Matrix models and gauge theories

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1. Recap

Recall: the aim of the *QFT and BV formalism* learning seminar is to somehow give sense to the expectation values

$$\langle f_1, \dots, f_m \rangle_U = \frac{1}{Z_U} \int_{\mathcal{F}} e^{-S(\phi)} f_1(\phi) \cdots f_m(\phi)\, \mathcal{D}\phi, \tag{1.1}$$

where $Z_U = \int e^{-S(\phi)} \mathcal{D}\phi$ is called the partition function factor, $\mathcal{F}$ is the space of fields with an inner product $\langle \cdot, \cdot \rangle$, and S is the (Euclidean) action depending on the fields ϕ:

$$S(\phi) = \frac{1}{2} \langle \phi, K\phi \rangle - \hbar U(\phi). \tag{1.2}$$

Here $K\colon \mathcal{F} \to \mathcal{F}$ is a linear operator. Often, $\mathcal{F}$ are sections of a vector bundle $E \to X$, where X is interpreted as the space-time, and K is a linear differential operator acting on sections of E.

PROBLEM. How to properly define measure the above quantities?

EXAMPLE (0-dimensional case). In the well-defined 0-dimensional case, where we have $X = \{ p_1, \dots, p_d \}$, $\mathcal{F} = C^\infty(X, \mathbb{R}) = \mathbb{R}^d$, $K = A$ is a symmetric, positive-definite matrix, $U = 0$ and $f_j(x) = x_{i_j}$, we found (Wick's theorem, Proposition (2.7) in [7])

$$\langle x_{i_1}, \dots, x_{i_m} \rangle_0 = \frac{1}{Z_0} \partial_{b_{i_1}} \cdots \partial_{b_{i_m}} Z(b) \Big|_{b=0}, \tag{1.3}$$

where we have set

$$Z(b) = \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle x, Ax \rangle + \langle b, x \rangle} dx. \tag{1.4}$$

In physical jargon, b is called a source field. As a consequence, for U and f_j polynomials, we are lead to define the values

$$Z_U \cdot \langle f_1, \dots, f_m \rangle_U = e^{\hbar U(\partial_b)} f_1(\partial_b) \cdots f_m(\partial_b)\, Z(b) \Big|_{b=0} \tag{1.5}$$

and

$$Z_U = e^{\hbar U(\partial_b)}\, Z(b) \Big|_{b=0} \tag{1.6}$$

as elements in $\mathbb{R}[[\hbar]]$. After some manipulations, we found the following expression in terms of graph, which makes sense also for potentials given by power series. More precisely, consider for a fixed potential

$$U(x) = \sum_{k \geqslant 3} U_{j_1 \cdots j_k} x_{j_1} \cdots x_{j_k}, \tag{1.7}$$

where $U_{j_1 \cdots j_k}$ is symmetric. Define $\mathcal{G}^0_n$ to be the set of graphs with vertices of valence $k \geqslant 3$. Denote the number of vertices of $\Gamma \in \mathcal{G}^0$ as $|\Gamma|$. A label of a graph $\mathcal{G}^0$ is the assignment of a label to every half-edge, so that every vertex of valence k is identified by an unordered k-uple $v = \{ j_1, \dots, j_k \}$ and every vertex by an unordered couple $e = \{i, j\}$. So it makes sense to define $U_v = U_{j_1\cdots j_k}$ and $A_e^{-1} = A_{ij}^{-1}$. With this notation, we found (Proposition (2.17) in [7])

$$Z_U = Z_0 \sum_{\Gamma\in\mathcal{G}^0} \frac{\hbar^{|\Gamma|}}{|\operatorname{Aut}\Gamma|} \sum_{\text{labels}} \prod_v U_v \prod_e A_e^{-1} \in \mathbb{R}[[\hbar]]. \tag{1.8}$$

Similarly, one can define for every m-uple $(i_1, \ldots, i_m)$ the set of $\mathcal{G}^m$ of graphs with vertices of valence $k \geqslant 3$, called internal vertices, and m vertices of valence 1, called legs and labeled by $i_1, \ldots, i_m$, such that each connected component has at least one leg. Denote the number of internal vertices of $\Gamma \in \mathcal{G}^m$ as $|\Gamma|$. A label of a graph $\mathcal{G}^m$ is the assignment of a label to every half-edge that is not a leg, so that every vertex of valence k is identified by an unordered k-uple $v = \{j_1, \ldots, j_k\}$ and every vertex by an unordered couple $e = \{i, j\}$. With this notation, we found (Proposition (2.18) in [7])

$$\langle x_{i_1}, \ldots, x_{i_m}\rangle_U = \sum_{\Gamma\in\mathcal{G}^m} \frac{\hbar^{|\Gamma|}}{|\operatorname{Aut}\Gamma|} \sum_{\text{labels}} \prod_v U_v \prod_e A_e^{-1} \in \mathbb{R}[[\hbar]]. \tag{1.9}$$

As a consequence, we can compute the expectation value of formal power series $f_1, \ldots, f_m$ by multiplying them and, thanks to linearity, compute the expectation values for monomial. Note the following fact: the above relation makes sense for any symmetric, non-degenerate matrix A, even with values in $\mathbb{C}$. We can also make sense for a non-symmetric one, but in this case the edges should be oriented. The only assumption to which we can not renounce is the non-degeneracy of A.

EXAMPLE (ϕ^4-theory). Consider a pointed compact Riemannian manifold (X, g) and take $\mathcal{F} = C_0^\infty(X)$, that is the set of smooth functions vanishing at the marked point ∞, with the L^2 scalar product, $K = \Delta$ is the (geometric) Laplacian and $U(\phi) = \int_X \phi(x)^4 d\Omega_g$. Then we have

$$S(\phi) = \frac{1}{2}\langle\phi, \Delta\phi\rangle - \hbar U(\phi) = \int_X \underbrace{\left(\frac{1}{2}|d\phi(x)|^2 - \hbar\phi(x)^4\right)}_{=\mathcal{L}\left(\phi(x), d\phi(x)\right)} d\Omega_g.$$

Here K is symmetric, in the sense that $\langle\phi, K\psi\rangle = \langle K\phi, \psi\rangle$, and positive-definite, that is $\langle\phi, K\phi\rangle = \|d\phi\|$ is strictly positive for $\phi \not\equiv 0$. However, the path integral Z_U still does not make sense.

EXAMPLE (Electromagnetic field). Let us consider the free theory, that is $U \equiv 0$. For a pointed compact Riemannian manifold (X, g), we have $\mathcal{F} = \Omega_0^1(X)$, that is the space of differential forms vanishing at ∞, and scalar product on 1-forms given by

$$\langle\omega, \eta\rangle = \int_X \omega \wedge *\eta.$$

Define the free action taking $K = d^t d$, that is

$$S_{\text{free}}(A) = \frac{1}{2}\langle A, d^t dA\rangle = \int_X \frac{1}{2} dA \wedge *dA. \tag{1.10}$$

Here K is symmetric, in the sense that $\langle A, KB\rangle = \langle KA, B\rangle$, but not positive-definite. More precisely, the action is invariant under the transformation $A \to A + d\Lambda$ for a scalar function $\Lambda \in C_0^\infty(X)$. In particular, the group $\mathcal{G} = C_0^\infty(X)$ acts freely on $\mathcal{F} = \Omega_0^1(X)$ by $\Lambda.A = A + d\Lambda$ and the action functional S is invariant under such action.

PROBLEM. What should we do in case of symmetries? That is, how to handle the situation of an action invariant under the action of a group? We will see how to solve this problem in section 3.

2. A QFT matrix model for enumeration of maps

Let us present the example of Hermitian matrix model as presented in [4] and the connection with enumeration of maps, discovered by [1, 9]. This is a 0-dimensional QFT, where the matrix model on one side and the Feynman diagrams on the other side can be used to obtain results for both perspectives. More precisely, take $\mathbb{R}^d = \mathcal{H}_N$ as the set of Hermitian matrices, for $d = N^2$, with normalized measure

$$dM = \frac{1}{2^N}\left(\frac{N}{\pi\hbar}\right)^{\frac{N^2}{2}} \prod_{i=1}^N dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}. \tag{2.1}$$

Further, let us consider the action $S(M) = \frac{1}{2}\operatorname{tr} M^2 - \frac{1}{4}\operatorname{tr} M^4$, so that the partition function will be

$$Z = \int e^{-\frac{N}{\hbar}S(M)}\, dM. \tag{2.2}$$

The integral has to be intended as a formal power series, that is

$$Z = \sum_{k=0}^{\infty} \frac{1}{m!} \int_{\mathcal{H}_N} e^{-\frac{N}{2\hbar}\operatorname{tr} M^2} \left(\frac{N}{4\hbar}\operatorname{tr} M^4\right)^m dM. \tag{2.3}$$

We will see that Z is actually an element in $\mathbb{Q}[N, N^{-1}][[\hbar]]$. Let us rewrite it in such a way that we can apply Wick's theorem:

$$Z = \sum_{m=0}^{\infty} \frac{1}{4^m m!} \left\langle \left(\frac{N}{\hbar}\operatorname{tr} M^4\right)^m \right\rangle_0, \qquad \langle f \rangle_0 = \int_{\mathcal{H}_N} e^{-\frac{N}{2\hbar}\operatorname{tr} M^2} f(M)\, dM. \tag{2.4}$$

Note that $Z_0 = 1$ due to the normalization of dM. We can express the trace of M^4 as a sum of products of coordinates, that is $\operatorname{tr} M^4 = \sum_{i,j,k,l} M_{ij}M_{jk}M_{kl}M_{li}$, so that by Wick's theorem:

$$\begin{aligned} \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle_0 &= \langle \wick{\c1 M_{ij} \c1 M_{jk} \c2 M_{kl} \c2 M_{li}}\rangle_0 + \langle \wick{\c1 M_{ij} \c2 M_{jk} \c1 M_{kl} \c2 M_{li}}\rangle_0 + \langle \wick{\c1 M_{ij} \c2 M_{jk} \c2 M_{kl} \c1 M_{li}}\rangle_0 \\ &= \langle M_{ij}M_{jk}\rangle_0 \langle M_{kl}M_{li}\rangle_0 + \langle M_{ij}M_{kl}\rangle_0 \langle M_{jk}M_{li}\rangle_0 + \langle M_{ij}M_{li}\rangle_0 \langle M_{jk}M_{kl}\rangle_0 . \end{aligned} \tag{2.5}$$

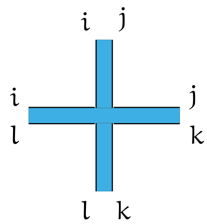
and the propagator is given by the inverse of the trace pairing:

$$\langle M_{ab}M_{cd}\rangle_0 = \frac{\hbar}{N}\delta_{ad}\delta_{bc}. \tag{2.6}$$

As a consequence, we find

$$\begin{aligned} \left\langle \frac{N}{\hbar}\operatorname{tr} M^4 \right\rangle_0 &= \frac{N}{\hbar}\sum_{i,j,k,l} \frac{\hbar}{N}\delta_{ik}\delta_{ll}\frac{\hbar}{N}\delta_{ki}\delta_{ll} + \frac{\hbar}{N}\delta_{ii}\delta_{jl}\frac{\hbar}{N}\delta_{jl}\delta_{kk} + \frac{\hbar}{N}\delta_{il}\delta_{jk}\frac{\hbar}{N}\delta_{ji}\delta_{kl} \\ &= 2\frac{N}{\hbar}\left(\frac{\hbar}{N}\right)^2 N^3 + \frac{N}{\hbar}\left(\frac{\hbar}{N}\right)^2 N \\ &= 2\hbar N^2 + \hbar. \end{aligned} \tag{2.7}$$

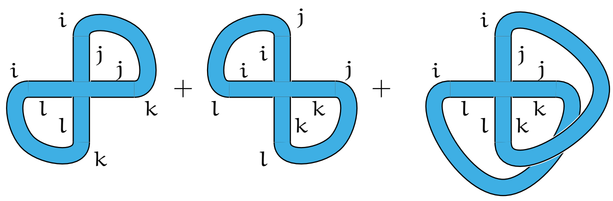
We can represent graphically the above computation as follows. Associate a 4-valent vertex with half double lines to each $\langle \operatorname{tr} M^4\rangle_0 = \sum_{i,j,k,l} \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle_0$



and associate a double line edge (ribbon) to each propagator $\langle M_{ab}M_{cd}\rangle_0$

In computing $\langle \frac{N}{\hbar}\operatorname{tr} M^4\rangle_0$, we have glued half double edge of the vertex with propagators in all possible ways, obtaining a ribbon graph:

$$\left\langle \frac{N}{\hbar}\operatorname{tr} M^4 \right\rangle_0 = \tag{2.8}$$



For a ribbon graph Γ, define its Euler characteristic as

$$\chi(\Gamma) = \#\text{vertices} - \#\text{ribbons} + \#\text{loops}. \tag{2.9}$$

Now, to take into account the powers of N and $\hbar$, note that we have a contribution of

- $\frac{N}{\hbar}$ for each vertex;
- $\frac{\hbar}{N}$ for each ribbon;
- N for each loop in the resulting ribbon graph.

As a consequence, we find in the more general case that

$$\left\langle \left(\frac{N}{\hbar} \operatorname{tr} M^4 \right)^m \right\rangle_0 = \sum_{\Gamma \in \bar{\mathcal{R}}^{4,m}} \hbar^{\ell(\Gamma)} \left(\frac{N}{\hbar} \right)^{\chi(\Gamma)}, \tag{2.10}$$

where $\ell(\Gamma)$ is the number of loops in Γ and $\bar{\mathcal{R}}^{4,m}$ is the set of labeled, (possibly disconnected) ribbon graphs with m, 4-valent vertices. Thanks to the orbit-stabilizer theorem, one can show that

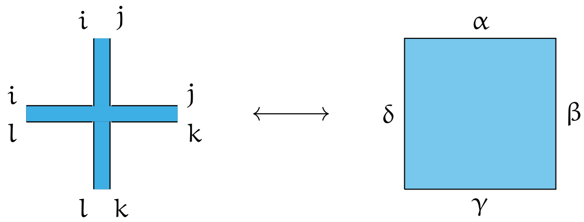
$$\left\langle \frac{1}{m!} \left(\frac{N}{4\hbar} \operatorname{tr} M^4 \right)^m \right\rangle_0 = \sum_{\Gamma \in \mathcal{R}^{4,m}} \frac{\hbar^{\ell(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \left(\frac{N}{\hbar} \right)^{\chi(\Gamma)}, \tag{2.11}$$

where $\mathcal{R}^{4,m}$ is the set of (possibly disconnected) ribbon graphs with m, 4-valent vertices. Thus, the partition function will be

$$Z = \sum_{\Gamma \in \mathcal{R}^4} \frac{\hbar^{\ell(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \left(\frac{N}{\hbar} \right)^{\chi(\Gamma)} \tag{2.12}$$

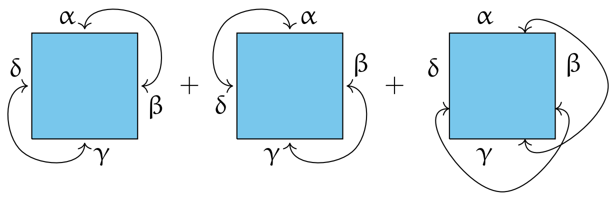
where $\mathcal{R}^4$ is the set of ribbon graphs with 4-valent vertices. The fact that the power of N is a topological invariant is due to the physics Nobel prize Gerard 't Hooft [9] and is the origin of the name *topological expansion* of Equation (2.18).

An equivalent description is ribbon diagrams is done in terms of dual graphs, called closed maps. In particular, to every 4-valent vertex we can associate a 4-gon

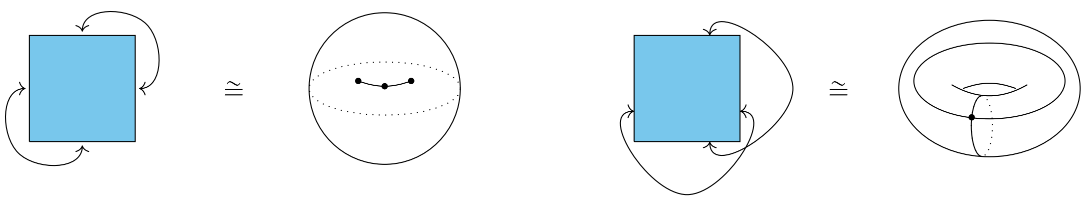


and glue together half double edges of the vertices corresponds to glue together sides of the faces. The resulting dual graph is called a closed map. As an example, Equation (2.8) will be written as

(2.13) $\left\langle \frac{N}{\hbar} \operatorname{tr} M^4 \right\rangle_0 =$



Note that every closed map (and likewise, every ribbon graph) can be drawn without intersections on a closed Riemann surface, whose genus is the same as that of the map, that is $g(\Sigma) = \frac{2-\chi(\Sigma)}{2}$. More precisely, maps are proper embeddings of graphs in Riemann surfaces. For example, we have



Note that in the duality ribbon graphs $\Gamma \longleftrightarrow$ closed maps Σ, we have the correspondence

- vertices of $\Gamma \longleftrightarrow$ faces of Σ,
- ribbons of $\Gamma \longleftrightarrow$ edges of Σ,
- loops of $\Gamma \longleftrightarrow$ vertices of Σ,

and the equality $\chi(\Gamma) = \chi(\Sigma)$, where

$$\chi(\Sigma) = \#\text{faces} - \#\text{edges} + \#\text{vertices}. \tag{2.14}$$

As a consequence, Equation (2.12) can be rewritten as

$$Z = \sum_{\Sigma \in \mathcal{M}^4} \frac{\hbar^{v(\Sigma)}}{|\operatorname{Aut}(\Sigma)|} \left(\frac{N}{\hbar}\right)^{\chi(\Sigma)}. \tag{2.15}$$

We can also be more general, considering the potential

$$U(M) = \sum_{k \geqslant 3} \frac{t_k}{k} \operatorname{tr} M^k. \tag{2.16}$$

The same arguments leads to the following theorem.

THEOREM (Brézin, Itzykson, Parisi, Zuber [1]). The partition function $Z_U = \int e^{-\frac{N}{\hbar}(\frac{1}{2}\operatorname{tr} M^2 - U(M))} dM$ is the generating series for (possibly disconnected) closed maps:

$$Z_U = \sum_{\Sigma \in \mathcal{M}} \frac{\hbar^{v(\Sigma)}}{|\operatorname{Aut}(\Sigma)|} \left(\frac{N}{\hbar}\right)^{\chi(\Sigma)} \prod_{k \geqslant 3} t_k^{n_k(\Sigma)}, \tag{2.17}$$

where χ is the Euler characteristic, $v(\Sigma)$ is the number of vertices of Σ and $n_k(\Sigma)$ is the number of k-gons in Σ. The logarithm $F_U = \ln(Z_U)$ is the generating series for connected maps, and it has a topological expansion

$$F_U = \sum_{g=0}^{\infty} \left(\frac{N}{\hbar}\right)^{2-2g} F_g, \qquad F_g = \sum_{v \geqslant 1} \hbar^v \sum_{\Sigma \in \mathcal{M}_g^{\text{conn}}(v)} \frac{1}{|\operatorname{Aut}(\Sigma)|} \prod_{k \geqslant 3} t_k^{n_k(\Sigma)} \tag{2.18}$$

and $\mathcal{M}_g^{\text{conn}}(v)$ is the set of connected closed maps of genus g with v vertices.

We can also compute expectation values with respect to the potential U. In the particular case of $\langle \operatorname{tr} M^{\ell_1} \cdots \operatorname{tr} M^{\ell_n} \rangle_U$, the resulting series is related to the generating series of maps with n fixed boundaries of lengths $\ell_1, \dots, \ell_n$. An interesting application of the QFT point-of-view is that we can easily prove some non-trivial equations satisfied by expectation values, known as loop equations, that on the enumerative side correspond to relations between generating series. In particular, for any matrix-valued polynomial function $g(M)$, we have

$$\begin{aligned} 0 = &\sum_{i=1}^{N} \int \frac{\partial}{\partial M_{ii}} \Big(g_{ii}(M) e^{-\frac{N}{\hbar} S(M)} \Big) dM \\ &+ \sum_{i<j} \int \frac{\partial}{\partial \mathrm{Re} M_{ij}} \Big(g_{ij}(M) e^{-\frac{N}{\hbar} S(M)} \Big) dM \\ &- i \sum_{i<j} \int \frac{\partial}{\partial \mathrm{Im} M_{ij}} \Big(g_{ij}(M) e^{-\frac{N}{\hbar} S(M)} \Big) dM. \end{aligned} \tag{2.19}$$

For the particular case of $g(M) = M^{\ell_1} \prod_{k=2}^{n} \operatorname{tr} M^{\ell_n}$, we obtain the famous Tutte's equation, relating the generating of maps Euler characteristic $2 - 2g + n$ (that is, genus g and n boundary components) with maps of higher Euler characteristic.

REMARK. With similar ideas, one can construct other different matrix models which leads to some enumerative problems. The following partial list is taken from [5].

- The Ising model. With space of field $\mathcal{H}_N \times \mathcal{H}_N$ and action given by

  $$S(A, B) = \frac{1}{2} \operatorname{tr}(A^2 + B^2 + cAB) - \frac{1}{3} \left(a \operatorname{tr} A^3 + b \operatorname{tr} B^3 \right)$$

  we obtain the Ising model, whose enumerative counterpart are maps with triangles colored in two different ways. It arises in connection with statistical physics and the description of particles with spin up and down.

- Non-Hermitian matrix model. With space of field $\mathcal{M}_N$ the space of $N \times N$ matrices over $\mathbb{C}$ with measure
$$dM = \frac{1}{Z_0^{nH}} \prod_{i,j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}$$
and action given by
$$S(M) = \frac{1}{2}\operatorname{tr}(MM^\dagger) - \left(\lambda \operatorname{tr} M^2(M^2)^\dagger + \mu \operatorname{tr} MM^\dagger MM^\dagger\right),$$
we obtain a non-Hermitian matrix model describing ribbon graphs with 4-valent vertices and oriented edges.
- Real and quaternionic matrix model. With space of field $\mathcal{S}_N$ the space of $N \times N$ symmetric matrices over $\mathbb{R}$ with measure
$$dM = \frac{1}{Z_0^{\mathbb{R}}} \prod_{i \leqslant j} dM_{ij}$$
and action given by
$$S(M) = \frac{1}{2}\operatorname{tr} M^2 - \frac{1}{4}\operatorname{tr} M^4,$$
we obtain a real matrix model describing ribbon graphs with 4-valent vertices and twisted and untwisted ribbons. Thus, this models describes properly embedded graphs composed of 4-gons in (not necessarily oriented) closed, compact surfaces. A similar model can be constructed for Hermitian quaternionic matrices.

3. Gauge theories

Let us motivate the expression for the gauge-fixed partition function following the exposition of [2, 7]. Consider the space of fields to be an n-dimensional manifold M endowed with a "nice" measure dx (for example, a measure induced by a top form or, more generally, by a density). Consider a compact Lie G of dimension l with an invariant measure dg, acting freely and measure-preserving on M. Take a smooth function $f\colon M \to \mathbb{R}$ (we can think of it as e^{-S}). We can associate to it a new function, denoted by $\int_G f$, which is G-invariant: it is the mean value at each orbit, that is

$$\left(\int_G f\right)(x) = \int_G f(g.x)\, dg. \tag{3.1}$$

In the above setting, there exists a unique measure $d\bar{x}$ on the quotient manifold $\overline{M} = M/G$ such that the following relation holds for every function f (see [3], section (3.13) as a reference):

$$\int_M f(x)\, dx = \int_{\overline{M}} \left(\int_G f\right)(\bar{x})\, d\bar{x}. \tag{3.2}$$

Note that in particular, if f is G-invariant, then we find

$$Z = \frac{1}{\mathrm{vol}(G)} \int_M f\, dx = \int_{\overline{M}} \bar{f}\, d\bar{x}, \tag{3.3}$$

where $\bar{f}$ are the induced function on the quotient manifold $\overline{M}$. Locally, dx is the product measure of dg and $d\bar{x}$.

EXAMPLE. Consider $M = \mathbb{R}^2 \setminus \{0\}$ and $G = \mathbb{S}^1$ acting by rotations. Then $\overline{M} = (0, +\infty)$. Choose $d\mu(x,y) = dx \wedge dy$ as the Lebesgue measure on M and $dg(\phi) = d\phi$. Note that in polar coordinates we can write $d\mu(r,\phi) = r dr \wedge d\phi$. Then $d\bar{\mu}(r) = r dr$, and for a $\mathbb{S}^1$-invariant function $f = f(r)$, we find

$$\frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \{0\}} f(x,y)\, dx \wedge dy = \int_0^{+\infty} f(r)\, r dr.$$

The aim of the section is to write the integral Z in a way suitable for the Feynman graph method. Suppose there exists a smooth function $F\colon U \to \mathfrak{g}$ defined in an open subset U of M, such that

- $\pi(U) = \overline{M}$,
- $F^{-1}(0) \cong \overline{M}$.

In physical literature, F is called a gauge-fixing function and $F = 0$ the gauge fixing. Now, point-wise we have the decomposition $T_x U = \mathfrak{g} \oplus T_{\bar{x}} \overline{M}$, so that

$$\Lambda(x) = d_x F|_{\mathfrak{g}} \in \operatorname{End}(\mathfrak{g}). \tag{3.4}$$

Denote by $J(x)$ its determinant, called the Faddeev-Popov determinant. Then we can use F as a coordinate system on $\overline{M}$, localizing the integral along the zero locus of F with a delta function and integrating along the orbits taking into account the change of variable

$$Z = \int_U f(x)\delta\big(F(x)\big)J(x)\, dx, \tag{3.5}$$

where δ is the Dirac delta function.

EXAMPLE (Continuation). Concluding our previous example, we can choose $U = \{\, (x, y) \in \mathbb{R}^2 \mid x > 0 \,\}$ and $F(x, y) = y$. Identifying

$$T_{(r,\phi)} U = \mathbb{R} \oplus T_r(0, +\infty),$$

we find $J(x, y) = \frac{\partial F}{\partial \phi}(x, y) = x$. As a consequence, we find the identity

$$Z = \int_U f(x, y)\delta(y)\, x\, dx \wedge dy = \int_0^{+\infty} f(x, 0)\, x dx.$$

Now, to apply the techniques used so far, we would like to write the delta function and the Faddeev-Popov determinant in exponential form.
To do so, fix a basis $c^1, \dots, c^l$ of $\mathfrak{g}$, that givs a volume form in $\Lambda^{\mathrm{top}}\mathfrak{g}$. Then we are able to integrate over $\mathfrak{g}$. The delta function can be written as an exponential thanks to the Fourier transform

$$\delta\big(F(x)\big) = \int_{\mathfrak{g}^*} e^{i\langle \xi, F(x)\rangle} đ\xi, \tag{3.6}$$

where $đ\xi = \frac{d\xi}{(2\pi)^l}$.
We can also write the determinant as a Berezin integral (see for instance [8]). Consider a real vector space V and choose a volume form $\Omega \in \Lambda^{\mathrm{top}} V^*$. Then we define the Berezin integral as the projection of $\Lambda^\bullet V^*$ onto $\Lambda^{\mathrm{top}} V^* \cong \mathbb{R}$:

$$\left(\int_{\Pi V} \omega\right) \Omega = \omega^{\mathrm{top}}. \tag{3.7}$$

Here Π stands for the parity operator applied to a super vector space. We want to consider the case of $V = \mathfrak{g} \oplus \mathfrak{g}^*$. To make it more concretely, consider the basis $c^1, \dots, c^l$ of $\mathfrak{g}$ and its dual basis $\bar{c}^1, \dots, \bar{c}^l$. The Grassmann algebra is given by

$$\Lambda^\bullet V^* = \Lambda_{\mathbb{R}}(\bar{c}^1, \cdots, \bar{c}^l, c^1, \dots, c^l,). \tag{3.8}$$

An element in $\omega \in \Lambda$ is the odd version of a polynomial:

$$\omega = \sum_{n,m} \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_m}} \frac{1}{n!m!} a_{i_1\dots i_n, j_1\dots j_m} \bar{c}^{i_1} \cdots \bar{c}^{i_n} c^{j_i} \cdots c^{j_m}. \tag{3.9}$$

with $a_{i_1\dots i_n, j_1\dots j_m}$ skew-symmetric in the indices $i_1, \dots, i_n$ and $j_1, \dots, j_m$. We can define the Berezin integral by setting

$$\begin{aligned} \int c^i\, dc^j &= \int \bar{c}^i\, d\bar{c}^j = \delta^{ij} \\ \int dc^i &= \int d\bar{c}^i = 0, \end{aligned} \tag{3.10}$$

extended to Λ by $\mathbb{R}$-linearity and iteration over $dc d\bar{c} = dc^l \cdots dc^1 d\bar{c}^l \cdots d\bar{c}^1$. Explicitly, for ω as above, we find

$$\int_{\Pi V} \omega\, dc d\bar{c} = a_{1\dots l, 1\dots l}. \tag{3.11}$$

This is in accordance with the previous definition, with volume form on $\Lambda^{\mathrm{top}}V^*$ given by $\bar{c}^1 \cdots \bar{c}^l c^1 \cdots c^l$. For a matrix $\Lambda \in M(l \times l, \mathbb{R})$, define the element

$$e^{\langle \bar{c}, \Lambda c \rangle} = \sum_{k \geqslant 0} \frac{1}{k!} \left(\sum_{i,j=1}^{l} \Lambda_{ij} \bar{c}^i c^j \right)^k. \tag{3.12}$$

Note that the coefficient of $\bar{c}^1 \cdots \bar{c}^l c^1 \cdots c^l$ is precisely $\det \Lambda$, so that

$$\int_{\Pi V} e^{\langle \bar{c}, \Lambda c \rangle} \, dc d\bar{c} = \det \Lambda. \tag{3.13}$$

With this new formalism, we can finally write $J(x)$ as the Berezin integral over the Grassmann algebra generated by a basis c^i of $\mathfrak{g}$ and its dual basis $\bar{c}^i$:

$$J(x) = \int_{\Pi(\mathfrak{g} \oplus \mathfrak{g}^*)} e^{\langle \bar{c}, \Lambda(x) c \rangle} \, dc d\bar{c}. \tag{3.14}$$

Putting everything together, we find the expression

$$Z = \int f(x) e^{i\langle \xi, F(x) \rangle} e^{\langle \bar{c}, \Lambda(x) c \rangle} \, dx đ_{\Omega} \xi dc d\bar{c}, \tag{3.15}$$

where the domain of integration can be written as $U \times \mathfrak{g}^* \times \Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$.

Consider now $f = e^{-S}$, with $S(x) = \frac{1}{2} \langle x, Ax \rangle - \hbar U(x)$ for M with a scalar product, and suppose that $F(x) = Bx$ is linear and defined on the whole M. Then $e^{-S(x) + i\langle \xi, F(x) \rangle + \langle \bar{c}, \Lambda(x) c \rangle}$ can be written as e^{-S_F}, where

$$S_F(X, c, \bar{c}) = \frac{1}{2} \langle X, A_F X \rangle - \hbar U(X) - \langle \bar{c}, \Lambda c \rangle \tag{3.16}$$

for $X = (x, \xi)$ and A_F is non-degenerate:

$$A_F = \begin{pmatrix} A & iB^t \\ iB & 0 \end{pmatrix}. \tag{3.17}$$

Here A_F is invertible because its determinant is $\det(B^t B)$, which is different from zero because $B = dF$ is of maximal rank.

Then we can write the Feynman diagram expansion of correlation function for the gauge fixed theory. The quadratic form now consists of two parts: A_F and Λ, so there are two types of edges. The first type presents A_F , with the labels x_i and ξ_i at the ends. The second type presents Λ, with the labels c^i and $\bar{c}^i$ at the ends. Note that since Λ is not symmetric, these edges are directed. Also, there are new vertices, presenting all higher degree terms of the Lagrangian (in particular some where edges of both types meet.

In the physics literature, the "new" fields c and $\bar{c}$ are called Faddeev-Popov ghost and anti-ghost respectively. This comes from the fact that they can be interpreted as virtual particles, that came out from the formal manipulation of the partition function, but they do not really exists. More precisely, ghost particles do not obey the spin-statistic theorem (see [6] for a physics reference) and the name ghosts actually highlight this feature.

EXAMPLE (Lorenz gauge). In the electromagnetism example, we can consider the gauge fixing function $F\colon \Omega^1_0(X) \to C^\infty_0(X)$ given by

$$F(A) = d^t A. \tag{3.18}$$

Then we have gauge-fixed operator K_F acting on the complexified space $\Omega^1_0(X) \oplus C^\infty_0(X)$ given by

$$K_F = \begin{pmatrix} d^t d & id \\ id^t & 0 \end{pmatrix} \tag{3.19}$$

which is non-degenerate (in this case, the term $B^t B$ is the Laplacian Δ acting on functions).

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