2, 6].

1. Motivation & history

In symplectic topology and algebraic geometry, Gromov-Witten (GW) invariants are rational numbers that count pseudo-holomorphic curves meeting prescribed conditions in a given symplectic manifold, or algebraic curves meeting prescribed conditions in a given algebraic variety. They also play a crucial role in closed type IIA string theory.

The invariants were first introduced by Gromov in [3] in the symplectic context. A striking application of it was the proof of the non-squeezing theorem: it is not possible to embed a ball into a cylinder via a symplectic map, unless the radius of the ball is less than or equal to the radius of the cylinder. This highlights how symplectic transformations are more restrictive than volume-preserving ones. Successively, the invariants were studied by Witten [7] in a string theory context, where the free energy of the theory corresponds to the GW invariants generating function. In the aforementioned paper, Witten conjectured some relations between such invariants, and in some sense built bridges to enumerative algebraic geometry.

In the following, we will adopt this last point of view, started with Kontsevich in [4]. Let us start with the most natural enumerative problem.

How many straight lines pass between two distinct points in the plane?

The answer is intuitively easy: only one. This is nothing but Euclid's 1st axiom. The second natural step is the following.

How many conics pass through five generic points in the plane?

The fact that this is one is a classically known fact (Apollonius, 50 AD). How can you show this? We can construct the solution explicitly. Given five points (a_i, b_i), the polynomial is constructed as the determinant

$$p(x,y) = \begin{vmatrix} 1 & x & y & x^2 & xy & y^2 \\ 1 & a_1 & b_1 & a_1^2 & a_1 b_1 & b_1^2 \\ & & & \vdots & & \\ 1 & a_5 & b_5 & a_5^2 & a_5 b_5 & b_5^2 \end{vmatrix}.$$

The natural generalization of these questions is the following.

How many rational nodal curves of degree d pass through $3d-1$ generic points in the projective plane?

Call N_d such number. After the discoveries $N_1 = N_2 = 1$ of the antiques, it took quite some time to prove that the number of nodal cubics passing through 8 general points in the plane is $N_3 = 12$. This was done by Steiner in 1848. The next step was done by Chasles, de Jonquières and Zeuthen around

1870, that is $N_4 = 620$. Immediately after, Schubert computed $N_5 = 87304$. The final answer, known as Kontsevich formula, solved in 1993 the problem at once:

$$N_d = \sum_{\substack{d_A, d_B > 0 \\ d_A + d_B = d}} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A-1} \right) N_{d_A} N_{d_B}$$

The aim of these notes is to give a proof of Kontsevich formula through Gromov-Witten theory.

A small remark: why do we need exactly $3d-1$ conditions for a degree d curve rational curve in the plane? A quick explanation goes as follows. The space of degree d curves in $\mathbb{P}^2$ consists of degree d polynomials in two variables, up to constants. This space has dimension $\binom{d+2}{2} - 1$. Further, a generic curve in such family will be smooth of genus $g = \frac{(d-1)(d-2)}{2}$. In order to have a nodal, rational curve, we have to impose $\frac{(d-1)(d-2)}{2}$ conditions (one for each node). Thus, the space of nodal, rational curves in the plane will have dimension

$$\binom{d+2}{2} - 1 - \frac{(d-1)(d-2)}{2} = 3d - 1.$$

2. Moduli space of stable maps and Gromov-Witten invariants

We see that a natural question is counting curves of a fixed degree and a prescribed genus into $\mathbb{P}^2$. Let us make it more general: consider a smooth projective variety X. For technical reasons, let us suppose that X has even cohomology only. The degree can be substituted with an element $\beta \in H_2(X, \mathbb{Z})$, and we will try to count curves by parameterising them.

DEFINITION 2.1 (*(Pre-)stable curves*). A pre-stable curve of type (g, n) is the data $(C, p_1, \dots, p_n)$ of

- a (possibly nodal) curve of arithmetic genus g (*i.e.* $\chi(\mathcal{O}_C) = 1 - g$),
- smooth distinct marked points $p_1, \dots, p_n$.

The curve is called stable if it has finitely many automorphisms.

DEFINITION 2.2 (*Stable maps*). Let X be a smooth projective variety. A stable map of type (g, n) is the data $(C, p_1, \dots, p_n, f)$ of

- a pre-stable curve $(C, p_1, \dots, p_n)$ of type (g, n),
- a map $f\colon C \to X$ with finitely many automorphisms.

A morphism between $f\colon (C, p_1, \dots, p_n) \to X$ and $g\colon (C', p'_1, \dots, p'_n) \to X$ is a morphism between marked curves $\varphi\colon (C, p_1, \dots, p_n) \to (C', p'_1, \dots, p'_n)$ such that $f = g \circ \varphi$.

The stability condition for a map, that is the condition of admitting finitely many automorphisms, is equivalent to requiring that, for any irreducible component C_0 of C contracted to a point by f, one has

$$2g(C_0) - 2 + n(C_0) > 0,$$

where $n(C_0)$ is the number of special points (marked points or nodes) on C_0.

We want to define the moduli space of stable maps. We can refine this space by the discrete data of a homology class $\beta \in H_2(X, \mathbb{Z})$.

DEFINITION 2.3 (*Moduli space of stable maps*). Define the moduli space of stable maps of type (g, n) as

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \left\{ (C, p_1, \dots, p_n, f) \;\middle|\; \begin{smallmatrix} f \text{ is stable of type } (g,n) \\ f_*[C] = \beta \end{smallmatrix} \right\} \Big/ \sim .$$

EXAMPLE 2.4. Consider the case of a zero-dimensional target: $X = \{*\}$. Then $\overline{\mathcal{M}}_{g,n}(X, \beta) = \overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves.

Another simple example is $X = \mathbb{P}^2$, $\beta = 1$ and type $(0, 0)$.* This parameterizes maps $\mathbb{P}^1 \to \mathbb{P}^2$ up to reparameterisation of the map. That is, this is nothing but the collection of lines in $\mathbb{P}^2$, or $\mathrm{Gr}(2, 3)$. More generally,

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1) = \mathrm{Gr}(2, r+1).$$

This is obviously smooth, compact, and irreducible. It is the best of all worlds.

*We will use the convention that, if $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, then we use an integer to represent the homology class which is that multiple of a generator. In the case of $\mathbb{P}^r$, $H_2(\mathbb{P}^r, \mathbb{Z}) = \mathbb{Z}.H$ where H is the class of a hyperplane.

Things rapidly degenerate from here, however. Let us consider the next simplest case, that of plane conics. Consider

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2).$$

This should be the space of conics in $\mathbb{P}^2$, but it is not. This moduli space is built up as follows.

- There is a locus of maps whose sources are smooth. This is denoted by $\mathcal{M}_{0,0}(\mathbb{P}^2, 2)$.
  - Generically, the image of such a map will be a smooth conic, which will be an open locus in the space of all conics, which itself is isomorphic to $\mathbb{P}^5$.
  - There is a sublocus consisting of those maps that are 2: 1 covers of a line in $\mathbb{P}^2$. This is a 4-dimensional locus, since we need two parameters to describe the target line, and two to describe the ramification points of the map. Each map in this locus also has $\mathbb{Z}_2$ as an automorphism group, coming from the exchange of the covering sheets.
- Consider the "boundary" $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2) \setminus \mathcal{M}_{0,0}(\mathbb{P}^2, 2)$.
  - There are those curves whose domains consist of a nodal curve with two components, each of which maps with degree 1 into $\mathbb{P}^2$. Within this, there is the locus of those maps with image two distinct lines (which necessarily join at one point). This is a four dimensional space, two for each line in $\mathbb{P}^2$.
  - Deeper into the boundary, there is the locus of those curves with nodal sources, but whose image are both the same line. This is three dimensional; two for the line, and one for the point on that line where the two components meet. Furthermore, every map in this locus also has $\mathbb{Z}_2$ as automorphism group, since there is an automorphism of the source curve which exchanges the two components.

We have to remark that, as a moduli space, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has much more structure than that of a set. By definition, a moduli space should be equipped with a geometry that encodes how objects can deform. This can be done, so that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has the structure of a Deligne-Mumford stack (the algebro-geometric version of a complex orbifold).

Notice from the previous example that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ does not necessarily have a well-defined dimension. However, it does have an "expected" or "virtual" dimension, calculated by studying the space of infinitesimal deformations of a stable map (the tangent space to the moduli space) as well as the obstructions to extending infinitesimal deformations to honest ones. Explicitly, the virtual (complex) dimension is

$$d_{g,n}(X, \beta) = (\dim X - 3)(1 - g) + \int_\beta c_1(\mathcal{T}_X) + n,$$

where $c_1(\mathcal{T}_X)$ is the first Chern class of the holomorphic tangent bundle to X. Intuitively, one should understand the virtual dimension by imagining that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the zero locus of a section s of a rank-r vector bundle $E \to B$ on some nonsingular ambient space B. If s does not intersect the zero section of E transversally, then the dimension of the zero locus $Z(s)$ could be larger than expected, but generically, one expects its dimension to be $\dim(B) - r$. This is the "virtual" dimension of $Z(s)$.

Equipped with a replacement for the notion of dimension, it is a difficult fact [1] that there also exists a replacement for the fundamental class, an element

$$\left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X, \beta))$$

known as the virtual fundamental class, which agrees with the fundamental class in the case where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is smooth of the expected dimension. Again, the idea can be explained intuitively by supposing that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the zero locus of a section of a vector bundle, which may not meet the zero section transversally. For example, consider the least transverse situation possible, when s is identically zero. Then $[Z(s)] = [B]$ lies in too-high dimension, but there is a natural way to achieve a homology class in the virtual dimension: take $[B] \cap e(E)$, where $e(E)$ is the Euler class of E. This amounts to perturbing $s \equiv 0$ to a transverse section and then taking its zero locus.

As an example, consider the case of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$. Since $\mathcal{T}_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(3)$, it follows that

$$\int_{dH} c_1(\mathcal{T}_{\mathbb{P}^2}) = 3d.$$

Thus, the virtual dimension of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ will be

$$\dim \mathbb{P}^2 - 3 + \int_{dH} c_1(\mathcal{T}_{\mathbb{P}^2}) + n = 3d - 1 + n.$$

Further, the formula for the virtual dimension explains why GW invariants are particularly interesting in the case of Lastly Calabi-Yau threefolds. In such a case, we see that most of the terms in the dimension formula vanish: $\dim X = 3$ covers the first term, while $c_1(\mathcal{T}_X) = 0$ covers the second. Thus, if X is a Calabi-Yau threefold, then $d_{g,0}(X, \beta) = 0$, and so we should generically expect finitely many curves of any genus in one of these varieties.

With this technical remarks, we want to use the previously discussed $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to count holomorphic curves in X. We first note that this space comes together with some evaluation maps to X. That is, there are maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ defined by

$$\mathrm{ev}_i(C, p_1, \ldots, p_n, f) = f(p_i).$$

We also have forgetful morphisms. Assuming the $\overline{\mathcal{M}}_{g,n}(X, \beta)$ exists, these are given by the maps $\pi_n \colon \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ where we set

$$\pi_n(C, p_1, \ldots, p_{n+1}, f) = (C, p_1, \ldots, p_n, f)^{\mathrm{stab}}.$$

That is, we forget the $(n + 1)$-st marked point of the source curve, and we stabilize the resulting map, *i.e.* we collapse any components of the curve that is unstable. We also have a forgetful morphism $\pi \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$

$$\pi(C, p_1, \ldots, p_n, f) = (C, p_1, \ldots, p_n)^{\mathrm{stab}}$$

which forgets the map (and the target space), provided again that the latter moduli space exists, that is $2g - 2 + n > 0$.

We are now ready to give the definition of GW invariants. Consider $\gamma_1, \ldots, \gamma_n \in H^\bullet(X)$. We set

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \prod_{i=1}^n \mathrm{ev}_i^* \gamma_i,$$

where $\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ stands for

$$\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \alpha = \begin{cases} \left[\overline{\mathcal{M}}_{g,n}(X, \beta)\right]^{\mathrm{vir}} \cap \alpha & \text{if } \alpha \in H^{2\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X, \beta)), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = 0$ unless

$$\sum_{i=1}^n \deg \gamma_i = 2d_{g,n}(X, \beta).$$

How should we interpret these numbers? Consider the case in which $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a smooth, compact orbifold with components all of the same dimension (the expected dimension, of course), and suppose that each γ_i is the Poincaré dual of a subvariety Y_i of X. The Poincaré dual of $\mathrm{ev}_i^* \gamma_i$ represents the collection of maps $f \colon (C, p_1, \ldots, p_n) \to X$ such that $f(p_i) \in Y_i$. Moreover, since the cup product is Poincaré dual to intersection, we have that $\prod_i \mathrm{ev}_i^* \gamma_i$ represents (in a suitably generic setting) those maps $f \colon (C, p_1, \ldots, p_n) \to X$ such that $f(p_i) \in Y_i$ for all $1 \leqslant i \leqslant n$. Since the location of the points on C is arbitrary (*i.e.* varies over the moduli space), we can read this as follows.

The cohomology class $\prod_i \mathrm{ev}_i^* \gamma_i$ represents the collection of morphisms $f \colon (C, p_1, \ldots, p_n) \to X$ such that the image $f(C)$ intersects Y_i for all $1 \leqslant i \leqslant n$.

If this is a finite number (which should generically occur if this class is a top class in $H^\bullet(\overline{\mathcal{M}}_{g,n}(X, \beta))$), then by pairing it with the fundamental class we should get the number of such curves. That is, if we consider the integral

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \prod_{i=1}^n \mathrm{ev}_i^* \gamma_i,$$

then this number can be interpreted as follows.

The GW invariant $\langle\gamma_1\cdots\gamma_n\rangle^X_{g,\beta}$ is the number of genus g curves in X such that they have non-zero intersection with the subvarieties $Y_1,\dots,Y_n$.

Now, we have assumed for the purposes of this discussion that the moduli space is smooth, compact, and finite-dimensional. Unfortunately, this is not necessarily true. It is proper (compact), but it is often not smooth, and it often has many different components of varying dimensions as we saw before. In particular, it is not clear whether GW invariants have always the associated enumerative interpretation. This is in general a hard problem, which require *ad hoc* techniques that will not be discussed here. We just remark that for our main example, that is $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2,d)$, the GW invariants coincide with the enumerative interpretation. That is,

$$\langle\underbrace{\mathrm{pt}\cdots\mathrm{pt}}_{3d-1}\rangle^{\mathbb{P}^2}_{0,d}=N_d$$

is the number of nodal rational curves of degree d in the projective plane. In the next section we will introduce one of the basic tools in GW theory, the GW potential and the WDVV equation. This will allow us to compute all GW invariants of the plane, and thus prove Kontsevich formula. Before going on with the GW potential, let us state some of the fundamental properties of GW invariants, that are consequences of the properties of the virtual fundamental class.

FUNDAMENTAL CLASS AXIOM. We have the equality

$$\langle\gamma_1\cdots\gamma_{n-1}\cdot 1\rangle^X_{g,\beta}=\langle\gamma_1\cdots\gamma_{n-1}\rangle^X_{g,\beta}$$

We can think of this as saying that imposing the constraint that a point on our curve be incident to X is no condition at all. This has the further consequence that

$$\langle\gamma_1\cdots\gamma_{n-1}\cdot 1\rangle^X_{g,\beta}=0$$

provided that $(g,n,\beta)\neq(0,3,0)$. This is because the moduli spaces in question on the left- or right-hand side have different dimension. It thus follows that if the forgetful map exists, then we must have that the GW invariants are zero.

DIVISOR AXIOM. If $[D]\in H^2(X)$ is a divisor, then

$$\langle\gamma_1\cdots\gamma_{n-1}\cdot[D]\rangle^X_{g,\beta}=\left(\int_\beta[D]\right)\langle\gamma_1\cdots\gamma_{n-1}\rangle^X_{g,\beta},$$

provided that $(g,n,\beta)\neq(0,3,0)$. In this case, this is morally due to the fact that the possible number of points that a curve $f(C)$ with $f_*[C]=\beta$ may intersect a divisor D is exactly $\int_\beta[D]$.

POINT MAPPING AXIOM. The invariants with $\beta=0$ satisfy

$$\langle\gamma_1\cdots\gamma_n\rangle^X_{g,0}=\begin{cases}\int_X\gamma_1\cup\gamma_2\cup\gamma_3 & \text{if } (g,n)=(0,3),\\ 0 & \text{otherwise.}\end{cases}$$

This follows from the fact that $\overline{\mathcal{M}}_{0,3}(X,0)=X$, since there is a unique isomorphism $C\cong\mathbb{P}^1$ sending the three marked points to 0, 1, and ∞, so all that must be chosen to specify a point in $\overline{\mathcal{M}}_{0,3}(X,0)$ is the image point of the constant map $f\colon C\to X$. Furthermore, the virtual class actually is the ordinary fundamental class in this case, so no deformation-theoretic argument is required.

3. Gromov-Witten potential

The key to working with GW invariants to their full potential is to do what one should always do when confronted with an infinite collection of numbers depending on discrete data: arrange them into a generating function. In order to do so, we need to fix some notation. As before, let X be a smooth projective variety with even cohomology only, and let $\alpha_0,\alpha_1,\dots,\alpha_m$ be a basis (as a $\mathbb{Q}$-vector space) of $H^\bullet(X)$ such that

- $\alpha_0=1\in H^0(X)$,
- $\alpha_1,\dots,\alpha_r\in H^2(X)$ be a basis of $H^2(X)$.

DEFINITION 3.1 (*Gromov-Witten potential*). We define the genus g Gromov-Witten potential function of X to be the formal series

$$\Phi_g^X(t_0,\dots,t_m;q) = \sum_{k_0,\dots,k_m\geqslant 0}\sum_{\beta\in H_2(X,\mathbb{Z})} \langle \alpha_0^{k_0}\cdots\alpha_m^{k_m}\rangle_{g,\beta}^X \left(\prod_{i=0}^m \frac{t_i^{k_i}}{k_i!}\right) q^\beta.$$

A small clarification: the symbol γ^k in the GW invariants stands for

$$\langle\cdots\gamma^k\cdots\rangle_{g,\beta}^X = \langle\cdots\underbrace{\gamma\cdots\gamma}_{k\text{ times}}\cdots\rangle_{g,\beta}^X.$$

Further, the q^β term might look a little odd, as β is a homology class. To make this precise, we can look at it in the following way. Let $\beta_1,\dots,\beta_r$ be a basis of $H_2(X,\mathbb{Z})$. For convenience, it is sometimes nice to choose it to be dual to the basis $\alpha_1,\dots,\alpha_r$ of $H^2(X)$ in the sense that

$$\int_{\beta_p}\alpha_q = \delta_{pq}.$$

In this case, we can write any $\beta=\sum_{i=1}^r d_i\beta_i$ and, for a collection of formal variables $q_1,\dots,q_r$, we set

$$q^\beta = \prod_{i=1}^r q_i^{d_i}.$$

As q^β can be manipulated similarly, that is $q^{\beta_1+\beta_2}=q^{\beta_1}q^{\beta_2}$, it does not really matter. Writing q^β is more invariant (*i.e.* does not rely on a choice of basis), which is one reason that it may be preferred.

From a physics standpoint (and from a mirror symmetry standpoint) we should not really consider q as a formal variable at all. We should instead consider it as a coordinate on the Kähler moduli space of X, which we denote by $\mathcal{K}_X$. That is, if we consider the function

$$q\colon \mathcal{K}_X\times H_2(X,\mathbb{Z})\to\mathbb{C},\qquad (\omega,\beta)\mapsto q^\beta = e^{2\pi i\int_\beta\omega},$$

then we interpret the GW potential as a function

$$\Phi_g^X\colon \mathcal{K}_X\to\mathbb{C}[\![t_0,\dots,t_m]\!].$$

However, we must then contend with issues of convergence. To avoid these, one can consider it to be a purely formal series $\mathbb{Q}[\![t_0,\dots,t_m,q_1,\dots,q_r]\!]$.

The first thing that we can do is to simplify it by using the divisor axiom. Let us focus, for fixed $\beta\neq 0$ and for $1\leqslant s\leqslant r$, on the sum

$$\sum_{k_s\geqslant 0}\langle\alpha_0^{k_0}\cdots\alpha_s^{k_s}\cdots\alpha_m^{k_m}\rangle_{g,\beta}^X\frac{t_s^{k_s}}{k_s!}.$$

Repeated use of the divisor axiom yields

$$\sum_{k_s\geqslant 0}\langle\alpha_0^{k_0}\cdots\widehat{\alpha_s^{k_s}}\cdots\alpha_m^{k_m}\rangle_{g,\beta}^X\left(\int_\beta\alpha_s\right)\frac{t_s^{k_s}}{k_s!} = \langle\alpha_0^{k_0}\cdots\alpha_{s-1}^{k_{s-1}}\cdot\alpha_{s+1}^{k_{s+1}}\cdots\alpha_m^{k_m}\rangle_{g,\beta}^X\, e^{t_s\int_\beta\alpha_s}.$$

Thus, we find that the GW potential with $\beta\neq 0$ is given by

$$\sum_{\beta\in H_2(X,\mathbb{Z})}\langle\alpha_0^{k_0}\cdot\alpha_{r+1}^{k_{r+1}}\cdots\alpha_m^{k_m}\rangle_{g,\beta}^X\frac{t_0^{k_0}}{k_0!}\left(\prod_{s=1}^r e^{t_s\int_\beta\alpha_s}\right)\left(\prod_{i=r+1}^m\frac{t_i^{k_i}}{k_i!}\right)q^\beta.$$

There is a further simplification due to the fundamental class axiom and the point mapping axiom. Let us demonstrate it by computing the genus zero GW potential of $\mathbb{P}^2$. Let us fix the basis $\alpha_0=1,\alpha_1=H,\alpha_2=\mathrm{pt}$. We have

$$\Phi_0^{\mathbb{P}^2}(t_0,t_1,t_2;q) = \sum_{k_0,k_1,k_2}\langle 1^{k_0}\cdot H^{k_1}\cdot \mathrm{pt}^{k_2}\rangle_{0,0}^{\mathbb{P}^2}\frac{t_0^{k_0}}{k_0!}\frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!} + \sum_{k_0,k_2}\sum_{d\in\mathbb{Z}^\times}\langle 1^{k_0}\cdot\mathrm{pt}^{k_2}\rangle_{0,d}^{\mathbb{P}^2}\frac{t_0^{k_0}}{k_0!}\left(e^{t_1\int_{dH}H}\right)\frac{t_2^{k_2}}{k_2!}q^d.$$

Notice now few facts. We can simplify the second term by noticing that the sum is over $d > 0$, since $d < 0$ cannot support the class of a curve. Further, $e^{t_1 \int_{dH} H} = e^{dt_1}$ and from the fundamental class axiom

$$\langle 1^{k_0} \cdot \mathrm{pt}^{k_2} \rangle_{0,d}^{\mathbb{P}^2} = \begin{cases} N_d & \text{if } k_0 = 0,\ k_2 = 3d-1 \\ 0 & \text{otherwise.} \end{cases}$$

Further, from the point mapping axiom, we have that the second term is different form zero, unless $k_0 + k_1 + k_2 = 3$ and $2k_1 + 4k_2 = 2d_{0,0}(\mathbb{P}^2, 0) = 4$. In this case, the invariant $\langle 1^{k_0} \cdot H^{k_1} \cdot \mathrm{pt}^{k_2} \rangle_{0,0}^{\mathbb{P}^2}$ will be given by

k_0	k_1	k_2	$\langle 1^{k_0} \cdot H^{k_1} \cdot \mathrm{pt}^{k_2} \rangle_{0,0}^{\mathbb{P}^2}$
2	0	1	$\int_{\mathbb{P}^2} \mathrm{pt} = 1$
1	2	0	$\int_{\mathbb{P}^2} H^2 = 1$

Thus, the genus zero GW potential of $\mathbb{P}^2$ will be

$$\Phi_0^{\mathbb{P}^2}(t_0, t_1, t_2; q) = \frac{1}{2}\left(t_0^2 t_2 + t_0 t_1^2\right) + \sum_{d=1}^{\infty} N_d e^{dt_1} \frac{t_2^{3d-1}}{(3d-1)!} q^d.$$

4. Quantum Product

Using the GW potential in genus zero, we define a deformation of the usual cup product on cohomology. We will keep using the above conventions for the basis of $H^\bullet(X)$.

DEFINITION 4.1 (*Quantum products*). Define $g_{ij} = \int_X \alpha_i \cup \alpha_j$ and let g^{ij} be the inverse matrix. Set $\alpha^i = \sum_{j=1}^m g^{ij} \alpha_j$. Define the big quantum product to be $*\colon H^\bullet(X)[\![t_0, \ldots, t_m, q]\!]^{\otimes 2} \to H^\bullet(X)[\![t_0, \ldots, t_m, q]\!]$ as

$$\alpha_i * \alpha_j = \sum_{k=0}^{m} \frac{\partial^3 \Phi_0^X(t_0, \ldots, t_m, q)}{\partial t_i \partial t_j \partial t_k} \Phi_0^X(t_0, \ldots, t_m; q) \alpha^k.$$

We define the small quantum product $*\colon H^\bullet(X)[\![q]\!]^{\otimes 2} \to H^\bullet(X)[\![q]\!]$ by setting

$$\alpha_i * \alpha_j = \sum_{k=0}^{m} \left. \frac{\partial^3 \Phi_0^X(t_0, \ldots, t_m; q)}{\partial t_i \partial t_j \partial t_k} \right|_{t_0 = \cdots = t_m = 0} \alpha^k.$$

Define the small quantum cohomology ring as $QH^\bullet(X) = (H^\bullet(X)[\![q]\!], *)$.

LEMMA 4.2. Setting $q = 0$ in the small quantum product, we get the standard intersection pairing.

PROOF. Firstly, notice that

$$\Phi_0^X(t_0, \ldots, t_m; 0) = \sum_{k_0, \ldots, k_m \geqslant 0} \langle \alpha_0^{k_0} \cdots \alpha_m^{k_m} \rangle_{0,0}^X \left(\prod_{i=0}^{m} \frac{t_i^{k_i}}{k_i!} \right).$$

Then from the point mapping axiom, we see that

$$\Phi_0^X(t_0, \ldots, t_m; 0) = \sum_{k_a + k_b + k_c = 3} \int_X \alpha_a^{k_a} \cup \alpha_b^{k_b} \cup \alpha_c^{k_c} \frac{t_a^{k_a}}{k_a!} \frac{t_b^{k_b}}{k_b!} \frac{t_c^{k_c}}{k_c!}$$

and as a consequence

$$\left. \frac{\partial^3 \Phi_0^X(t_0, \ldots, t_m; q)}{\partial t_i \partial t_j \partial t_k} \right|_{t_0 = \cdots = t_m = q = 0} = \int_X \alpha_i \cup \alpha_j \cup \alpha_k.$$

□

EXAMPLE 4.3. On $\mathbb{P}^2$ we have the following small quantum multiplication table. Note that $\alpha_0 = \alpha^2 = 1$, $\alpha_1 = \alpha^1 = H$ and $\alpha_2 = \alpha^0 = \mathrm{pt}$. We see that

$$QH^\bullet(\mathbb{P}^2) \cong \mathbb{Q}[H]/(H^3 - N_1 q)$$

whereas

$$H^\bullet(\mathbb{P}^2) \cong \mathbb{Q}[H]/H^3.$$

$*$	1	H	pt
1	1	H	pt
H	H	pt	N_1q
pt	pt	N_1q	N_1qH

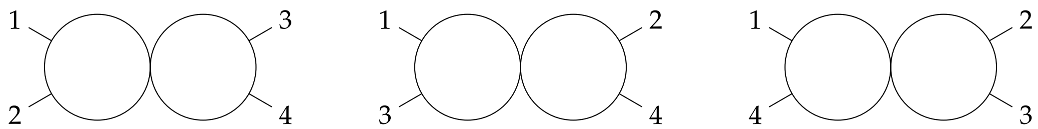
Notice that the big quantum product is clearly commutative. The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations express the associativity of the product.

THEOREM 4.4 (*WDVV equations*). The genus zero GW potential Φ_0^X satisfies the WDVV equations

$$\sum_{a,b} \Phi_{ija} g^{ab} \Phi_{bkl} = \sum_{a,b} \Phi_{jka} g^{ab} \Phi_{bil}$$

IDEA OF PROOF. The idea the proof is similar to the standard proof of associativity of the product defined by a 2d-TQFT. We want to consider an invariant associated to the four holed sphere, and then use different decompositions into three holed spheres to actually calculate. This should be independent of the decomposition into three holed spheres.

More precisely, recall that $\mathcal{M}_{0,4} \cong \mathbb{P}^1 - \{0, 1, \infty\}$ and $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, where we have three boundary divisors, denoted by $D(12|34)$, $D(13|24)$ and $D(14|23)$, which using the identification above are 0 1 and ∞. These correspond to the nodal curves



Notice that we have a linear equivalence $0 \equiv 1 \equiv \infty$ and therefore $D(12|34) \equiv D(13|24) \equiv D(14|23)$. Now consider the map $\Pi \colon \overline{\mathcal{M}}_{0,n}(X, \beta) \to \overline{\mathcal{M}}_{0,4}$ defined by $\Pi(C, p_1, \ldots, p_n, f) = (C, p_1, \ldots, p_4)^{\mathrm{stab}}$. Then $\Pi^* D(12|34) \equiv \Pi^* D(13|24) \equiv \Pi^* D(14|23)$.

To relate this to Gromov-Witten invariants we use the the fact that integrating over boundary divisors can be written in terms of Gromov-Witten invariants higher Euler characteristic. That is, it can be shown that $\Pi^* D(ab|cd) \subseteq \overline{\mathcal{M}}_{0,n}(X, \beta)$ is a divisor that for some $n = n_1 + n_2$ and $\beta = \beta_1 + \beta_2$ decomposes as

$$D = \overline{\mathcal{M}}_{0,n_1+1}(X, \beta_1) \times_X \overline{\mathcal{M}}_{0,n_2+1}(X, \beta_2),$$

where the fibre product identifies the last marked point of each component. Interpreted in the right way, this can be lifted to the level of virtual fundamental classes and we obtain

$$\int_D \mathrm{ev}_1^*(\alpha_1) \cup \cdots \cup \mathrm{ev}_n^*(\alpha_n) = \sum_{i=0}^m \langle \alpha_1 \cdots \alpha_{n_1} \alpha_i \rangle_{0,\beta_1}^X \langle \alpha^i \alpha_{n_1+1} \cdots \alpha_n \rangle_{0,\beta_2}^X .$$

The linear equivalence gives the thesis. □

COROLLARY 4.5. The big quantum product $*$ on $H^\bullet(X)[\![t_0, \ldots, t_m, q]\!]$ is an associative and commutative product.

5. Proof of Kontsevich formula

THEOREM 5.1 (*Kontsevich formula*). Let N_d denote the number of degree d rational nodal curves passing through $3d - 1$ generic points in $\mathbb{P}^2$. Then N_d satisfies the recurrence relation

$$N_d = \sum_{d_1+d_2=d} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2},$$

with initial condition $N_1 = 1$.

PROOF. Denote simply $\partial_{ijk} \Phi_0^{\mathbb{P}^2}$ by Φ_{ijk}. The WDVV equation for $\mathbb{P}^2$ is given by

$$\Phi_{ij0}\Phi_{2kl} + \Phi_{ij1}\Phi_{1kl} + \Phi_{ij2}\Phi_{0kl} = \Phi_{jk0}\Phi_{2il} + \Phi_{jk1}\Phi_{1il} + \Phi_{jk2}\Phi_{0il},$$

which for $(i, j, k, l) = (1, 2, 2, 1)$ becomes

$$\Phi_{120}\Phi_{221} + \Phi_{121}\Phi_{121} + \Phi_{122}\Phi_{021} = \Phi_{220}\Phi_{211} + \Phi_{221}\Phi_{111} + \Phi_{222}\Phi_{011}$$

and reduces to

$$\Phi_{222} = \Phi_{112}^2 - \Phi_{111}\Phi_{122}.$$

This gives

$$\sum_{d=1}^{\infty} N_d e^{dt_1} \frac{t_2^{3d-4}}{(3d-4)!} q^d = \left(\sum_{d=1}^{\infty} d^2 N_d e^{dt_1} \frac{t_2^{3d-2}}{(3d-2)!} q^d \right)^2 + \\ + \left(\sum_{d=1}^{\infty} d^3 N_d e^{dt_1} \frac{t_2^{3d-1}}{(3d-1)!} q^d \right) \left(\sum_{d=1}^{\infty} d N_d e^{dt_1} \frac{t_2^{3d-3}}{(3d-3)!} q^d \right).$$

Collecting the coefficient of q^d, we find

$$N_d e^{dt_1} \frac{t_2^{3d-4}}{(3d-4)!} = \sum_{d_1+d_2=d} d_1^2 N_{d_1} e^{d_1 t_1} \frac{t_2^{3d_1-2}}{(3d_1-2)!} N_{d_2} e^{d_2 t_1} \frac{t_2^{3d_2-2}}{(3d_2-2)!} + d_1^3 N_{d_1} e^{d_1 t_1} \frac{t_2^{3d_1-1}}{(3d_1-1)!} d_2 N_{d_2} e^{d_2 t_1} \frac{t_2^{3d_2-3}}{(3d_2-3)!}$$

from which we conclude

$$\begin{aligned} N_d &= \sum_{d_1+d_2=d} d_1^2 N_{d_1} N_{d_2} \frac{(3d-4)!}{(3d_1-2)!(3d_2-2)!} + d_1^3 N_{d_1} d_2 N_{d_2} \frac{(3d-4)!}{(3d_1-1)!(3d_2-3)!} \\ &= \sum_{d_1+d_2=d} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}. \end{aligned}$$

□

6. Descendants

In the terminology of Witten, the GW invariants are correlation functions for some topological σ-model. The idea of (gravitational) descendants is that they should correspond to coupling the topological σ-model to gravity. Mathematically, this corresponds to the insertion of ψ-classes into GW invariants:

$$\langle \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \rangle_{g,\beta}^X = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \prod_{i=1}^{n} \pi^* \psi_i^{a_i} \, ev_i^* \gamma_i,$$

where ψ_i is the first Chern class of the line bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$, whose fiber over $[C, p_1, \ldots, p_n]$ is $T_{p_i}^* C$. Such GW invariants can be computed in many examples using the following equations.

STING EQUATION:

$$\langle \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \cdot \tau_0 1 \rangle_{g,\beta}^X = \sum_{i=1}^{n} \langle \tau_{a_1}\gamma_1 \cdots \tau_{a_i-1}\gamma_i \cdots \tau_{a_n}\gamma_n \rangle_{g,\beta}^X.$$

DILATON EQUATION:

$$\langle \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \cdot \tau_1 1 \rangle_{g,\beta}^X = (2g-2+n) \langle \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \rangle_{g,\beta}^X.$$

TOPOLOGICAL RECURSION RELATIONS:

$$\langle \tau_{a_1}\gamma_1 \cdots \tau_{a_n}\gamma_n \cdot \tau_{k+1}\alpha_a \cdot \tau_l \alpha_b \cdot \tau_m \alpha_c \rangle_{0,\beta}^X \\ = \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ \beta_1+\beta_2=\beta}} \sum_{i=0}^{m} \left\langle \prod_{i\in I} \tau_{a_i}\gamma_i \cdot \tau_k \alpha_a \cdot \tau_0 \alpha_i \right\rangle_{0,\beta_1}^X \left\langle \tau_0 \alpha^i \cdot \prod_{j\in J} \tau_{a_j}\gamma_j \cdot \tau_l \alpha_b \cdot \tau_m \alpha_c \right\rangle_{0,\beta_2}^X$$

As before, α_i is a basis of $H^\bullet(X)$ and α^i is the dual basis with respect to the Poincaré pairing.

We can now form the generating function for the descendant invariants. Set

$$t(z) = \sum_{a\geqslant 0} \sum_{i=0}^{m} t_a^i z_a \alpha_i$$

and define

$$F_0^X(t) = \sum_{n\geqslant 0} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle t(\tau)^n \rangle_{0,\beta}^X .$$

For the $X = \{*\}$, this is already a non-trivial generating function in infinitely many variables. It was conjectured by Witten, and proved by Kontsevich, that such generating function satisfies an integrable hierarchy, the KdV hierarchy.

THEOREM 6.1 (*Kontsevich theorem [5], Witten conjecture [7]*). The function

$$F^*(t_0, t_1, \dots) = \sum_{\substack{g \geqslant 0, n \geqslant 1 \\ 2g-2+n>0}} \sum_{a_1+\dots+a_n = 3g-3+n} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} t_{a_i} \psi_i^{a_i}$$

is a τ-function for the KdV hierarchy.

In particular the string equation is

$$\frac{\partial F^*}{\partial t_0} = \frac{1}{2} t_0^2 + \sum_{k=0}^{\infty} t_{k+1} \frac{\partial F^*}{\partial t_k}$$

and it also satisfies the KdV equation

$$\frac{\partial^2 F^*}{\partial t_0 \partial t_1} = \frac{1}{2} \left(\frac{\partial^2 F^*}{\partial t_0^2} \right)^2 + \frac{1}{12} \frac{\partial^4 F^*}{\partial t_0^4}.$$

Noting that $F^* = t_0^3/6 + \cdots$ along with these equations is enough to completely determine the generating function. There are some conjectures relating the more general generating series to integrable hierarchies as well.

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MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, 53111 BONN
Email address: agiacche@mpim-bonn.mpg.de, cjwh@mpim-bonn.mpg.de