Hurwitz theory, with (a) spin

Overview:
1) (Spin) Hurwitz #s & rep. theory
2) Fermion formalism & top. recursion
3) ELSV & applications to GW theory

LECTURE 1

§1.1) (SPIN) H#s:

"Hurwitz theory is the enumeration of branched covers of Riemann surfaces"

WHY?
1) Natural counting (maps between curves)
2) Relations to combinat.
   - counting of permutations
   - rep. theory of symm. group
3) Relations to math-ph
   - integreble hierarchies, matrix model
   - topological recursion
4) Relations to alg. geom
   - moduli space of curves (ELSV)
   - GW theory of $\mathbb{P}^1$
   - Masur-Veech volumes of holom. diffs

cmct, conn, RS

Defn. Fix a base $B \ni x_1, \dots, x_k$, and $\mu^1, \dots, \mu^k \vdash d$. The **Hurwitz number** $H_d(B; \mu^1, \dots, \mu^k)$ is the number of iso classes of covers $f: C \to B$ where
- C is cmct, conn, RS
- f is ramified over x_i with ram. profile μ^i, unramified elsewhere, $d:1$ — of type $(\mu^1, \dots, \mu^k)$
- $(f: C \to B) \sim (f': C' \to B)$ iff $\exists\ C \xrightarrow{\varphi} C'$ bihol. st.
$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr] & \square & C' \arrow[dl] \\ & B & \end{tikzcd}$$

- the weight is $\dfrac{1}{|\mathrm{Aut}(f)|}$

$$\rightsquigarrow \quad H_d(B;\mu^1,\dots,\mu^k) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}$$

Notation. Disconnected HHs are denoted as $H_d^{\bullet}(B;\mu^1,\dots,\mu^k)$.
↑ C possibibly disc.

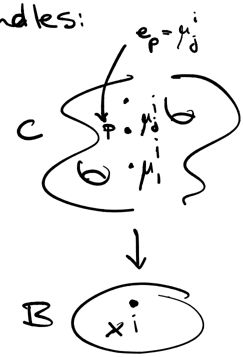
Fact (Riemann-Hurwitz formula). $f: C \to B$ of type $(\mu^1,\dots,\mu^k)$

$$2g(C)-2 = d(2g(B)-2) + \sum_{i=1}^{k}(d-\ell(\mu^i))$$

Consequence of a general formula between line bundles:

canonical bundle = cotangent bundle

$$\omega_C \cong f^*\omega_B \otimes \mathcal{O}\Big(\sum_{p\in \mathrm{Ram}(f)} (e_p-1)p\Big)$$



holom diffs on C = holom diffs on C with zeros of order e_p-1 at p × merom fncts on C with poles of order e_p-1 at p

Example.

① $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$, $z \mapsto z^2$ $\rightsquigarrow$ gen. 2:1
ramified at $0,\infty$
$\mathrm{Aut}(f) = \mathbb{Z}_2 = \{z \mapsto \pm z\}$

$\Rightarrow \quad H_2(\mathbb{P}^1;(2),(2)) = \frac{1}{2}$ $\quad\{\ 2g(C)-2 = 2(-2)+[(2-1)+(2-1)] = -2$
$\Rightarrow g(C) = 0$ ✓

② $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$, $z \mapsto z^d$ $\rightsquigarrow$ gen $d:1$
ramified at $0,\infty$
$\mathrm{Aut}(f) = \mathbb{Z}_d = \{z \mapsto \zeta z \mid \zeta^d = 1\}$

$\Rightarrow \quad H_d(\mathbb{P}^1;(d),(d)) = \frac{1}{d}$

③ $T = \{(x,y) \mid y^2 = x^3+ax+b\} \xrightarrow{f} \mathbb{P}^1$, $(x,y) \mapsto x$ $\rightsquigarrow$ gen. 2:1
ram at 3 roots of x^3+ax+b and $x=\infty$

$\mathrm{Aut}(\ell) = \mathbb{Z}_2 = \{(x,y) \mapsto (x, \pm y)\} \Rightarrow H_2(\mathbb{P}^1; (2),(2),(2),(2)) = \frac{1}{2}$

WHAT ABOUT SPIN?

"Spin Hurwitz theory is the signed enumeration of branched covers of RS."

Defn. A spin structure on B is a holomorphic line bundle $\vartheta \to B$ s.t. $\vartheta^{\otimes 2} \cong \omega_B$

Ex. Spin structures on $\mathbb{P}^1$? Recall:

$$\begin{cases} \mathbb{P}^1 = \{ V \subseteq \mathbb{C}^2 \mid V = \text{1-dim vector subspace} \} \\ \mathcal{O}(-1) \to \mathbb{P}^1 \quad \text{taut. line bundle,} \qquad \mathcal{O}(-1)_V = V \end{cases}$$

Fact ①: if L is a line bundle over $\mathbb{P}^1$, $\exists d \in \mathbb{Z}$ s.t.

$L \cong \mathcal{O}(-1)^{\otimes d} =: \mathcal{O}(-d)$

Fact ②: $\omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$

$\Rightarrow$ $\mathcal{O}(-1)$ is the only spin structure over $\mathbb{P}^1$.

Defn. $\vartheta \to B$ a spin structure. Define parity

$p(\vartheta) = h^0(B, \vartheta) \pmod 2$ where $h^0(B,\vartheta) = \dim_{\mathbb{C}} \underbrace{H^0(B,\vartheta)}_{\text{space of global sections of } \vartheta \to B}$

$\rightsquigarrow$ ϑ is $\begin{cases} \text{pos. if } p(\vartheta) = 0 \\ \text{neg. if } p(\vartheta) = 1 \end{cases}$

Take $f: C \to B$ of type $(\mu^1, \dots, \mu^k)$

$$\omega_C \cong f^*\omega_B \otimes \mathcal{O}\Big(\sum_{p \text{ ram.}} (\underbrace{e_p}_{\text{parts of } \mu^i} - 1)p\Big)$$

Suppose $\vartheta \to B$ is a spin structure:

$$\vartheta^{\otimes 2} \cong \omega_B$$

IF all parts of μ^i are **ODD**, then $e_p - 1$ is even!

$$\vartheta_f := f^*\vartheta \otimes \mathcal{O}\Big(\sum_{p \text{ ram}} \frac{e_p - 1}{2}\, p\Big)$$

is a spin structure on C:

$$\vartheta_f^{\otimes 2} = f^* \underbrace{\vartheta^{\otimes 2}}_{\cong \omega_B} \otimes \mathcal{O}\Big(\sum_p (e_p - 1)p\Big) \cong \omega_C$$

Defn. Fix $B \ni x_1, \dots, x_k$, $\mu^1, \dots, \mu^k \vdash d$ ODD partitions. Fix $\vartheta \to B$ spin strct. Define **spin Hurwitz numbers**

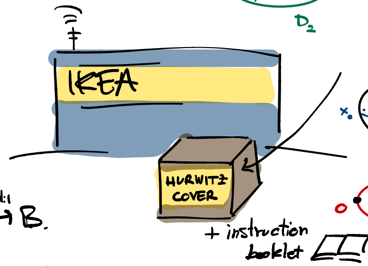
$$H_d(B, \vartheta; \mu^1, \dots, \mu^k) = \sum_{[f]} \frac{(-1)^{p(\vartheta_f)}}{|\mathrm{Aut}(f)|}$$

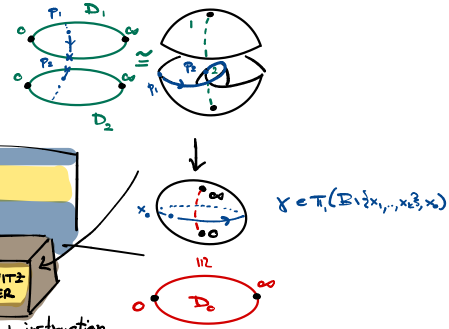
Rmk. Spin H#s are related to vol("positive" holom diffs) $-$ vol("negative" holom diffs)

Rmk. $(p, n) \leftrightarrow (p+n, p-n)$

behaves well

§ 1.1) GEOMETRY $\overset{\text{IKEA}}{\longleftrightarrow}$ ALGEBRA





0) IKEA employee takes order $f: C \xrightarrow{d:1} B$.

1) Cut B into a topological disk D_0, with cuts containing branch pnts $\Rightarrow$ C cut into d disks $D_1, \dots, D_d$

2) Choose $x_0 \in B^* = B - \{x_1, \dots, x_k\}$ $\Rightarrow$ $p_j \in f^{-1}(x_0)$ in D_j, $j = 1, \dots, d$

$\forall \gamma \in \pi_1(B^*, x_0)$, $\exists!$ lift $\tilde{\gamma}_j$ that starts at $\tilde{\gamma}_j(0) = p$

It will end at $\tilde{\gamma}_j(1) = p_{\sigma_\gamma(j)}$, $\sigma_\gamma \in S_d$

$$\Rightarrow \quad \rho: \pi_1(B^*, x_0) \to S_d \qquad \text{monodromy representation}$$
$$\gamma \mapsto \sigma_\gamma$$

If γ is a loop arond x_i $\Rightarrow$ $\rho(\gamma)$ has cycle type μ^i

We say that ρ has type $(\mu^1, \dots, \mu^k)$.

Thm. There is an automorphism preserving bijection

$$\left\{ \begin{array}{c} \text{iso class of covers} \\ \text{(labeled) } f: C \to B \; + \vartheta \\ C \text{ is possibly disc.} \\ \text{ram. at } x_1, \dots, x_k \\ \text{ram profile } \mu^1, \dots, \mu^k \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{monodr. reprs} \\ \rho: \pi_1(B^*, x_0) \to S_d \\ \text{of type } \mu^1, \dots, \mu^k \end{array} \right\}$$

ϑ_f is positive/neg. odd

w/ or w/o a lift to $\tilde{S}_d$

$\boxed{\text{SPIN?}}$ Spin symmetric group: $0 \to \mathbb{Z}_2 \to \tilde{S}_d \to S_d \to 0$

Defn. Define the group algebra

$$\mathbb{C}[S_d] = \left\{ \sum_{\sigma \in S_d} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{C} \right\}$$

with addition and mult. by sclars

$$\sum_\sigma a_\sigma \sigma + \sum_\sigma b_\sigma \sigma = \sum_\sigma (a_\sigma b_\sigma), \qquad \lambda \sum_\sigma a_\sigma \sigma = \sum_\sigma (\lambda a_\sigma) \sigma$$

and multipl. given by mult. on S_d, extended linearly.

Ex If $\mu \vdash d$, $\quad C_\mu = \sum_{\substack{\sigma \text{ of cycle} \\ \text{type } \mu}} \sigma$

For $d=3$:

id	$\mu = (1\,1\,1)$	$C_{(1,1,1)} = \mathrm{id}$
(12)	$\mu = (2\,1)$	$C_{(2\,1)} = (12) + (13) + (23)$
(13)	$\mu = (3)$	$C_{(3)} = (123) + (132)$
(23)		
(123)		
(132)		

Thm. If $B = \mathbb{P}^1$,

coeff of identity

$$H^\bullet_d(\cancel{\mathbb{P}^1}\, E; \mu^1, \dots, \mu^k) = \frac{1}{d!} [\mathrm{id}].\, K^{g(E)}\, C_{\mu^k} \cdots C_{\mu^1}$$

Proof. $\pi_1(\mathbb{P}^1 \setminus \{x_1, \dots, x_k\}, x_0) \cong \langle \gamma_1, \dots, \gamma_k \mid \gamma_1 \cdots \gamma_k \rangle$.

Thus, a monodromy rep of type $\mu^1, \dots, \mu^k$ is a choice of

$$\rho(\gamma_i) = \sigma_i \text{ of cycle type } \mu^i, \quad \text{w/} \quad \sigma_k \cdots \sigma_1 = \mathrm{id}$$

Ex: $H^\bullet_3(\mathbb{P}^1; (3), (3)) = \frac{1}{3!} [\mathrm{id}] \underbrace{C_{(3)} \cdot C_{(3)}}_{} = \frac{2}{3!} = \frac{1}{3}$

$$[(123) + (132)]^2 = (132) + (123) + 2\,\mathrm{id}$$

Defn. Class algebra of S_d is the centre of $\mathbb{C}[S_d]$

$$Z_d = \{ x \in \mathbb{C}[S_d] \mid xy = yx \quad \forall y \in \mathbb{C}[S_d] \}$$

Facts: ① $\forall \mu \vdash d$, $C_\mu \in Z_d$ and form a basis (as vector space)

$$Z_d = \bigoplus_{\mu \vdash d} \mathbb{C} \cdot C_\mu$$

② (Maschke) $Z\mathbb{C}[S_d]$ is a semisimple algebra: $\exists$ basis $(e_\lambda)_{\lambda \vdash d}$ st.

$$e_\lambda \cdot e_{\lambda'} = \delta_{\lambda, \lambda'} e_\lambda$$

and the change of basis $(C_\mu)_{\mu \vdash d} \leftrightarrow (e_\lambda)_{\lambda \vdash d}$ is given by characters of S_d:

$$\begin{cases} C_\mu = \sum_\lambda |C_\mu| \dfrac{\chi_\lambda(\mu)}{\dim(\lambda)} e_\lambda \\ e_\lambda = \dfrac{\dim(\lambda)}{d!} \sum_\mu \chi_\lambda(\mu) C_\mu = \dfrac{\dim(\lambda)^2}{d!} \mathrm{id} + \cdots \end{cases}$$

V_λ irrep of S_d

$\Phi_\lambda : S_d \to GL(V_\lambda)$

- $\chi_\lambda(\mu) = \mathrm{tr}(\Phi_\lambda(\sigma))$ σ of cycle type μ
- $\dim(\lambda) = \dim(V_\lambda) = \chi_\lambda(1^d)$
- $|C_\mu| = \# \{ \sigma \in S_d$ of cycle type $\mu \}$

Thm (Burnside formula)

$$H_d^\bullet(B; \mu^1, \dots, \mu^k) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^{\chi(B)} \prod_{i=1}^{k} |C_{\mu^i}| \frac{\chi_\lambda(\mu^i)}{\dim \lambda}$$

Proof. For $\mathbb{P}^1$:

$$H_d^\bullet(\mathbb{P}^1, \mu^1, \dots, \mu^k) = \frac{1}{d!} [\mathrm{id}] C_{\mu^k} \cdots C_{\mu^1}$$

$$= \frac{1}{d!} [\mathrm{id}] \sum_{\lambda^1, \dots, \lambda^k} \prod_{i=1}^{k} |C_{\mu^i}| \frac{\chi_{\lambda^i}(\mu^i)}{\dim(\lambda^i)} e_{\lambda^i}$$

$$= \frac{1}{d!} [id] \sum_{\lambda \vdash d} \left(\prod_{i=1}^{k} |C_{\mu^i}| \frac{\chi_\lambda(\mu^i)}{\dim \lambda} \right) e_\lambda$$

$$= \sum_{\lambda \vdash d} \left(\frac{\dim(\lambda)}{d!} \right)^2 \prod_{i=1}^{k} |C_{\mu^i}| \frac{\chi_\lambda(\mu^i)}{\dim(\lambda)}$$

$\lambda_i > \lambda_{i-1}$

$\lambda = (\lambda_1, \dots \lambda_e)$

$\boxed{\text{SPIN?}}$ Facts: ① $\{\text{irreps of } \tilde{S}_d\} \overset{1:1}{\longleftrightarrow} \{\lambda \vdash d \mid \lambda \text{ is strict}\}$

② A basis of the spin class algebra is

$$\tilde{Z}_d = \bigoplus_{\substack{\mu \vdash d \\ \text{odd}}} \mathbb{C}. \tilde{C}_d$$

$\Downarrow$

$\tilde{\chi}_\lambda(\mu)$ spin charact.

Thm (Gunningham' formula)

$$H_d^\bullet(B, \vartheta; \mu^1, \dots, \mu^k) = 2^{\frac{\sum_i (\ell(\mu^i) - d) - 2\chi(B)}{2}} \sum_{\substack{\lambda \vdash d \\ \text{strict}}} (-1)^{p(\vartheta)\ell(\lambda)} \left(\frac{\dim(\lambda)}{2^{\frac{p(\lambda)}{2}} d!} \right)^{\chi(B)} \prod_{i=1}^{k} |\tilde{C}_{\mu^i}| \frac{\tilde{\chi}_\lambda(\mu^i)}{\dim(\lambda)}$$

$$p(\lambda) = \begin{cases} 1 & \text{if } \ell(\lambda) \text{ is odd} \\ 0 & \text{if } \ell(\lambda) \text{ is even} \end{cases}$$

LECTURE 2

From now on:

- $\begin{cases} B = \mathbb{P}^1 & \text{in ordinary case} \\ B = (\mathbb{P}^1, \mathcal{O}(-1)) & \text{in spin case} \end{cases}$
- $\mu^1 = \mu$ $\begin{cases} \text{a generic partition of } d \text{ in ord. case} \\ \text{'' —— '' \underline{odd} partition of } d \text{ in spin case} \end{cases}$
- $\mu^2 = \mu^3 = \dots \overset{?}{=} (r+1, 1, \dots, 1)$ + lower order terms with $\begin{cases} r \geq 1 \text{ int. in ord case} \\ r \geq 1 \text{ even integer spin case} \end{cases}$

In ord. case:

① $\mathcal{Z}_d \xrightarrow{\phi} \mathbb{C}^{\mathcal{P}_d}$ $\qquad \mathcal{P}_d = \{\lambda \vdash d\}$ is an iso of algebras $\qquad \tilde{\mathcal{Z}}_d \xrightarrow{\tilde\phi} \mathbb{C}^{S\mathcal{P}_d}$

$$C_\mu \mapsto |C_\mu| \frac{\chi_\cdot(\mu)}{\dim(\cdot)} =: f_\mu \qquad\qquad \tilde C_\mu \mapsto |\tilde C_\mu| \frac{\tilde\chi_\cdot(\mu)}{\dim(\cdot)} =: \tilde f_\mu$$

② $f_\mu = \frac{1}{\prod_i \mu_i} \mathbf{p}_\mu + \text{l.o.t.}$ with $\mathbf{p}_\mu = \prod \mathbf{p}_{\mu^i}$ $\qquad p_\mu = \prod_i p_{\mu^i}$

$\tilde f_\mu = \frac{1}{\prod_i \mu_i} p_\mu$ $\qquad$ and $\mathbf{p}_m(\lambda) = \sum_{k>0}\left[\left(\lambda_k - k + \frac12\right)^k - \left(-k+\frac12\right)\right]$

$\uparrow$ shifted symm. fncts $\qquad p_m =$ power sums

$$\rightsquigarrow \quad \mu^2 = \mu^3 \cdots = \phi^{-1}\left(\frac{1}{r+1}\mathbf{p}_{r+1}\right) =: \overline{C}_{r+1}. \qquad \boxed{\text{SPIN?}} \quad \text{Similar construction } \overline{\tilde C}_{r+1}$$

$\uparrow$ completed $(r+1)$-cycles

Defn. For $r \geq 1$, $\mu \vdash d$ $\qquad rb = 2g-2+\ell(\mu)+d$

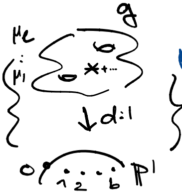
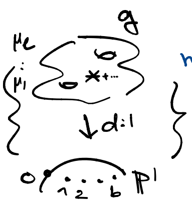
$$h^{\bullet,r}_{g;\mu} = \frac{|\mathrm{Aut}\,\mu|}{b!} H^\bullet_d(\mathbb{P}^1; \mu, (\overline{C}_{r+1})^b)$$

$$= \frac{|\mathrm{Aut}\,\mu|}{b!} \sum_\lambda \left(\frac{\dim(\lambda)}{d!}\right)^2 f_\mu(\lambda) \left(\frac{\mathbf{p}_{r+1}(\lambda)}{r+1}\right)^b = \#\left\{ \right.$$  $$\left.\right\}$$

For $r \geq 1$ even, $\mu \vdash d$ odd

$$\tilde h^{\bullet,r}_{g,\mu} = \frac{|\mathrm{Aut}(\mu)|}{b!} H^\bullet_d(\mathbb{P}^1, \mathcal{O}(-1); (\overline{\tilde C}_{r+1})^b)$$

$$\stackrel{!}{=} \frac{|\mathrm{Aut}(\mu)|}{b!} 2^{\cdots} \sum_{\lambda \text{ strict}} \left(\frac{\dim(\lambda)}{2^{\frac{\ell(\lambda)}{2}} d!}\right)^2 \tilde f_\mu(\lambda) \left(\frac{p_{r+1}(\lambda)}{r+1}\right)^b$$

$= \#\{$  $\}$ positive $-\ \#\{$  $\}$ negative

§ 2.1) KP & FERMION FORMALISM

Def. Define $\{\tau\text{-fncts}\}$ of KP as $\tau \in \mathbb{C}[t_1, t_2, \dots]$ s.t. $u = 2\partial_{t_1}^2 \log \tau$

$$\begin{cases} 3\partial_{t_2}^2 u + \partial_{t_1}(-4\partial_{t_3} u + 6u\partial_{t_1} u + \partial_{t_1}^3 u) = 0. & \text{KP eqn} \\ \vdots \end{cases}$$

?

Defn. Define $Gr_2(\mathbb{C}^4)$ as $[x_0 : \dots : x_5] \in \mathbb{P}^5$ s.t.

$x_0 x_1 - x_2 x_3 + x_4 x_5 = 0$

$$\begin{array}{ccc} W & & Gr_2(\mathbb{C}^4) = \{ W \subseteq \mathbb{C}^4 \mid V = \text{2-dim subspace} \} \\ \parallel & & \downarrow \\ \mathrm{span}(e, e') & & \mathbb{P}^5 = \mathbb{P}\Lambda^2 \mathbb{C}^4 \\ \downarrow & & \\ e \wedge e' & & \end{array}$$

Take $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$.

$$V = \bigoplus_{s \in \mathbb{Z}'} \mathbb{C}.e_s, \qquad V_+ = \bigoplus_{s \in \mathbb{Z}'_+} \mathbb{C}.e_s$$

Defn. Fock space

$$\mathcal{F} = \mathrm{span}\left\{ e_{s_1} \wedge e_{s_2} \wedge \dots \;\middle|\; \begin{array}{l} \exists c \in \mathbb{Z} \text{ s.t. for } k \gg 0 \\ s_k - (k - \tfrac{1}{2}) = \underbrace{c}_{\text{charge}} \end{array} \right\} \Big/ \sim \;\supseteq\; \underbrace{\mathcal{F}^0}_{\text{charge}=0 \text{ subsp.}}$$

$$\dots \wedge e_s \wedge e_r \wedge \dots = (-1) \dots \wedge e_r \wedge e_s \wedge \dots$$

Ex. $|0\rangle = e_{1/2} \wedge e_{3/2} \wedge e_{5/2} \wedge \dots \in \mathcal{F}^0$ $\qquad s_k = k - 1/2 \quad \forall k$

$\llcorner$ vacuum

$\forall \lambda \vdash d \quad \rightsquigarrow \quad |\lambda\rangle = e_{1/2 - \lambda_1} \wedge e_{3/2 - \lambda_2} \wedge \dots$

Defn. Sato Grassmannian as

$$Gr(V) = \{ W \subseteq V \text{ subspace} \mid \pi_W : W \to V_+ \text{ is an iso} \}$$
$$\overset{\text{morally}}{=} \{\tau\text{-fncts}\}$$

Thm. Define $\quad Gr(V) \to \mathbb{P}\mathcal{F}^0$

$$W \mapsto \pi_W^{-1}(e_{1/2}) \wedge \pi_W^{-1}(e_{3/2}) \wedge \cdots$$

Then $|W\rangle \in \mathbb{P}\mathcal{F}^0$ is in the image iff

$$\sum_{s\in\mathbb{Z}'} \psi_s |W\rangle \otimes \psi_s^+ |W\rangle = 0$$

where $\begin{cases} \psi_s = e_s \wedge \\ \psi_s^+ = \iota\, e^*_{-s} \end{cases}$ creation operators $(s<0)$ & annihilation operators $(s>0)$.

$\rightsquigarrow$ repr. of Clifford alg: $\{\psi_r, \psi_s^+\} = \delta_{r+s,0}$

$\{\psi_r, \psi_s\} = \{\psi_r^+, \psi_s^+\} = 0$

(fermions)

How to go to fncts $\tau \in \mathbb{C}[t_1, t_2, \ldots]$ satisfying PDEs?

Thm (Boson-Fermion corresp). Set $\quad J_n = \sum_{s\in\mathbb{Z}'} :\psi_{-s}\psi^+_{s+n}: \quad n\in\mathbb{Z}.$

$\uparrow$ currents

$$BF:\ \mathcal{F}^0 \to \mathbb{C}[t_1, t_2, \ldots]$$

$$|W\rangle \mapsto \langle 0| e^{H(\underline{t})} |W\rangle, \qquad H(\underline{t}) = \sum_{n>0} t_n J_n$$

$\uparrow$ take coeff of vacuum

is an isomorphism.

Corollary. $\tau \in \mathbb{P}\mathbb{C}[t_1, t_2, \ldots]$ lies in $BF(\mathrm{Im}(Gr(V) \to \mathbb{P}\mathcal{F}_0))$ iff

$$\oint \exp\Big(2\sum_{k>0} z^k t_k'\Big) \exp\Big(-2\sum_{k>0} \frac{z^{-k}}{k}\frac{\partial}{\partial t_k'}\Big) \tau(t-t')\tau(t+t')\, dz$$

order by order in Taylor expansion of t_k'.

$\downarrow$ taking coeffs

PDEs satisfied by τ

Fact: Grassmannians are homogeneous spaces: $GL(\mathbb{C}^n) \curvearrowright Gr_k(\mathbb{C}^n)$ transitively.

Q: Is this true for $Gr(V)$?

Defn. $\widehat{gl}(\infty) = \left\{ c + \sum_{r,s\in\mathbb{Z}'} X_{rs} :\psi_{-r}\psi_s^+: \;\middle|\; c\in\mathbb{C},\; \begin{array}{l} X_{rs}\in\mathbb{C} \\ =0 \quad \text{for } |r-s|\gg 0 \end{array} \right\}$

$$\widehat{GL}(\infty) = \{ e^{g_1} \cdots e^{g_k} \mid g_i \in \widehat{gl}(\infty) \}$$

Fact: $\widehat{GL}(\infty) \curvearrowright Gr(V)$ transitively, i.e.

$$Gr(V) = \{ e^{g_1}\cdots e^{g_k}|0\rangle \mid g_i \in \widehat{gl}(\infty),\ k\in\mathbb{N} \}$$

Cor. $\tau \in \mathbb{P}\mathbb{C}[t_1,t_2,\dots]$ is a tau-fnct iff

$$\tau(\underline{t}) = \langle 0|e^{H(\underline{t})} e^{g_1}\cdots e^{g_k}|0\rangle \qquad \text{for some } g_i \in \widehat{gl}(\infty)$$

Q: Natural elements in $\widehat{gl}(\infty)$?

- $J_n = \sum_{s\in\mathbb{Z}'} :\psi_{-s}\psi_{s+n}^+: \;\in \widehat{gl}(\infty)$

- $\mathcal{F}_m = \sum_{s\in\mathbb{Z}'} s^m :\psi_{-s}\psi_s^+: \;\in \widehat{gl}(\infty)$

How do they act?

$$J_{\mu_1}\cdots J_{\mu_n}|\lambda\rangle = \chi_\mu(\lambda)|0\rangle$$

shifted symmetric fnct

$$\mathcal{F}_m|\lambda\rangle = p_m(\lambda)|\lambda\rangle, \qquad p_m(\lambda) = \sum_{i=1}^{\infty}\left[(\lambda_i - i + \tfrac{1}{2})^m - (-i+\tfrac{1}{2})^m\right]$$

$$(J_{-1})^d|0\rangle = \sum_{\lambda\vdash d} \dim(\lambda)|\lambda\rangle$$

Thm (Okounkov). Consider the tau-fnct

$$\tau^r(\beta;\underline{t}) = \langle 0| e^{H(\underline{t})} e^{\beta\frac{\mathcal{F}_{r+1}}{r+1}} e^{J_{-1}}|0\rangle$$

Then

$$\tau^r(\beta;\underline{t}) = \sum_{n,b\geq 0}\frac{1}{n!}\sum_{\mu_1,\dots,\mu_n=0}^{\infty} \underbrace{a^r_m(b;\mu)}\frac{\beta^b}{b!}\prod_{i=1}^{n} t_{\mu_i}$$

$$a^r_m(b;\mu) = \sum_{\lambda\vdash|\mu|}\chi_\lambda(\mu)\left(\frac{\mathbb{P}_{r+1}(\lambda)}{r+1}\right)^b \frac{\dim\lambda}{|\mu|!} \overset{?}{=} \text{Hurwitz } \#\text{s} \,?$$

Proof. $e^{J_{-1}}|0\rangle = \sum_{d\geq 0}\frac{1}{d!}(J_{-1})^d|0\rangle$

$$\overset{!}{=} \sum_{d\geq 0}\frac{1}{d!}\sum_{\lambda\vdash d}\dim(\lambda)\,|\lambda\rangle$$

$$e^{\beta\frac{\mathcal{F}_{r+1}}{r+1}} e^{J_{-1}} = \sum_{b\geq 0}\frac{\beta^b}{b!}\left(\frac{\mathcal{F}_{r+1}}{r+1}\right)^b$$

$$\overset{!}{=} \sum_{b\geq 0}\frac{\beta^b}{b!}\sum_{d\geq 0}\frac{1}{d!}\sum_{\lambda\vdash d}\dim(\lambda)\left(\frac{\mathbb{P}_{r+1}(\lambda)}{r+1}\right)^b|\lambda\rangle$$

Apply $\langle 0|e^{H(\underline{t})}$ to get the thesis.

Cor. The generating series of H# is a tau-fnct of the KP hierarchy.

WHY USEFUL? 1) Cut-and-join eqn: cut-and-join operator

$$\frac{\partial\tau^r(\beta,\underline{t})}{\partial\beta} = e^{\beta W^r}\,1$$

← fastest way to compute H#s

2) Allows for explicit computations:

$$h^{\bullet,r}_{g;\mu} = \langle 0|\left(\prod_{i=1}^{\ell(\mu)}\frac{J_{\mu_i}}{\mu_i}\right)\left(\frac{\mathcal{F}_{r+1}}{r+1}\right)^b J_{-1}|0\rangle$$

$\rightarrow$ commute J_{μ_i} until they hit $|0\rangle$ $\rightsquigarrow$ $J_{\mu_i}|0\rangle = 0$

$\Downarrow$

known comm. relations

Corollary.

$$h^r_{g=0,\mu=(d)} = \frac{d^{\frac{d-1}{r}-2}}{\left(\frac{d}{r}\right)!}, \qquad h^r_{g=0,\mu=(\mu_1,\mu_2)} = \frac{r}{\mu_1+\mu_2} \frac{\mu_1^{\lfloor \frac{\mu_1}{r} \rfloor} \mu_2^{\lfloor \frac{\mu_2}{r} \rfloor}}{\lfloor \frac{\mu_1}{r} \rfloor! \lfloor \frac{\mu_2}{r} \rfloor!}$$

$\boxed{\text{SPIN?}}$ $\begin{cases} \text{B-Fock space, B-Grassmannian} = \hat{O}(\infty)|0\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad \hat{O}(\infty) = \text{skew-symm matrices} \\ \text{B-KP hierarchy} \end{cases}$

$\rightsquigarrow$
1. gen ser of spin H# is a BKP tau-fnct
2. explicit formulae for $(g=0, \ell(\mu)=1,2)$ i.e. cap and tube

§ 2.2) TOPOLOGICAL RECURSION

$$\text{spectral curve } S \xrightarrow{TR} (\omega_{g,n})_{\substack{g\geq 0 \\ n\geq 1}}$$

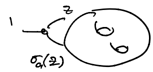
i) $\Sigma = RS$ (not necessarily cmct nor conn)

ii) $x, y : \Sigma \to \mathbb{C}$, $\begin{cases} x \text{ simple ram } a_1, \dots, a_r \\ y \text{ holom. around ram pnts} \end{cases}$

iii) B symm bidiff with double pole along diagonal

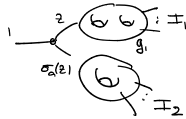
$$B(z_1,z_2) \underset{\substack{z_1\to a \\ z_2 \to b}}{\sim} \left(\frac{\delta_{a,b}}{(z_1-z_2)^2} + \text{holom} \right) dz_1 dz_2$$

Recursion: $\omega_{0,1} = y\,dx$, $\omega_{0,2} = B$



$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{a \in \mathrm{Ram}(x)} \operatorname*{Res}_{z=a} K_a(z_1,z)\Big(\omega_{g-1,n+2}(z,\sigma_a(z),z_2,\dots,z_n)$$

$$+ \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\dots,n\}}} \omega_{g_1,1+|I_1|}(z,z_{I_1})\,\omega_{g_2,1+|I_2|}(\sigma_a(z),z_{I_2})\Big)$$



$\omega_{g,n}$ have specific pole structure:

$\rightsquigarrow$ $\omega_{g,n}$ are unique solution to loop eqns

Defn. $x(z) = \log(z) - z^r$, $y(z) = z$, $B(z_1,z_2) = \dfrac{dz_1\,dz_2}{(z_1-z_2)^2}$.

$$\omega_{0,1}(z) = d \sum_{d \geq 1} h^r_{0,(d)}\, e^{\mu x(z)}, \quad \omega_{0,2}(z_1,z_2) = d_1 d_2 \sum_{\mu_1,\mu_2 \geq 1} h^r_{0,(\mu_1,\mu_2)}\, e^{\mu_1 x(z_1)} e^{\mu_2 x(z_2)}$$

using explicit $h^r_{0,(d)}$

Thm. $\omega_{g,n}(z_1,\dots,z_n) = d_1 \cdots d_n \sum_{\mu_1,\dots,\mu_n} h^r_{g,\mu} \prod_{i=1}^{n} e^{\mu_i x(z_i)}$ satisfy loop eqns.

Bouchard-Mariño conj $(r=2)$ $\Rightarrow$ $\omega_{g,n}$ are given by TR

Proved by Eynard-Safnuk-Mulase

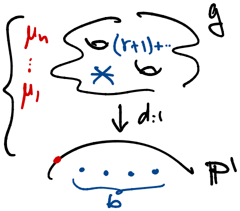
General r: Shadrin + school

Modern approach: Bychkov-Dunin-Barkowski-Kazarian-Shadrin

LECTURE 3

Recap. $r \geq 1$, $\mu = (\mu_1, \dots, \mu_n) \vdash d$, $g \geq 0$

$$h^r_{g;\mu} = \# \left\{ \quad \right\}$$



1) $$h^{\bullet, r}_{g;\mu} \sim \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \mathbf{f}_\mu(\lambda) \left(\frac{\mathbf{p}_{r+1}(\lambda)}{r+1} \right)^b$$

$$\sim \left\langle 0 \left| \prod_{i=1}^n \frac{J_{\mu_i}}{\mu_i} \left(\frac{\mathcal{F}_{r+1}}{r+1} \right)^b (J_{-1})^d \right| 0 \right\rangle \quad \Longrightarrow \text{KP}$$

2) $$\begin{cases} x = \log(z) - z^r \\ y = z \\ B(z_1, z_2) = \dfrac{dz_1 dz_2}{(z_1 - z_2)^2} \end{cases} \xrightarrow{\text{TR}} \omega_{g,n}(z_1, \dots, z_n) = d_1 \cdots d_n \sum_{\mu_1, \dots, \mu_n} h^r_{g;\mu} \prod_{i=1}^n e^{\mu_i x(z_i)}$$

§ 3.1) MODULI SPACE OF CURVES

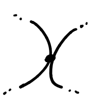
Fix $g, n \geq 0$ s.t. $2g-2+n > 0$

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \dots, p_n) \;\middle|\; \begin{array}{l} C \text{ genus } g, \cancel{\text{smooth}}\ \mathbb{C}\text{-crv} \\ p_1, \dots, p_n \text{ labeled sm pnts} \end{array} \right\} \Big/ \sim$$

C at worst nodal

$|\mathrm{Aut}| < \infty$

nodal:



locally: $xy = 0$

$(C, p_1, \dots, p_n) \sim (C', p_1', \dots, p_n')$ iff

$\exists\, \varphi : C \to C'$ bihlom., $\varphi(p_i) = p_i'$

Good: cmlx orbifold, $\dim_{\mathbb{C}} = 3g-3+n$

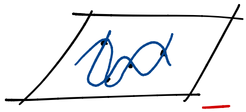
~~Bad: non-cmct~~

Natural maps (gluing maps)

$$q: \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$$r: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

WHAT FOR?

$\# \left\{ \begin{array}{l} \text{curves of} \\ \text{genus } g \\ \text{w/ } n \text{ pnts} \\ + \text{structure} \end{array} \right\} \overset{?}{=}$  $= \int_{\overline{\mathcal{M}}_{g,n}}$ inters. of coh. classes

$\overline{\mathcal{M}}_{g,n}$

Q: Can we understand $H^\bullet(\overline{\mathcal{M}}_{g,n})$?

Defn. Natural cohomology classes:

- $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ line bundle $\quad \Rightarrow \quad \psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$

  $\mathcal{L}_i \,|_{(C,p_1,\dots,p_n)} = \omega_C|_{p_i} \qquad \forall i = 1,\dots,n,$

- $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}$ $\mathrm{rk}_{\mathbb{C}} = g$ bundle

  $\mathbb{E}\,|_{(C,p_1,\dots,p_n)} = H^0(C,\omega_C) \underset{\uparrow}{=}$ holomorpic diffs on C

  if C is smooth

  $\Rightarrow \quad \Lambda(t) = \sum_{k=0}^{g} c_k(\mathbb{E})\, t^k$ Chern poly of $\mathbb{E}$

Main property: $\Lambda(t)$ behaves well when gluing RS

- $q^* \Lambda_{g,n}(t) = \Lambda_{g-1,n+2}(t)$
- $r^* \Lambda_{g,n}(t) = \Lambda_{g_1,n_1+1}(t) \otimes \Lambda_{g_2,n_2+1}(t)$

} cohomological version of 2d ~~TFQF~~

$\uparrow$ terms appearing in TR!

Thm (Dunin-Barkowski–Orantin–Spitz–Shadrin)

$$\{\text{topological recursion}\} \longleftrightarrow \{\text{CohFTs}\}$$

Q. What is the CohFT associated to $\begin{cases} x = \log z - z^r \\ y = z \\ B = \dfrac{dz_1\, dz_2}{(z_1 - z_2)^2} \end{cases} \rightsquigarrow$ $r=1$ $\Lambda(-1)$; $r \geq 2$ a generaliz. rth roots of holom. diffs

Thm (ELSV)

$$h^{r=1}_{g;\mu} = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(-1)}{\prod_{i=1}^{n}(1-\mu_i \psi_i)}$$

$\uparrow$ simple H#

Thm (spin ELSV)

$r+1$ odd!

$$\tilde{h}^{r=2}_{g;\mu} = 2^{g-1+n} \prod_{i=1}^{n} \frac{\mu_i^{\frac{\mu_i-1}{2}}}{\left(\frac{\mu_i-1}{2}\right)!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(2)\Lambda(-1)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}$$

§3.2) APPLICATIONS TO GW THEORY

Fix X smooth cmct cmlx mnfld (target).

GW invariants of X = counting RS in X meeting prescribed conditions

Ex: How many $g=0$ curves of $\deg = d$ pass through $3d-1$ pnts in $\mathbb{P}^2$?

$N_d \rightsquigarrow$

d	1	2	3	4	5
N_d	1	1	12	620	87304

N_d = recursion (Kontsevich)

GW invariants of X = path integrals in topological string theory (X = space-time)

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \left\{ (C, p_1, \dots, p_n, f) \;\middle|\; \begin{array}{l} C, p_1, \dots, p_n \text{ as before} \\ f : C \to X, \ \mathrm{Aut}(f) \text{ finite} \\ f_*[C] = \beta \end{array} \right\} \Big/ \sim$$

$$\underset{\beta \in H_2(X)}{\uparrow}$$

Fact: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a DM stack, virtual $\dim_{\mathbb{C}} = (\dim X - 3)(1-g) + \int_\beta c_1(X) + n$

+ virtual fund. class

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$
$$[C, p_1, \dots, p_n, f] \mapsto f(p_i)$$

$$\pi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n} \qquad \rightsquigarrow \qquad \psi_i := \pi^*$$

Defn. $\gamma_i \in H^\bullet(X)$

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_g^{X,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n (ev_i^* \gamma_i)\, \psi_i^{k_i}$$

Idea: $\langle \tau_0(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_g^{X,\beta}$ = genus g curves in X, deg β intersecting $[\gamma_i]^{PD}$

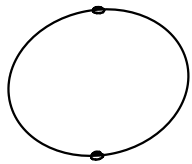
Ex: $$\langle \underbrace{\tau_0(pt) \cdots \tau_0(pt)}_{3d-1} \rangle_0^{\mathbb{P}^2, d[H]} = N_d$$

Question: understand GW invariants when $X = \underset{\text{0-dim}}{pt}, \underset{\text{1-dim}}{C}, \underset{\text{2-dim}}{S}, \dots$

- 0-dim: Witten's conj / Kontsevich thm (1992)
  → recursion for GW invariants: Virasoro constraints

- 1-dim: Okounkov – Pandharipande (2006)

1) Complete study of $\mathbb{P}^1$ by localisation $\qquad \frac{1}{1-x} = \sum x^k$



$\mathbb{C}^\times \circlearrowleft$

$$\int_{[\overline{M}_{g,n}(\mathbb{P}^1, d[H])]^{vir}} \prod_i ev_i^* pt\, \psi_i^{k_i} \rightsquigarrow \int_{\overline{M}_{g,n}} \Lambda(-1) \prod_i d_i^{k_i} \psi_i^{k_i}$$

moduli space of maps into fixed pnts $0, \infty$

$\Updownarrow$

Hurwitz numbers

$\Updownarrow$

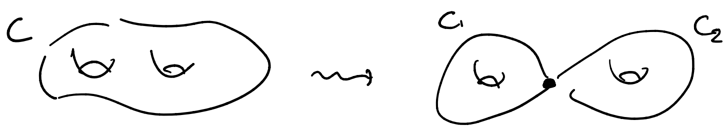
vacum expect values in Fock space

$$\langle \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) \rangle_g^{\mathbb{P}^1, d[H]} =$$

$$= H_d(\mathbb{P}^1, \overline{C}_{(k_1+1, 1, \dots, 1)}, \dots, \overline{C}_{(k_n+1, 1, \dots, 1)})$$

$$= \sum_{\lambda \vdash d} (\quad)^{2-2g} \prod_{i=1}^n \frac{p_{k_i+1}(\lambda)}{k_i+1}$$

2) Complete study of any curve by degeneration to $\mathbb{P}^1$



GW(C) $\qquad$ GW(C1) + GW(C2)

Thm. The generating series of GW inv of any curve satisfies Virasoro constraints.

$1, \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, pt \rightsquigarrow$

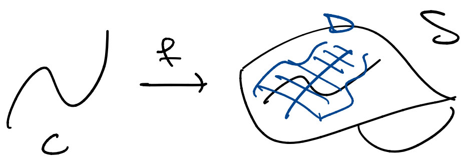
$t_k^0 \leftrightarrow ev^* 1\, \psi^k$

$s_k^i, \bar{s}_k^i \leftrightarrow ev^* \alpha_i\, \psi^k$, $ev^* \beta_i\, \psi^k$

$t_k^1 \leftrightarrow ev^* pt\, \psi^k$

- 2-dim (w/ Reinier + Dani + Adrien Sauvaget)

0) For some surfaces S (smooth canonical div D) GW invariants localise around the canonical div D (a curve inside S)



- $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_g^{X, \beta} = 0$

  unless $\beta = d[D]$

- $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_g^{X, d[D]}$ depends on D, N only

$GW(S) \leftsquigarrow GW(\underline{D}, N)$

$\uparrow$ normal bundle

spin GW inv of (D, N)

$N^{\otimes 2} \cong \omega_D$ is a spin strucr!

UP<u>SH</u>OT: If we understand spin GW invariants, we understand GW invts of many surfaces.

1) Complete study of $(\mathbb{P}^1, \mathcal{O}(-1))$ by loc.

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d[H])]^{loc, \mathcal{O}(-1)}} \prod_i ev_i^* pt \, \psi_i^{k_i} \rightsquigarrow \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(2)\Lambda(-1) \prod_i d_i \psi_i^{k_i}$$

$\Updownarrow$

spin Hurwitz numbers

$$\langle \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) \rangle_g^{\mathbb{P}^1, \mathcal{O}(-1), d[H]} = H_d(\mathbb{P}^1, \mathcal{O}(-1); \overline{\widetilde{C}}_{(k_1+1, 1, \dots, 1)}, \dots, \overline{\widetilde{C}}_{(k_n+1, 1, \dots, 1)})$$

2) Degeneration + Virasoro: <u>WIP</u>