SUSY and Morse Theory

Review of the article by E. Witten
Supersymmetry and Morse Theory

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1 Morse Theory

In this section a review of Morse Theory will be given, following (Milnor, 1963). In the following, M will be a compact oriented smooth m-manifold.

The basic insight of Morse Theory is that a smooth function $f\colon M \to \mathbb{R}$ can provide us with information about the underlying topological structure of the manifold. In particular, for each $p \in M$ we have that $f(p)$ is either a regular value (*i.e.* $df_p\colon T_pM \to \mathbb{R}$ is non zero), or $f(p)$ is a critical value. The local behaviour of f around a regular point can be completely understood up to diffeomorphism by the inverse function theorem. The question remains how to understand the set of critical points.

The content of Morse Theory is to shed light on the relationship between this set and the topology of M. We make the following assumption before beginning our analysis. In this work, we will restrict our study to functions whose Hessian in every critical point is non-degenerate.

Definition 1.1. Let $p \in M$ be a critical point for f. Define the Hessian in p as

$$\begin{aligned} d^2f_p\colon T_pM \times T_pM &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto v(\widetilde{w}f) \end{aligned} \tag{1.1}$$

where $\tilde{w}$ is a vector field which extends w.

As expected, the Hessian is a symmetric bilinear form. This comes from the fact that

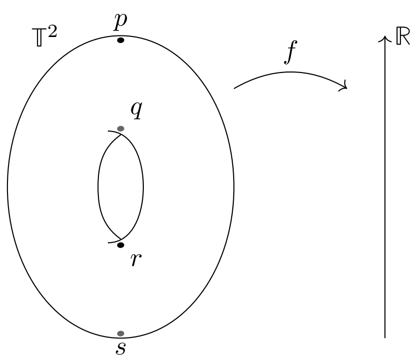
$$[v, w]_p f = d_p f[v, w] = 0,$$

since p is a critical value. This also proves that the definition does not depend on the extension of w.

Definition 1.2. A smooth function $f\colon M \to \mathbb{R}$ is called a **Morse function** such that the Hessian in every critical point is non-degenerate. Define the **Morse index** of a critical point p as

$$\mu_p = \#\left\{ \text{negative eigenvalues of } d^2f_p \right\}. \tag{1.2}$$

A classical example of Morse function is the height function on a torus $\mathbb{T}^2$.



In this case, we have four critical points: p with Morse index $\mu_p = 2$, q and r with Morse indices are $\mu_q = \mu_r = 1$ and s with Morse index $\mu_s = 0$.

Because of the non-degeneracy character of the Hessian, one can give a local characterization of Morse functions around any one of the critical points.

Lemma 1.1 (Morse lemma)**.** Let $f\colon M \to \mathbb{R}$ be a Morse function, $p \in M$ a critical point. Then there exists a system of coordinates (U, x^i) near p such that in these coordinates

$$f(x) = f(p) - \sum_{i=1}^{\mu_p} (x^i)^2 + \sum_{i=\mu_p+1}^{m} (x^i)^2. \tag{1.3}$$

In particular, every Morse function on a compact manifold has a finite number of critical points.

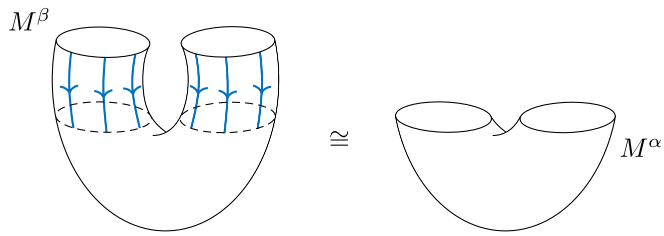
The two fundamental results which allow to reconstruct the underlying topological structure of the manifold M from a Morse function f are the following.

- If one considers the set
$$M^\alpha = \{\, p \in M \mid f(p) \leq \alpha \,\}, \tag{1.4}$$
the topology of M^α does not change as we vary α, unless upon varying we pass through a critical point of f.
- When passing through a critical point, the Morse index of the critical point completely determines how the topology of M^α changes.

More precisely:

Theorem 1.2. Let $f\colon M \to \mathbb{R}$ be a Morse function, $\alpha < \beta$ real numbers and suppose that $\{\, \alpha \leq f \leq \beta \,\}$ does not contain critical point of f. Then M^α is a deformation retract of M^β.

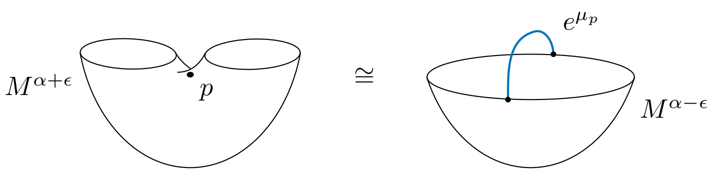
The idea of the proof is somehow simple. If we endow M with a Riemannian structure, then we can define the gradient ∇f and the associated flow. The deformation retraction is defined as the identity on $M^\alpha \subset M^\beta$ and through the flow on $M^\beta \setminus M^\alpha$. The following picture illustrates this procedure.



Now if upon varying the level sets we do encounter a critical point p, the fundamental result in this direction tells us that the resulting level set after deforming has the homotopy type of the original level set with a μ_p-dimensional cell attached.

Theorem 1.3. Let $f\colon M \to \mathbb{R}$ be a Morse function and p a critical point with $f(p) = \alpha$ and $\epsilon > 0$ such that f does not have critical points in $(\alpha - \epsilon, \alpha + \epsilon)$ rather than p. Then $M^{\alpha+\epsilon}$ is homotopically equivalent to $M^{\alpha-\epsilon} \cup e^{\mu_p}$.

One can construct the handle e^{μ_p} using the local description of the Morse function, where the attaching map between the boundary of the μ_p-dimensional cell and $\partial M^{\alpha-\epsilon}$. One uses homotopy-type arguments using technical results of Whitehead to obtain a cell complex. The picture below shows an example on the height function of the torus.



As a corollary we get:

Corollary. Let p be a non-degenerate critical point of f. Let $f(p) = \alpha$ and assume it is the only critical point with level α. Then for sufficiently small ϵ

$$H_k(M^{\alpha-\epsilon}, M^{\alpha-\epsilon}) = \begin{cases} \mathbb{Z} & \text{if } q = \mu_p \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Note that these results can be strengthened to the case where the function f has $p_1, \dots, p_N$ critical points at the level α with Morse indices $\mu_1, \dots, \mu_N$. Then $M^{\alpha+\epsilon}$ has the homotopy type of $M^{\alpha-\epsilon}$ with N cells attached, each having the dimension of the respective Morse index of the critical point under consideration. Moreover, for ϵ sufficiently small, $H_k(M^{\alpha+\epsilon}, M^{\alpha-\epsilon}) = \mathbb{Z}^{n_k}$ where n_k is the number of critical points $p_1, \dots, p_N$ with Morse index k.

We now arrive at the important Morse inequalities, which actually were the original Morse's point of view of the subject: the relationship between the topology of M and the critical point of a Morse function is described by a collection of inequalities.

Theorem 1.4 (Morse inequalities). Let $f\colon M \to \mathbb{R}$ be a Morse function and β_k be the kth Betti number of the singular homology of M. Set

$$M_k = \#\{\text{ critical points with Morse index } k\,\}. \tag{1.6}$$

- Weak Morse inequality:
$$\beta_k \le M_k. \tag{1.7}$$
- Strong Morse inequality. For every $n \in \{0, \dots, m\}$,
$$\sum_{k=0}^{n} (-1)^{n-k} \beta_k \le \sum_{k=0}^{n} (-1)^{n-k} M_k. \tag{1.8}$$
- Morse index theorem. Let $\chi(M)$ be the Euler-Poincaré characteristic of M. Then
$$\chi(M) = \sum_{k=0}^{m} (-1)^k M_k. \tag{1.9}$$

Proof. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ denote the critical values of f, which are only finitely many by the compactness of M. Choose real numbers $\beta_0, \beta_1, \dots, \beta_N$ such that $\beta_i < \alpha_i < \beta_{i+1}$ for all $0 \le i \le N-1$. In particular, we have $M^{\beta_0} = \varnothing$ and $M^{\beta_N} = M$. By Theorem 1.2, we have that for any integers i and k and any small $\epsilon > 0$,

$$H_k(M^{\alpha_i+\epsilon}, M^{\alpha_i-\epsilon}) \cong H_k(M^{\beta_i}, M^{\beta_{i-1}}).$$

Hence applying the corollary and the subadditivity properties of the Betti numbers (see the following remark, Equation (1.10)), we conclude the proof. □

Remark. Remember that the Betti number is a subadditive function of pairs of topological spaces: if we have $X \supset Y \supset Z$, then for any integer k

$$\beta_k(X, Z) \le \beta_k(X, Y) + \beta_k(Y, Z),$$

where $\beta_k(A, B)$ is the kth Betti number of the relative homology $H_\bullet(A, B)$. The same holds true for

$$B_n(A, B) = \sum_{k=0}^{n} (-1)^{n-k} \beta_k(A, B).$$

Finally, the Euler-Poincaré characteristic is an additive function:

$$\chi(X, Z) = \chi(X, Y) + \chi(Y, Z).$$

As a consequence, from the inclusions $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_N = X$, we obtain

$$\begin{aligned} \beta_k(X) &\leq \sum_{i=0}^{N} \beta_k(X_i, X_{i-1}) \\ B_n(X) &\leq \sum_{i=0}^{N} B_n(X_i, X_{i-1}) \\ \chi_k(X) &= \sum_{i=0}^{N} \chi(X_i, X_{i-1}). \end{aligned} \tag{1.10}$$

2 Supersymmetric Quantum Mechanics

In this section a brief introduction on Supersymmetric Quantum Mechanics will be given.

A particular class of quantum mechanical theories are the supersymmetric ones. **Supersymmetry** (SUSY) is an extension of the Poincaré symmetry group and it establishes a relation between bosons and fermions. As we know from Noether's theorem, we can associate to every symmetry of the Lagrangian a set of conserved charges. In the case of supersymmetry, these are Hermitian operators, often denoted by Q, which map bosons to fermions and vice versa:

$$\text{bosons} \overset{Q}{\longleftrightarrow} \text{fermion}.$$

More formally, a SUSY theory consists of a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{b}} \oplus \mathcal{H}_{\mathrm{f}}$, decomposed into the spaces of bosonic and fermionic states, together with a set of Hermitian SUSY operators Q_i, $i = 1, \dots, N$ mapping $\mathcal{H}_{\mathrm{b}}$ to $\mathcal{H}_{\mathrm{f}}$ and vice versa. In a SUSY quantum mechanical theory, they must satisfy the SUSY algebra

$$[H, Q_i] = 0 \qquad \{Q_i, Q_j\} = 2\delta_{ij} H. \tag{2.1}$$

In the simplest case, where we have two SUSY operators Q_1 and Q_2, we can simply set

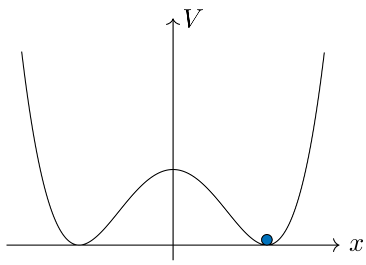
$$Q = \frac{1}{2}(Q_1 + iQ_2) \qquad Q^\dagger = \frac{1}{2}(Q_1 - iQ_2), \tag{2.2}$$

so that SUSY algebra (2.1) reduces to

$$Q^2 = (Q^\dagger)^2 = 0 \qquad H = QQ^\dagger + Q^\dagger Q. \tag{2.3}$$

The operators Q and $Q^\dagger$ are called supercharges. One of the major physical implication of such a theory, as we will see in a bit, is the existence of so-called superpartners: for every elementary particle of the bosonic type, there exists a particle of the fermionic type with equal energy. Dually, every fermionic particle has a bosonic superpartner of the same energy and mass. This one-to-one correspondence between bosons and fermions, however, has not been observed in nature. So, for SUSY to play a role in nature it must be *spontaneously broken*.

In general, a symmetry is said to be spontaneously broken when the vacuum $|0\rangle$ of the theory, the state of lowest energy, is not invariant under the symmetry. Although the equations of motion themselves are symmetric, the symmetry is hidden: there are no states which are left invariant by the symmetry. An example of system with spontaneous symmetry breaking is the double-well potential: although the potential is symmetric in $x \to -x$, the ground states of the system, centred at one of the minima, is not. In defining the vacuum we must choose between one of the minima.



In the case of supersymmetry, we would like to know if the vacuum is invariant under a supersymmetry transformation. In other words, we would like to know if the vacuum $|0\rangle$ is annihilated by the SUSY operators:

$$Q|0\rangle = Q^\dagger|0\rangle = 0. \tag{2.4}$$

The requirement of $|0\rangle$ to be annihilated by the SUSY operators comes from the fact that they are the generators of the symmetry: if Equation (2.4) holds, then

$$e^{i\epsilon Q}|0\rangle = (1 + i\epsilon Q)|0\rangle = |0\rangle.$$

The special form of the SUSY algebra with two generators allows us to rewrite the condition for supersymmetry breaking. In particular,

$$\langle\psi|H|\psi\rangle = \langle\psi|QQ^\dagger|\psi\rangle + \langle\psi|Q^\dagger Q|\psi\rangle = \|Q|\psi\rangle\|^2 + \|Q^\dagger|\psi\rangle\|^2$$

so that H is positive definite and

$$H|\psi\rangle = 0 \qquad \Longleftrightarrow \qquad Q|\psi\rangle = Q^\dagger|\psi\rangle = 0.$$

As a consequence, we have a natural lower bound on the energy. If we have a state of zero energy, it must be the ground state of the system. From the above discussion we draw the important conclusion that the supersymmetry is broken if and only if the vacuum has energy $E > 0$.

Summarizing the above discussion, one has a very straightforward procedure for determining whether supersymmetry is spontaneously broken or not: one must check if the vacuum energy is zero. In attacking this problem a physicist might be inclined to use perturbation theory: it is often much easier to work with a local approximation of the potential. However, as it turns out, having found that supersymmetry is unbroken in perturbation theory does not allow one to conclude that the supersymmetry is also unbroken in the exact system. If in some approximation the minimum of the potential is zero, an arbitrarily small quantum effect, shifting the potential by a tiny amount, could lift the minimum to a small but non-zero value. Then, the supersymmetry would be spontaneously broken. The effects that break the symmetry here are non-perturbative, *i.e.* they do not show up in perturbation theory to any finite order. Only a calculation which reveals the existence of more than one zero of the potential could possibly tell us something about the supersymmetry being broken or not. These calculations are so-called instanton calculations, in which the tunneling from one zero to another is evaluated. The non-perturbative effect, in which a quantum mechanical particle tunnels through a potential barrier, is often responsible for lifting the vacuum energy slightly above zero. The result is a spontaneous breakdown of supersymmetry.

3 Morse Inequalities via SUSY

This is the main section of the project. The original work is due to Edward Witten (Witten, 1982). The basic idea of the article is to study a SUSY quantum mechanical model, built on a Riemannian manifold (M, g) through a Morse function $f\colon M \to \mathbb{R}$. The classical procedure in the analysis of the Hamiltonian will give us the Morse inequalities relating the topology of M and the critical points of f: firstly, perturbation theory for the Hamiltonian will lead to the weak Morse inequality, while the non-perturbative study via instanton analysis will give us the strong Morse inequality.

The underlying Hilbert space of our Quantum Mechanics is the (completion of the) exterior algebra of the manifold: $\Omega^\bullet(M)_\mathbb{C} = \Omega^\bullet(M) \otimes \mathbb{C}$. As shown in Section A.4, we have a natural definition of scalar product on $\Omega^\bullet(M)_\mathbb{C}$ as soon as M is an orientable compact smooth m-manifold, with

$$\langle\omega, \eta\rangle = \int_M \overline{\omega} \wedge *\eta \tag{3.1}$$

if $\omega, \eta \in \Omega^k(M)_\mathbb{C}$. Forms of different degree are defined as being orthogonal. The natural operator which should substitute the Laplace operator in the classical Hamiltonian is the Laplace-de Rham operator

$$\Delta = -(d + d^\dagger)^2. \tag{3.2}$$

Here $d^\dagger$ is the adjoint of the exterior derivative d, which is given on $\Omega^k(M)_\mathbb{C}$ by

$$d^\dagger = (-1)^{k(m-k)} * d* \tag{3.3}$$

In this way, Δ is a self-adjoint negative-definite linear operator on the exterior algebra. So the free Hamiltonian will be

$$H_0 = -\frac{\hbar^2}{2m}\Delta. \tag{3.4}$$

Note the H_0 is a SUSY Hamiltonian, with

$$Q_{1,0} = \frac{\hbar}{\sqrt{2m}}(d + d^\dagger), \qquad Q_{2,0} = \frac{i\hbar}{\sqrt{2m}}(d - d^\dagger),$$

To simplify the following expressions, let us take $\hbar = 1$ and $m = \frac{1}{2}$. Consider now a Morse function $f\colon M \to \mathbb{R}$. We define a deformation of the exterior derivative as a sort of "Euclidean evolution" driven by f from time 0 to s:

$$d \longrightarrow d_s = e^{-fs}\, d\, e^{fs}. \tag{3.5}$$

The action of d_s of the exterior algebra is simply

$$\begin{aligned} d_s\omega &= e^{-fs}\, d\big(e^{fs}\omega\big) \\ &= e^{-fs}\big(e^{fs}\, s\, df \wedge \omega + e^{fs} d\omega\big) \\ &= d\omega + s\, \epsilon_{df}\omega, \end{aligned}$$

where $\epsilon_{df} = df\wedge$. Taking into account that the adjoint of the exterior multiplication by a 1-form is the interior multiplication by the associated vector field, we have

$$d_s = d + s\,\epsilon_{df} \qquad d_s^\dagger = d^\dagger + s\, i_{\nabla f}. \tag{3.6}$$

The deformed Hamiltonian will be

$$\begin{aligned} H_s = (d_s + d_s)^2 &= (d + s\,\epsilon_{df})(d^\dagger + s\, i_{\nabla f}) + (d^\dagger + s\, i_{\nabla f})(d + s\,\epsilon_{df}) \\ &= -\Delta + s\big(d i_{\nabla f} + i_{\nabla f} d + d^\dagger \epsilon_{df} + \epsilon_{df} d^\dagger\big) + s^2\big(\epsilon_{df} i_{\nabla f} + \epsilon_{df} i_{\nabla f}\big) \\ &= -\Delta + s\big(\mathcal{L}_{\nabla f} + \mathcal{L}^\dagger_{\nabla f}\big) + s^2\{\epsilon_{df}, i_{\nabla f}\}. \end{aligned}$$

Let us write explicitly the last term:

$$\begin{aligned} \{\epsilon_{df}, i_{\nabla f}\}\omega &= df \wedge i_{\nabla f}\omega + i_{\nabla f}(df \wedge \omega) \\ &= df \wedge i_{\nabla f}\omega + (i_{\nabla f} df)\omega - df \wedge i_{\nabla f}\omega \\ &= i_{\nabla f}(df)\,\omega, \end{aligned}$$

so that the Hamiltonian reads

$$H_s = -\Delta + s\big(\mathcal{L}_{\nabla f} + \mathcal{L}^\dagger_{\nabla f}\big) + s^2\, i_{\nabla f}(df) \tag{3.7}$$

The Hamiltonian is still supersymmetric, with

$$Q_{1,s} = (d_s + d_s^\dagger), \qquad Q_{2,s} = i(d_s - d_s^\dagger).$$

The link between H_s and the topology of M comes from the classical Hodge theorem: In fact, note that d_s differs from d only by conjugation by the invertible operator e^{fs}, defining an isomorphism of complex chains.

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Omega^k(M)_\mathbb{C} & \xrightarrow{\;d\;} & \Omega^{k+1}(M)_\mathbb{C} & \longrightarrow \cdots \\ & {\scriptstyle e^{-fs}}\big\downarrow & & \big\downarrow{\scriptstyle e^{-fs}} & \\ \cdots \longrightarrow & \Omega^k(M)_\mathbb{C} & \xrightarrow{\;d_s\;} & \Omega^{k+1}(M)_\mathbb{C} & \longrightarrow \cdots \end{array}$$

From $H_0 = -\Delta$, we have that

$$\dim\ker\big(H_s\colon \Omega^k(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}\big) = \dim\ker\big(\Delta\colon \Omega^k(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}\big) = \beta_k, \tag{3.8}$$

where the last equality is due to Hodge theorem A.3. On the other hand, it is clear that for large time the potential $V = s\big(\mathcal{L}_{\nabla f} + \mathcal{L}^{\dagger}_{\nabla f}\big) + s^2\, i_{\nabla f}(df)$ tends to become a infinite well near the critical point of f. We will see how to place upper bounds on β_k in terms of the critical points of f, by studying the spectrum of the Hamiltonian for large s via asymptotic expansion for the eigenvalues in powers of $1/s$.

3.1 Weak Morse inequality via perturbation theory

In this section, we will use Einstein summation convention only for repeated upper and lower indices. For example, $a^i\xi^i\xi_i$ means

$$\sum_i a^i\xi^i\xi_i.$$

Further, we set

$$\begin{aligned}(a^j)^\dagger &= \epsilon_{dx^j}\\ a^j &= i_{\partial_j}.\end{aligned} \tag{3.9}$$

Note that in our SUSY model, $(a^j)^\dagger$ can be interpreted as a fermion creation operator, while a_j as a fermion annihilation operator. Further, this notation breaks the index convention of upper and lower indices (the annihilator operator should be a_j rather that a^j). However, we preferred this notation in order to make some formulae more clear.

The expansion of the Hamiltonian can be explicitly calculated in terms of local data at the critical points. In particular, choose an orthogonal system of coordinates in the neighbourhood of a critical point p, so that

$$g_{ij} = \delta_{ij} + O(|x|^2). \tag{3.10}$$

Further, with can perform an orthogonal rotation, so that the Hessian in p becomes diagonal:

$$f(x) = f(p) + \frac{1}{2}\sum_i \lambda^i (x^i)^2 + O(|x|^3). \tag{3.11}$$

With $x^i \to x^i/\sqrt{s}$, we can take into account the higher order terms as powers of $1/\sqrt{s}$. In particular, we find

$$\begin{aligned} g_{ij} &= \delta_{ij} + O\left(\frac{1}{s}\right)\\ f(x) &= f(p) + \frac{1}{2s}\sum_i \lambda^i (x^i)^2 + O\left(\frac{1}{s^{3/2}}\right). \end{aligned} \tag{3.12}$$

The Laplace-de Rham operator applied to a k-form $\omega = \frac{1}{k!}\omega_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ will be

$$\Delta\omega = \frac{s}{k!}\sum_i \frac{\partial^2\omega_{i_1\cdots i_k}}{\partial x^i \partial x^i} dx^{i_1}\wedge\cdots\wedge dx^{i_k} + O\big(\sqrt{s}\big). \tag{3.13}$$

The computations can be find in (Felsager, 1998), with a slighly modification for taking into account the corrections in $1/\sqrt{s}$. On the other hand

$$\nabla f = \left(\frac{1}{\sqrt{s}}\lambda^i x^i + O\left(\frac{1}{s}\right)\right)\frac{\partial}{\partial x^i}, \tag{3.14}$$

so that the Lie derivative along ∇f acts on $\Omega^k(M)_{\mathbb{C}}$ as

$$\begin{aligned}\mathcal{L}_{\nabla f}\omega = \frac{1}{k!}&\left(\frac{1}{\sqrt{s}}\lambda^i x^i + O\left(\frac{1}{s}\right)\right)\frac{\partial\omega_{i_1\cdots i_k}}{\partial x^i} dx^{i_1}\wedge\cdots\wedge dx^{i_k} +\\ &+\frac{1}{k!}\sum_{n=1}^{k}(-1)^{n-1}\sqrt{s}\,\frac{\partial(\nabla f)^{i_n}}{\partial x^n}\omega_{i_1\cdots i_k} dx^n\wedge dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_n}}\wedge\cdots\wedge dx^{i_k}.\end{aligned}$$

The derivatives of the components of ∇f are

$$\frac{\partial(\nabla f)^{i_n}}{\partial x^n} = \frac{1}{\sqrt{s}}\,\lambda^{i_n}\delta^{i_n}_n + O\left(\frac{1}{s}\right),$$

so that

$$\begin{aligned}\mathcal{L}_{\nabla f}\omega &= \frac{1}{k!}\sum_{n=1}^{k}(-1)^{n-1}\lambda^{i_n}\delta^{i_n}_n\omega_{i_1\cdots i_k}dx^n\wedge dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_n}}\wedge\cdots\wedge dx^{i_k} + O\left(\frac{1}{\sqrt{s}}\right)\\ &= \frac{1}{k!}\lambda^{i_n}\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} + O\left(\frac{1}{\sqrt{s}}\right)\end{aligned}$$

On the other hand, consider the operator $\sum_i \lambda^i\,(a^i)^\dagger a^i$. On $\Omega^k(M)_\mathbb{C}$ it acts as

$$\begin{aligned}\sum_i \lambda^i\,(a^i)^\dagger\big(a^i\omega\big) &= \frac{1}{k!}\lambda^i\,dx^i\left(\sum_{n=1}^{k}(-1)^{n-1}\delta^{i_n}_i\omega_{i_1\cdots i_k}\wedge dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_n}}\wedge\cdots\wedge dx^{i_k}\right)\\ &= \frac{1}{k!}\lambda^{i_n}\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}.\end{aligned}$$

In particular,

$$\mathcal{L}_{\nabla f} = \sum_i \lambda^i\,(a^i)^\dagger a^i + O\left(\frac{1}{\sqrt{s}}\right).$$

In the same way, it is possible to prove that

$$\mathcal{L}^\dagger_{\nabla f} = -\sum_i \lambda^i\,a^i(a^i)^\dagger + O\left(\frac{1}{\sqrt{s}}\right).$$

The first term in the potential becomes

$$s\big(\mathcal{L}_{\nabla f}+\mathcal{L}^\dagger_{\nabla f}\big) = s\sum_i \lambda^i\left[(a^i)^\dagger, a^i\right] + O\big(\sqrt{s}\big).$$

Finally, the last term of the potential will be

$$\begin{aligned}s^2\,i_{\nabla f}(df) = g^{ij}\partial_i f\partial_j f &= s^2\delta_{ij}\left(\frac{1}{\sqrt{s}}\lambda^i x^i + O\left(\frac{1}{s}\right)\right)\left(\frac{1}{\sqrt{s}}\lambda^j x^j + O\left(\frac{1}{s}\right)\right)\\ &= s\sum_i(\lambda^i x^i)^2 + O\big(\sqrt{s}\big).\end{aligned}$$

The Hamiltonian will be

$$H_s = s\sum_{i=1}^{m}\left(-\frac{\partial^2}{\partial x^i\partial x^i} + (\lambda^i x^i)^2 + \lambda^i\left[(a^i)^\dagger, a^i\right]\right) + O\big(\sqrt{s}\big). \tag{3.15}$$

We see that in leading approximation, the problem separates into m one-dimensional problems, each with Hamiltonian sH' where

$$H' = -\frac{\partial^2}{\partial x^2} + \lambda^2 x^2 + \lambda\left[\epsilon_{dx}, i_\partial\right]. \tag{3.16}$$

Note that H' is the sum of a harmonic-oscillator Hamiltonian and an operator of the form $K = \lambda\left[\epsilon_{dx}, i_\partial\right]$. Further, the two operators commute, so that they can be simultaneously diagonalised. The eigenvalues of the harmonic Hamiltonian are well-known:

$$|\lambda|(1+2N), \qquad N = 0,1,2,\ldots$$

and the eigenforms are

$$\omega = \varphi\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}, \tag{3.17}$$

where φ is an eigenfunction of the harmonic oscillator Hamiltonian. The operator K has eigenvalues $\pm\lambda$, as can be seen by

$$\left[(a^i)^\dagger, a^i\right]\varphi\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \begin{cases} \varphi\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} & \text{if } i = i_n \text{ for some } n \in \{1, \ldots, k\} \\ -\varphi\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} & \text{if } i \neq i_n\ \forall\, n \in \{1, \ldots, k\}\,. \end{cases} \tag{3.18}$$

In other words, the form $\omega = \varphi\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is an eigenstate for $\left[(a^i)^\dagger, a^i\right]$ of eigenvalue $+1$ if the fermion dx^i is present in ω, -1 if it is not. As a consequence, the eigenvalues of H_s in the leading order will be

$$t\sum_{i=1}^{m}\Big(|\lambda^i|(1+2N^{(i)}) \pm \lambda^i\Big), \qquad N^{(i)} = 0, 1, 2, \ldots \tag{3.19}$$

It is clear now that there is just one possibility to have zero eigenvalue at leading order: all the $N^{(i)}$ must be zero and the ±1 must be chosen in the following way: $+1$ if $\lambda^i < 0$, -1 if $\lambda^i > 0$. From Equation (3.18), it turns out that the eigenform associated to the zeroth eigenvalue is a μ_p-form, where μ_p is Morse index in p. As a consequence, at leading order the kernel of H_s on $\Omega^k(M)_{\mathbb{C}}$ is nothing but M_k.

To summarize, we have been discussing the states localized near one critical point, but the low-lying eigenform of H_s for large s may of course be localized near any critical point on the manifold. Taking account of all the critical points, we see that for every critical point p, H_s has just one eigenstate ω whose energy does not diverge with s. Moreover, ω is a k-form if p has Morse index k. It is not necessarily the case that H_s annihilates all the states ω: we have only shown that the leading coefficients in perturbation theory vanish. But H_s certainly does not annihilate any of the other states, whose energy is proportional to s for large s. So at most the number of zero energy k-forms equals the number of critical points of Morse index k: we have established the weak Morse inequality

$$\beta_k \leq M_k. \tag{3.20}$$

Further, note that the perturnative ground state (*i.e.* the ground state of H') is proportional to the μ_p-form

$$\omega_p = e^{-s\sum_i |\lambda^i|(x^i)^2} \bigwedge_{j\,:\,\lambda^j<0} dx^j, \tag{3.21}$$

In particular, ω_p tends to a delta function concentrated in p as $s \to +\infty$.

3.2 Morse-Smale-Witten cochain complex via instanton analysis

We have just proved the weak Morse inequality computing the leading contribution to the spectrum of H_s. From a more accurate calculation of the spectrum, one can hope to get a better upper bound on the number of zero eigenvalues and thereby to strengthen the Morse inequality. However, all other terms in the asymptotic expansion vanish for all those states whose energy vanishes in lowest order thanks to SUSY. This really follows from the fact that the coefficients in perturbation theory can all be calculated in terms of local data at the critical points. From local data one cannot tell whether a given critical point is required by the topology or not. So all of the states which have zero energy in the first approximation remain at zero energy to all orders in s. In particular, the strong Morse inequality establishes an interplay of the different Morse indices.

To learn something new we must perform a calculation which is sensitive to the existence on M of more than one critical point. Since the potential energy in our problem, $V = s(\mathcal{L}_{\nabla f} + \mathcal{L}^\dagger_{\nabla f}) + s^2 i_{\nabla f}(df)$, has more than one minimum (one for each critical point), we must allow for the possibility of "tunnelling" from one critical point to another. This non-perturbative effect can be calculated via semiclassical trajectories in imaginary time called instantons.

Before computing the non-perturbative contributions, we have to construct the action which determines the dynamic. We begin by considering a particle propagating on the manifold (M, g). A massless bosonic particle is denoted by $\phi\colon \mathbb{R} \to M$, where $\mathbb{R}$ represents the time parametrized by t. Moreover, we have Grassmann fields ψ and $\bar{\psi}$ tangent to M which are complex conjugates of each other describing the fermionic superpartners of ϕ. More formally, $\phi \in C^\infty(\mathbb{R}, M)$ and $\psi, \bar{\psi} \in \Gamma^\infty(\mathbb{R}, \phi^*TM \otimes \mathbb{C})$. In local coordinates (x^i) we thus have the bosonic variables ϕ^i and fermionic variables ψ^i, $\bar{\psi}^i$. The dynamics of the system is described by the **supersymmetric non-linear sigma model** Lagrangian

$$L(\phi,\bar{\psi},\psi) = \frac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j + \frac{\mathrm{i}}{2}g_{ij}(\bar{\psi}^i D_t\psi^j - D_t\bar{\psi}^i\psi^j) + \frac{1}{4}R_{ijln}\bar{\psi}^i\psi^j\bar{\psi}^l\psi^n - s\frac{\partial^2 f}{\partial x^i\partial x^j}\bar{\psi}^i\psi^j - \frac{1}{2}s^2 g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}, \tag{3.22}$$

where

$$D_t\psi^i = \dot{\psi}^i + \Gamma^i{}_{jk}\dot{\phi}^j\psi^k.$$

is the pull-back of the covariant derivative, $\Gamma^k{}_{ij}$ are the Christoffel symbols of the Levi-Civita connection and R_{ijln} is the Riemann curvature tensor. This Lagrangian can be naturally constructed from a simpler one, imposing supersymmetry. See for istance (Freedman and Townsend, 1981). In order to simplify the computations, let us assume to be working in flat space. The Lagrangian simplifies substantially in

$$L(\phi,\bar{\psi},\psi) = \frac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j + \frac{\mathrm{i}}{2}g_{ij}(\bar{\psi}^i\dot{\psi}^j - \dot{\bar{\psi}}^i\psi^j) - s\frac{\partial^2 f}{\partial x^i\partial x^j}\bar{\psi}^i\psi^j - \frac{1}{2}s^2 g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}. \tag{3.23}$$

It can be shown that the above action is invariant under supersymmetry transformations

$$\begin{aligned} \delta\phi^i &= \epsilon\bar{\psi}^i - \bar{\epsilon}\psi^i \\ \delta\psi^i &= \epsilon\big(\mathrm{i}\dot{\phi}^i + g^{ij}\partial_j f\big) \\ \delta\bar{\psi}^i &= \bar{\epsilon}\big(-\mathrm{i}\dot{\phi}^i + g^{ij}\partial_j f\big), \end{aligned} \tag{3.24}$$

which by Noether's procedure gives conserved charges

$$Q = \bar{\psi}^i(\mathrm{i}g_{ij}\dot{\phi}^j + \partial_i f), \qquad \bar{Q} = \psi^i(-\mathrm{i}g_{ij}\dot{\phi}^j + \partial_i f). \tag{3.25}$$

Let us quantize the above theory. The conjugate momenta are

$$\begin{aligned} p_i &= \frac{\partial L}{\partial\dot{\phi}^i} = g_{ij}\dot{\phi}^j \\ \pi_i &= \frac{\partial L}{\partial\dot{\psi}^i} = \frac{\mathrm{i}}{2}g_{ij}\bar{\psi}^j. \end{aligned} \tag{3.26}$$

The canonical (anti)commutation relations are

$$\begin{aligned} [\phi^i, p_j] &= \mathrm{i}\delta^i_j \\ \{\psi^i, \bar{\psi}^j\} &= g^{ij} \end{aligned} \tag{3.27}$$

and all other relations vanishing. The supercharges become

$$Q = \bar{\psi}^i(\mathrm{i}p_i + \partial_i f), \qquad \bar{Q} = \psi^i(-\mathrm{i}p_i + \partial_i f). \tag{3.28}$$

The ordering ambiguity in the Hamiltonian is fixed by setting

$$H = \frac{1}{2}\{Q,\bar{Q}\}. \tag{3.29}$$

Using the anticommutation relations, we find

$$\begin{aligned} H &= \frac{1}{2}\Big(\bar{\psi}^i(\mathrm{i}p_i + \partial_i f)\psi^i(-\mathrm{i}p_i + \partial_i f) + \psi^i(-\mathrm{i}p_i + \partial_i f)\bar{\psi}^i(\mathrm{i}p_i + \partial_i f)\Big) \\ &= \frac{1}{2}\Big(p_i p_j + \mathrm{i}(\partial_j f p_i - \partial_i f p_j) + \partial_i f\partial_j f\Big)\Big(\bar{\psi}^i\psi^j + \psi^j\bar{\psi}^i\Big) \\ &= \frac{1}{2}g^{ij}\Big(p_i p_j + \mathrm{i}(\partial_j f p_i - \partial_i f p_j) + \partial_i f\partial_j f\Big). \end{aligned}$$

Quantization is not complete unless we specify the representation of the above algebra. The natural choice is the complexified space of differential forms

$$\mathcal{H} = \Omega^\bullet(M)_{\mathbb{C}} \tag{3.30}$$

with the Hermitian inner product

$$(\omega, \eta) = \int_M \overline{\omega} \wedge *\eta. \tag{3.31}$$

The observables are

$$\begin{aligned} \phi^k &= x^k \cdot \qquad & p_k &= -\mathrm{i}\partial_k \\ \bar{\psi}^k &= \epsilon_{dx^k} \qquad & \psi^k &= i_{\partial_k}. \end{aligned} \tag{3.32}$$

The supercharge Q becomes

$$\begin{aligned} Q\omega &= dx^i \wedge (\partial_i + \partial_i f)\omega \\ &= d\omega + \epsilon_{df}\omega \end{aligned} \tag{3.33}$$

and taking the adjoint, $\bar{Q} = Q^\dagger = d^\dagger + i_{\nabla f}$.

We are now ready to derive the instanton solutions from the action. In doing so we will closely follow (Hori, 2003). Recall that we have previously found a perturbative ground state ω_p for every critical point p of f. Here ω_p is a μ_p-form when p has index μ_p. However, it is not necessarily the case that each ω_p determines a supersymmetric ground state in the full theory. In other words,

$$Q\omega_p = 0$$

does not necessarily have to be satisfied. Although the above equation holds to all orders in perturbation theory, in the full theory we should expect an expansion

$$Q\omega_p = \sum_{\substack{\text{critical} \\ \text{points}}} \langle \omega_q, Q\omega_p \rangle\, \omega_q + \cdots, \tag{3.34}$$

where the ignored terms correspond to the expansions in the non-zero energy states. They can be neglected, since their energy is proportional to s, and they differ from the first terms by powers of s^{-1}. The elements

$$\langle \omega_q, Q\omega_p \rangle = \int_M \overline{\omega}_q \wedge *\big(Q\omega_p\big) \tag{3.35}$$

we want to compute are the non-perturbative corrections to the matrix elements of Q, representing the amplitudes associated to tunnelling paths between the critical points p and q. If the tunnelling amplitudes do not cancel, the sum gives a non-zero contribution, revealing the perturbative ground state ω_p to be a state with non-zero energy in the exact system. Notice that, since ω_q is a μ_q-form and $Q\omega_p$ is a $(\mu_p + 1)$-form, the above matrix element is zero only if

$$\mu_q \neq \mu_p + 1. \tag{3.36}$$

Thus, tunnelling corrections are only between states with relative Morse index one.

To go on with the instanton analysis, we would like to rewrite the matrix elements $\langle \omega_q, Q\omega_p \rangle$ using the path integral formalism. Recall that in the large s limit the ground state wave functions are sharply peaked near the critical points of f, *i.e.* ω_p is an approximate delta function at p for large s. In this limit the Morse function f may be viewed as an operator acting on the ground states by

$$f\omega_p = f(p)\omega_p + O\left(\frac{1}{t}\right). \tag{3.37}$$

This is a consequence of the Gaussian form of the perturbative ground states (see Equation (3.21)). Then

$$\begin{aligned}\langle \omega_q, Q\omega_p\rangle &= \frac{1}{f(p)-f(q)}\,\langle \omega_q, \big(Qf(p)-f(p)Q\big)\omega_p\rangle \\ &= \frac{1}{f(p)-f(q)}\,\langle \omega_q, \big[Q,f\big]\omega_p\rangle + O\left(\frac{1}{t}\right) \\ &= \frac{1}{f(p)-f(q)}\lim_{t\to+\infty}\langle \omega_q, e^{-tH}\big[Q,f\big]e^{-tH}\omega_p\rangle + O\left(\frac{1}{t}\right).\end{aligned}$$

The operators e^{-tH} in the limit $t\to+\infty$ project a state to the perturbative ground states. In this case, they do nothing (because ω_q and ω_p are already perturbative ground states), but it will let us write the expression as an Euclidean path integral. Note that

$$\begin{aligned}\big[Q,f\big]\omega &= (d+df\wedge)(f\omega) - f(d+df\wedge)\omega \\ &= df\wedge\omega + fd\omega + f\,df\wedge\omega - fd\omega - f\,df\wedge\omega \\ &= \epsilon_{df}\,\omega,\end{aligned}$$

so that the path integral representation will be

$$\langle \omega_q, Q\omega_p\rangle = \frac{1}{f(p)-f(q)}\int \mathcal{D}\phi\mathcal{D}\bar\psi\mathcal{D}\psi\;\bar\psi^k\frac{\partial f}{\partial x^k}\,e^{-S_E[\phi,\psi,\bar\psi]} \tag{3.38}$$

The integration is performed over bosonic fields ϕ such that $\phi(-\infty)=p$ and $\phi(+\infty)=q$, while ψ and $\bar\psi$ fall off sufficiently fast. The Euclidean action can be constructed from the Lagrangian (3.22) integrating in t and setting $t\to -it$:

$$S_E[\phi,\bar\psi,\psi] = \int_{\mathbb{R}} dt\left(\frac{1}{2}g_{ij}\dot\phi^i\dot\phi^j + \frac{1}{2}g_{ij}(\bar\psi^i\dot\psi^j - \dot{\bar\psi}^i\psi^j) + s\frac{\partial^2 f}{\partial x^i\partial x^j}\bar\psi^i\psi^j + \frac{1}{2}s^2 g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right). \tag{3.39}$$

In order to compute the transition amplitude (3.38), we can separately analyse the contribution from the bosonic action and from the fermionic one. The bosonic part can be written as

$$\begin{aligned}S_{\mathrm{b}}[\phi] &= \int_{\mathbb{R}} dt\left(\frac{1}{2}\left|\dot\phi^i \pm s\,g^{ij}\frac{\partial f}{\partial x^j}\right|^2 \mp s\,\dot\phi^i\frac{\partial f}{\partial x^i}\right) \\ &= \int_{\mathbb{R}} dt\left(\frac{1}{2}\left|\dot\phi^i \pm s\,g^{ij}\frac{\partial f}{\partial x^j}\right|^2 \mp s\,\dot f\right) \\ &= \frac{1}{2}\int_{\mathbb{R}} dt\left|\dot\phi^i \pm s\,g^{ij}\frac{\partial f}{\partial x^j}\right|^2 \mp s\big(f(q)-f(p)\big).\end{aligned}$$

Here the square has to be intended as the as contraction with the metric. In particular, we have a lower bound for S_E if

$$\dot\phi^i = \mp s\,g^{ij}\frac{\partial f}{\partial x^j}, \tag{3.40}$$

where the plus sign corresponds to $f(q)>f(p)$ and the minus sign to $f(q)<f(p)$. These are the instantons and anti-instantons and are interpreted as the steepest ascending and steepest descending paths from p to q. It can be shown, thanks to supersymmetry, that the dominant contributions come from the paths of steepest ascend: the path-integral is non-vanishing only if $f(q)>f(p)$. Thus, the relevant instanton for the present computation is an ascending gradient flow which starts from p and ends on q. The bosonic action for such paths is

$$I = s\big(f(q)-f(p)\big), \tag{3.41}$$

so that the contribution to the transition amplitude will decrease exponentially fast as $s\to+\infty$. Further, its non-analyticity for large s explains why this contribution was not captured in the perturbative analysis. The first order variation of the instanton equations reads

$$\delta\dot\phi^i = s\,g^{ij}\frac{\partial^2 f}{\partial x^j\partial x^k}\delta\phi^k. \tag{3.42}$$

In particular, the differential operators

$$\mathcal{D}_{\pm}\chi^i = \dot{\chi}^i \pm s\, g^{ij} \frac{\partial^2 f}{\partial x^j \partial x^k} \chi^k \tag{3.43}$$

turn out to be very important also in the fermionic scenario. Indeed, the fermionic action can be written as

$$\begin{aligned} S_{\rm f}[\bar{\psi}, \psi] &= \int_{\mathbb{R}} dt \left(\frac{1}{2} g_{ij} (\bar{\psi}^i \dot{\psi}^j - \dot{\bar{\psi}}^i \psi^j) + s \frac{\partial^2 f}{\partial x^i \partial x^j} \bar{\psi}^i \psi^j \right) \\ &= \int_{\mathbb{R}} dt\, g_{ij} \left(\bar{\psi}^i \dot{\psi}^j + s\, g^{jk} \frac{\partial^2 f}{\partial x^k \partial x^l} \bar{\psi}^i \psi^l \right) \\ &= \int_{\mathbb{R}} dt\, g_{ij} \bar{\psi}^i \mathcal{D}_+ \psi^j = - \int_{\mathbb{R}} dt\, g_{ij} \mathcal{D}_- \bar{\psi}^i \psi^j . \end{aligned} \tag{3.44}$$

For the path integral in (3.38) to be non-vanishing, the number of $\bar{\psi}$ zero modes must be larger than the number of ψ zero modes by one, since there is a single insertion of $\bar{\psi}$. Thus, the path integral is non-vanishing if and only if

$$\dim \ker \mathcal{D}_- - \dim \ker \mathcal{D}_+ = 1.$$

Noting that $\mathcal{D}_-$ are adjoint, we find

$$\operatorname{Ind} \mathcal{D}_- = \dim \ker \mathcal{D}_- - \dim \operatorname{coker} \mathcal{D}_- = \dim \ker \mathcal{D}_- - \dim \ker \mathcal{D}_+ = 1,$$

where $\operatorname{Ind} \mathcal{D}_-$ is the index of $\mathcal{D}_-$. This index can be computed by means of Hessian spectral flow as in (Hori, 2003). It turns out that, for paths connecting the critical points p and q, the index is

$$\operatorname{Ind} \mathcal{D}_- = \mu_q - \mu_p, \tag{3.45}$$

so that we find the previous result $\mu_q = \mu_p + 1$. We are finally ready to evaluate the path integral. Changing the variable $\phi = \gamma + \xi$, where γ is the insatanton solution of (3.42) and ξ is the fluctuation, the action in the quadratic approximation reads

$$S_E = I + \int_{\mathbb{R}} dt \big(|\mathcal{D}_- \xi|^2 - g_{ij} \mathcal{D}_- \bar{\psi}^i \psi^j \big). \tag{3.46}$$

We have made here the assumption that the Morse function f is generic in the sense that

$$\dim \ker \mathcal{D}_+ = 0 \tag{3.47}$$

for any instanton from p to q with $\Delta\mu = 1$. Then the only deformation of the instanton γ is a time translation $\gamma_\tau(t) = \gamma(t + \tau)$. As a consequence, an instanton can be determined by its "centre" τ and will be indicated as γ_τ. There is only one-dimensional kernel of $\mathcal{D}_-$ given by

$$\frac{d\gamma_\tau}{d\tau}\Big|_{t=0} = \frac{d\gamma}{dt}\Big|_{t=\tau} \tag{3.48}$$

and there is no kernel of $\mathcal{D}_+$. Thus, there is one ξ zero mode, one $\bar{\psi}$ zero mode and no ψ zero mode. The integration variable for the ξ zero mode is τ and we denote by $\bar{\psi}_0$ the variable for the $\bar{\psi}$ zero mode. In particular the variable $\bar{\psi}$ is expanded as

$$\bar{\psi}^i = \frac{d\gamma_\tau}{d\tau}\Big|_{t=0} \bar{\psi}_0 + \text{non-zero modes.} \tag{3.49}$$

In particular, we have to take into account the non-zero mode contribution an the zero mode one. The first contribution is given by the determinant ratio

$$\frac{\det{}' \mathcal{D}_-}{\sqrt{\det{}' (\mathcal{D}_-)^2}} = \pm 1, \tag{3.50}$$

where det′ do not take into account the zero mode contribution. This is given by

$$\int_{\mathbb{R}} d\tau \int d\bar{\psi}_0 \, \frac{d\gamma^i_\tau}{d\tau}\Big|_{t=0} \bar{\psi}_0 \frac{\partial f}{\partial x^i}\Big|_{t=0} = \int_{\mathbb{R}} d\tau \frac{d\gamma^i(\tau)}{d\tau} \frac{\partial f(\gamma(\tau))}{\partial x^i} = f(q) - f(p). \tag{3.51}$$

Thus, we obtain the following contribution from the instanton γ:

$$\pm \big(f(q) - f(p)\big) e^I. \tag{3.52}$$

Summing up the instantons and and including the prefactor from (3.38), we find

$$\langle \omega_q, Q\omega_p \rangle = \sum_\gamma n_\gamma e^{-I}, \qquad n_\gamma = \pm 1, \; I = s(f(q) - f(p)). \tag{3.53}$$

The sign of n_γ can be determined as follows. The integral $\int_M \overline{\omega}_q \wedge *Q\omega_p$ receives dominant contributions along the steepest ascents. For each steepest ascent γ, n_γ is 1 or -1 depending on whether the orientation determined by $\overline{\omega}_q \wedge *Q\omega_p$ along γ matches with the orientation of M or not. The form ω_p defines an orientation of the μ_p-dimensional plane $T_p^- M$ of negative eigenvectors of $d_p^2 f$. This plane can be transported along the steepest ascent and we obtain a subbundle E_p of $\gamma^* TM$ with the orientation determined by ω_p. Starting with the space of negative eigenvectors at q, we obtain another subbundle E_q with the orientation determined by ω_q. In the generic situation, only a single eigenvalue goes from positive to negative along the ascent and the eigenvector is the tangent vector v_γ to γ. Then E_p is a subbundle of E_q and the complement is spanned by v_γ. Now, $Q\omega_p$ defines an orientation of $\mathbb{R}v_\gamma \oplus E_p$. Thus, $n_\gamma = 1$ if this matches with the orientation determined by ω_p and $n_\gamma = -1$ otherwise.

From what we have seen by the path-integral analysis, we conclude that in the one-instanton approximation

$$Q\omega_p = \sum_{\mu_q = \mu_p + 1} e^{-s(f(q) - f(p))} \sum_\gamma n_\gamma \omega_q. \tag{3.54}$$

The exponential can be eliminated by rescaling the wave functions ω. This is the action of the supercharge Q on the perturbative ground states. Since the original supercharge Q is nilpotent, it should also be nilpotent when acting on ω_p's. Thus, if we define the graded space of perturbative ground states

$$C^k = \bigoplus_{\mu_p = k} \mathbb{R}\langle \omega_p \rangle, \tag{3.55}$$

we find the cochain complex with the coboundary operator given by the supercharge

$$0 \longrightarrow C^0 \xrightarrow{Q} \cdots \xrightarrow{Q} C^m \longrightarrow 0$$

The pair $(C^\bullet, Q)$ is called the **Morse-Smale-Witten cochain complex** of f. From

$$\beta_k = \dim\ker\big(H_s \colon \Omega^k(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}\big) \tag{3.56}$$

and the fact that $H\omega = 0$ if and only if $Q\omega = 0$, we find that the cohomology of $(C^\bullet, Q)$ is nothing but the de Rham cohomology of M.

As a first example, consider the height function on a horned sphere. We have four approximate vacua, one for each critical point. In Figure 1 the instanton paths γ are given with corresponding signs n_γ. With the correctly normalised perturbative ground states, we have

$$0 \longrightarrow \mathbb{R}\langle \omega_s \rangle \xrightarrow{0} \mathbb{R}\langle \omega_r \rangle \xrightarrow{\omega_p - \omega_q} \mathbb{R}\langle \omega_p \rangle \oplus \mathbb{R}\langle \omega_q \rangle \longrightarrow 0.$$

As a consequence,

$$H^0\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle \omega_s \rangle \xrightarrow{0} \mathbb{R}\langle \omega_r \rangle\big)}{\mathrm{Im}\big(0 \to \mathbb{R}\langle \omega_s \rangle\big)} = \frac{\mathbb{R}\langle \omega_s \rangle}{(0)} \cong \mathbb{R}$$

$$H^1\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle \omega_r \rangle \xrightarrow{\omega_p - \omega_q} \mathbb{R}\langle \omega_p \rangle \oplus \mathbb{R}\langle \omega_q \rangle\big)}{\mathrm{Im}\big(\mathbb{R}\langle \omega_s \rangle \xrightarrow{0} \mathbb{R}\langle \omega_r \rangle\big)} = \frac{(0)}{(0)} \cong 0$$

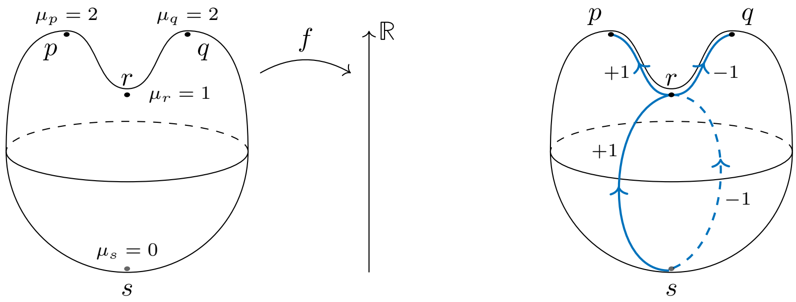


Figure 1: Height function on the horned sphere with instantons.

$$H^2\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle\omega_p\rangle \oplus \mathbb{R}\langle\omega_q\rangle \to 0\big)}{\mathrm{Im}\big(\mathbb{R}\langle\omega_r\rangle \xrightarrow{\omega_p-\omega_q} \mathbb{R}\langle\omega_p\rangle \oplus \mathbb{R}\langle\omega_q\rangle\big)} = \frac{\mathbb{R}\langle\omega_p\rangle \oplus \mathbb{R}\langle\omega_q\rangle}{\mathbb{R}\langle\omega_p - \omega_q\rangle} \cong \mathbb{R}$$

and we have recovered the cohomology of the sphere.

As a second example, consider the height function on the torus. Due to the many symmetries, this Morse function is not generic. However, the height function on a tilted torus works. We have four approximate vacua, one for each critical point. In Figure 2 the instanton paths γ are given with corresponding signs n_γ. With the correctly normalised perturbative ground states, we have

$$0 \longrightarrow \mathbb{R}\langle\omega_s\rangle \xrightarrow{\ 0\ } \mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle \xrightarrow{\ 0\ } \mathbb{R}\langle\omega_p\rangle \longrightarrow 0.$$

As a consequence,

$$H^0\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle\omega_s\rangle \xrightarrow{0} \mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle\big)}{\mathrm{Im}\big(0 \to \mathbb{R}\langle\omega_s\rangle\big)} = \frac{\mathbb{R}\langle\omega_s\rangle}{(0)} \cong \mathbb{R}$$

$$H^1\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle \xrightarrow{0} \mathbb{R}\langle\omega_p\rangle\big)}{\mathrm{Im}\big(\mathbb{R}\langle\omega_s\rangle \xrightarrow{0} \mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle\big)} = \frac{\mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle}{(0)} \cong \mathbb{R}^2$$

$$H^2\big((C^\bullet, Q)\big) = \frac{\ker\big(\mathbb{R}\langle\omega_p\rangle \to 0\big)}{\mathrm{Im}\big(\mathbb{R}\langle\omega_q\rangle \oplus \mathbb{R}\langle\omega_r\rangle \xrightarrow{0} \mathbb{R}\langle\omega_p\rangle\big)} = \frac{\mathbb{R}\langle\omega_p\rangle}{(0)} \cong \mathbb{R}$$

and we have recovered the cohomology of the torus.

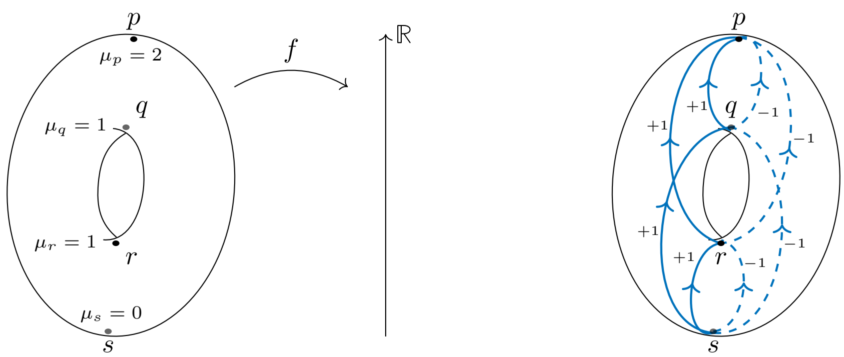


Figure 2: Height function on the tilted torus with instantons.

A Riemannian Geometry and Hodge Theory

A.1 Covariant derivative

This appendix is based on (Nakahara, 2003), where the reader can find deeper explanations and the proof of the stated theorems. In a differentiable manifold, a natural question is how to make the derivative of a vector field. In order to do that, we should be able to compare tangent vectors in different points p and q. A possibility is to "transport in parallel" a vector from p to q. Indeed, suppose p has coordinates $(x^1, \dots, x^m)$ and consider an infinitesimal displacement along the ath direction: q with coordinates $(x^1, \dots, x^a + \Delta x^a, \dots, x^m)$. Having fixed a vector field X, the "derivative" of X along the ath direction will be a vector, whose value in p is

$$(\nabla_a X)_p = \lim_{\Delta x^a \to 0} \frac{X_q^c - \widetilde{X}_q^c}{\Delta x^a} \frac{\partial}{\partial x^c}\Big|_p, \tag{A.1}$$

where $\widetilde{X}_q$ is the vector X_p parallel transported to q along the segment pq. Let us assume that

$$\widetilde{X}_q^c = X_p^c - X_p^b \Gamma^c{}_{ab} \Delta x^a, \tag{A.2}$$

so that $\widetilde{X}_q^c - X_p^c \propto \Delta x$ and the parallel transport is linear: $\widetilde{X+Y}_q^c = \widetilde{X}_q^c + \widetilde{Y}_q^c$. Then

$$(\nabla_a X)_p = \left(\frac{\partial X^c}{\partial x^a}\Big|_p + X_p^b \Gamma^c{}_{ab} \right) \frac{\partial}{\partial x^c}\Big|_p. \tag{A.3}$$

This idea leads to the definition of covariant derivative.

Definition A.1. Let M be a manifold. A **covariant derivative** (or affine connection) is a map

$$\begin{aligned} \nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ X, Y &\longmapsto \nabla_X Y \end{aligned} \tag{A.4}$$

such that the following conditions hold:

$$\begin{aligned} &\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \\ &\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z \\ &\nabla_{fX} Y = f \, \nabla_X Y \\ &\nabla_X f Y = X(f) Y + f \, \nabla_X Y, \end{aligned} \tag{A.5}$$

where $f \in C^\infty(M)$.

If (U, x^a) is a local chart of M, we set for complicity $e_a = \frac{\partial}{\partial x^a}$. Then we can determine the functions $\Gamma^c{}_{ab}$, called the **connection coefficients**, as

$$\nabla_{e_a} e_b = \Gamma^c{}_{ab} e_c. \tag{A.6}$$

In the following, we will write for simplicity $\nabla_a = \nabla_{e_a}$. With the connection coefficients, we are able to calculate in coordinates the action of ∇ on any vector. If $X = X^a e_a$, $Y = Y^a e_a$ then

$$\begin{aligned} \nabla_X Y &= X^a \, \nabla_a (Y^b e_b) = X^a \left(e_a(Y^b) \, e_b + Y^b \, \nabla_a e_b \right) \\ &= X^a \left(\frac{\partial Y^c}{\partial x^a} + \Gamma^c{}_{ab} Y^b \right) e_c \end{aligned} \tag{A.7}$$

The covariant derivative immediately generalises to tensor fields. First of all, let us define the covariant derivative for $f \in C^\infty(M)$ as the directional derivative:

$$\nabla_X f = X(f). \tag{A.8}$$

Note that in this notation the last equation in (A.5) is nothing but the Leibnitz rule. Then the covariant derivative of a generic tensor field is done by induction, after having required the Leibnitz rule to hold

$$\nabla_X (T \otimes S) = \nabla_X T \otimes S + T \otimes \nabla_X S. \tag{A.9}$$

For example, the covariant derivative of $\omega \in \Omega^1(M)$ is computed as

$$X(\langle \omega, Y\rangle) = \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle \quad \Longrightarrow \quad \langle \nabla_X \omega, Y\rangle = X(\langle \omega, Y\rangle) - \langle \omega, \nabla_X Y\rangle . \tag{A.10}$$

In coordinates and with $X = e_a$, we find

$$(\nabla_a \omega)_b = \partial_a \omega_b - \Gamma^c{}_{ab}\omega_c. \tag{A.11}$$

For a general tensor field,

$$(\nabla_a T)^{c_1\cdots c_p}_{b_1\cdots b_q} = \partial_a T^{c_1\cdots c_p}_{b_1\cdots b_q} + \sum_{i=1}^{p} \Gamma^{c_i}{}_{ad} T^{c_1\cdots d\cdots c_p}_{b_1\cdots b_q} - \sum_{j=1}^{q} \Gamma^d{}_{ab_j} T^{c_1\cdots c_p}_{b_1\cdots d\cdots b_q}. \tag{A.12}$$

The covariant derivative allows the **parallel transport** along a curve. Take $\gamma\colon I \to M$ and define the tangent vector as

$$V_{\gamma(t)} = \frac{d}{dt}\Big|_{\gamma(t)} = \frac{dx^a(\gamma(t))}{dt} e_a\Big|_{\gamma(t)}. \tag{A.13}$$

A vector field X (at least defined on γ) is parallel transported along γ if

$$\nabla_V X = 0. \tag{A.14}$$

In components

$$\frac{dx^a}{dt}\left(\frac{\partial X^c}{\partial x^a} + \Gamma^c{}_{ab} X^b\right) e_c = 0 \quad \Longrightarrow \quad \frac{dX^c}{dt} + \Gamma^c{}_{ab}\frac{dx^a}{dt} X^b = 0. \tag{A.15}$$

The notion of parallel transport along a curve allows us to define the concept of **geodesics**, *i.e.* the "straightest possible curves" on a manifold with affine connection. In particular, the geodesic is a curve whose tangent vector V is parallel transported along the curve itself:

$$\nabla_V V = 0. \tag{A.16}$$

In components, the geodesic equation reads

$$\frac{dx^c}{dt} + \Gamma^c{}_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt} = 0. \tag{A.17}$$

A weaker requirement is that any change of the tangent vector along the curve is still parallel to V: $\nabla_V V = fV$, with f a smooth function on γ. However, with the right parametrization, this Equation reduces to (A.16).

A.2 Riemannian manifolds

With the covariant derivative, we can give an affine structure to manifolds. The next step is the introduction of an Euclidean structure with the notion of Riemannian metric.

Definition A.2. Let M be a differentiable manifold. A **Riemannian metric** on M is a $(0,2)$ tensor field on M such that for every $p \in M$ and for every $v, w \in T_p(M)$ the following axioms holds true:

- $g_p(v, w) = g_p(w, u)$,
- $g_p(v, v) \geq 0$ and the equality holds if and only if $v = 0$.

The pair (M, g) is called a **Riemannian manifold**.

In the following, M will be a Riemannian manifold of dimension m with Riemannian metric g. If (U, x^a) is a chart of M, then for every $p \in U$ we can write g in coordinates as

$$g_p = g_{ab}(p)\, dx^a \otimes dx^b. \tag{A.18}$$

We will usually omit the dependence on p in $g_{ab}(p)$. Note that, since g_{ab} is positive definite, it is invertible, with inverse indicated as

$$(g_{ab})^{-1} = g^{ab}.$$

As usual, we will use g to rise or lower indices:

$$A^a = g_{ab}A^b, \qquad A_a = g^{ab}A_b.$$

In principle, covariant derivative and Riemannian metric have nothing to do with each other. Nevertheless, for a manifold M with both structures a natural condition arises: the metric should be covariantly constant, *i.e.* if two vector fields X and Y are parallel transported along any curve, then their inner product remains constant under parallel transport:

$$\nabla_V g(X,Y) = 0 \qquad \text{if } \nabla_V X = \nabla_V Y = 0, \tag{A.19}$$

for every curve and $X, Y \in \mathfrak{X}(M)$. In this case the covariant derivative is said to be metric compatible. In coordinate, Equation (A.19) becomes

$$0 = V^c\big((\nabla_c\, g)_{ab}\big)X^aY^b + g(\nabla_c X, Y) + g(X, \nabla_c Y\big) = V^c\big((\nabla_c\, g)_{ab}\big)X^aY^b, \tag{A.20}$$

which simplifies in

$$\big(\nabla_c\, g\big)_{ab} = \partial_c\, g_{ab} - \Gamma^d{}_{ca}g_{db} - \Gamma^d{}_{cb}g_{da} = 0 \tag{A.21}$$

since it must hold for every curve and vector fields. With cyclic permutation of $(a\ b\ c)$, we find

$$\partial_a\, g_{bc} - \Gamma^d{}_{ab}g_{dc} - \Gamma^d{}_{ba}g_{dc} = 0 \tag{A.22}$$

$$\partial_b\, g_{ca} - \Gamma^d{}_{bc}g_{da} - \Gamma^d{}_{ac}g_{db} = 0 \tag{A.23}$$

and the combination −(A.21)+(A.22)+(A.23) leads to

$$\partial_a\, g_{bc} + \partial_b\, g_{ca} - \partial_c\, g_{ab} + T^d{}_{ca}g_{db} + T^d{}_{cb}g_{da} - 2\Gamma^d{}_{(ab)}g_{cd} = 0. \tag{A.24}$$

Here $T^c{}_{ab} = 2\Gamma^c{}_{[ab]}$ is called the **torsion tensor**. Solving for $\Gamma^d{}_{(ab)}$, we obtain

$$\Gamma^d{}_{(ab)} = \begin{Bmatrix} d \\ ab \end{Bmatrix} + \frac{1}{2}\big(T_a{}^c{}_b + T_b{}^c{}_a\big), \tag{A.25}$$

where $\left\{ {d \atop ab} \right\}$ are the **Christoffel symbols**

$$\begin{Bmatrix} d \\ ab \end{Bmatrix} = \frac{1}{2}g^{dc}\big(\partial_a\, g_{bc} + \partial_b\, g_{ca} - \partial_c\, g_{ab}\big). \tag{A.26}$$

Finally, with $\Gamma^d{}_{ab} = \frac{1}{2}T^d{}_{ab} + \Gamma^d{}_{(ab)}$, we find the **metric compatible condition**

$$\Gamma^d{}_{ab} = \begin{Bmatrix} d \\ ab \end{Bmatrix} + \frac{1}{2}\big(T^d{}_{ab} + T_a{}^c{}_b + T_b{}^c{}_a\big), \tag{A.27}$$

In the next section we will further simplify Equation (A.27), arriving to the so-called Levi-Civita connection.

A.3 Torsion, Levi-Civita connection and Riemann curvature tensor

The connection coefficients do not form a tensor and, so that they cannot have an intrinsic geometric meaning. Nevertheless, its antisymmetric part, the torsion tensor, is a (1 2) tensor. Its intrinsic definition is $T\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{A.28}$$

In components, we find exactly the previous definition of $T^c{}_{ab}$:

$$\begin{aligned} T^c{}_{ab} &= \langle dx^c, \nabla_a e_b - \nabla_b e_a\rangle = \langle dx^c, (\Gamma^d{}_{ab} - \Gamma^d{}_{ba})e_d\rangle \\ &= (\Gamma^d{}_{ab} - \Gamma^d{}_{ba})\,\langle dx^c, e_d\rangle = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} \\ &= 2\Gamma^d{}_{[ab]}. \end{aligned} \tag{A.29}$$

Let us see the geometric meaning of the torsion. Consider a point p with coordinates (x^a) and two infinitesimal tangent vectors in T_pM: $X_p = \epsilon^a e_a(p)$, $Y_p = \delta^a e_a(p)$. They can be seen as small displacements on M to points q and r with coordinates $(x^a + \epsilon^a)$ and $(x^a + \delta^a)$ respectively. If we parallel transport Y along pq we obtain a vector of coordinates (see Equation (A.2))

$$\delta^c - \delta^a \Gamma^c{}_{ab} \epsilon^b. \tag{A.30}$$

With the same procedure for X along pr, we obtain

$$\epsilon^c - \epsilon^a \Gamma^c{}_{ab} \delta^b. \tag{A.31}$$

Viewing them as small displacements on M (see Figure **??**), we obtain two point q' and r' of coordinates $(x^c + \epsilon^c + \delta^c - \delta^a \Gamma^c{}_{ab} \epsilon^b)$ and $(x^c + \delta^c + \epsilon^c - \epsilon^a \Gamma^c{}_{ab} \delta^b)$ respectively. The difference between them will be

$$-\delta^a \Gamma^c{}_{ab} \epsilon^b + \epsilon^a \Gamma^c{}_{ab} \delta^b = \epsilon^a (\Gamma^c{}_{ab} - \Gamma^c{}_{ba}) \delta^b = T^c{}_{ab} \epsilon^a \delta^b. \tag{A.32}$$

In this sense, the torsion measures the failure in the closure of the parallelogram made up by small displacements and their parallel transports.

Definition A.3. A connection ∇ on a manifold M is called symmetric if the torsion tensor vanishes. In particular, the connection coefficients are symmetric:

$$\Gamma^c{}_{ab} = \Gamma^c{}_{ba}. \tag{A.33}$$

A natural requirement for a connection ∇ is the symmetric condition. In a Riemannian manifold we have seen a second condition of compatibility. it turns out that these two conditions uniquely determine a connection in a Riemannian manifold.

Theorem A.1 (The fundamental theorem of Riemannian geometry)**.** On a Riemannian manifold (M, g) there exists a unique symmetric connection which is compatible with the metric. This is called the **Levi-Civita connection** and its connection coefficients are given by the Christoffel symbols:

$$\Gamma^c{}_{ba} = \begin{Bmatrix} c \\ ab \end{Bmatrix}. \tag{A.34}$$

A.4 Hodge theory

One of the main properties of oriented Riemannian manifolds is the presence of a "natural" volume form. Since g_{ab} is positive definite, we can define locally the **invariant volume form** as

$$\Omega_M = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^m.$$

where $|g| = \det g_{ab}$. Let us check that Ω_M is well defined: if (U, x^a), (V, y^a) are charts of an oriented atlas with $U \cap V \neq \varnothing$, then

$$\begin{aligned}
\Omega_M &= \sqrt{\det\left(\frac{\partial y^a}{\partial x^c}\frac{\partial y^b}{\partial x^d}\, g_{ab}\right)}\, \det\left(\frac{\partial x}{\partial y}\right)\, dy^1 \wedge \cdots \wedge dy^m \\
&= \sqrt{|g|}\, \left|\det\left(\frac{\partial y}{\partial x}\right)\right| \det\left(\frac{\partial x}{\partial y}\right)\, dy^1 \wedge \cdots \wedge dy^m \\
&= \sqrt{|g|}\, dy^1 \wedge \cdots \wedge dy^m
\end{aligned}$$

since the sign of $\det \frac{\partial x}{\partial y}$ is positive.

In a Riemann manifold we can also define a natural isomorphism between $\Omega^p(M)$ and $\Omega^{m-p}(M)$. Let us define the Levi-Civita symbol

$$\epsilon_{a_1 \cdots a_p} = \begin{cases} +1 & \text{if } (a_1\ a_2\ \cdots\ a_p) \text{ is an even permutation of } (1\ 2\ \cdots\ m) \\ -1 & \text{if } (a_1\ a_2\ \cdots\ a_p) \text{ is an odd permutation of } (1\ 2\ \cdots\ m) \\ 0 & \text{otherwise.} \end{cases}$$

Definition A.4. Let M be an oriented Riemann manifold. The **Hodge dual operator** is the linear map $*\colon \Omega^p(M) \to \Omega^{m-p}(M)$ defined locally on the basis as

$$*(dx^{a_1} \wedge \cdots \wedge dx^{a_p}) = \frac{\sqrt{|g|}}{(m-p)!}\, \epsilon^{a_1 \cdots a_p}{}_{b_{p+1} \cdots b_m}\, dx^{b_{p+1}} \wedge \cdots \wedge dx^{b_m}$$

and then extended by linearity.

As in the case of the volume form, $*$ is well defined. It should be noted that $*1$ is the invariant volume form:

$$*1 = \Omega_M.$$

An important property of the Hodge dual operator is that it is a graded involution: if $\omega \in \Omega^p(M)$, then

$$**\,\omega = (-1)^{p(m-p)}\omega.$$

In particular, $*^{-1} = (-1)^{p(m-p)}*$. The Hodge dual operator allows us to define the inner product of p-forms as

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta.$$

Another tool coming from the $*$ operator is the adjoint derivative.

Definition A.5. Let M be an oriented Riemann manifold, $d\colon \Omega^{p-1}(M) \to \Omega^p(M)$ the exterior derivative. The **adjoint exterior derivative** is the linear map $d^\dagger\colon \Omega^p(M) \to \Omega^{p-1}(M)$ defined by

$$d^\dagger = (-1)^{mp+p+1} * d * .$$

Thus, the following diagram commute.

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{\quad d^\dagger \quad} & \Omega^{p-1}(M) \\ \big\downarrow * & & \big\uparrow * \\ \Omega^{m-p}(M) & \xrightarrow[(-1)^{mp+p+1}d]{} & \Omega^{m-p+1}(M) \end{array}$$

Note that the adjoint exterior derivative is nilpotent: $d^{\dagger 2} = *d *^2 d* = \pm * d^2 * = 0$. A consequence of Stokes theorem is that $d^\dagger$ is the adjoint operator of d with respect to the inner product introduced before: if M is a compact orientable Riemann manifold, then

$$\langle d\omega, \eta \rangle = \langle \omega, d^\dagger \eta \rangle .$$

The adjoint exterior derivative allows us to define the Laplacian on p-forms.

Definition A.6. Let M be an oriented Riemann manifold. The **Laplace-de Rham operator**[1] is defined as the linear operator $\Delta\colon \Omega^p(M) \to \Omega^p(M)$,

$$\Delta = -(d + d^\dagger)^2 = -dd^\dagger - d^\dagger d.$$

A p-form is said to be **harmonic** if $\Delta\omega = 0$. We denote set of harmonic p-forms on M by $\mathrm{Harm}^p(M)$.

The importance of the harmonic forms is given by the following theorems.

Theorem A.2 (Hodge decomposition theorem)**.** Let (M, g) be a compact orientable Riemann manifold. Then $\Omega^p(M)$ decomposes as

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus d^\dagger \Omega^{p+1}(M) \oplus \mathrm{Harm}^p(M)$$

Let us call $P\colon \Omega^p(M) \to \mathrm{Harm}^p(M)$ the projection operator given by the decomposition theorem.

Theorem A.3 (Hodge theorem)**.** Let (M, g) be a compact orientable Riemann manifold. Then the map

$$\begin{aligned} P\colon H^p(M) &\longrightarrow \mathrm{Harm}^p(M) \\ [\omega] &\longmapsto P\omega \end{aligned}$$

is an isomorphism.

[1] A more common definition of the operator is $\Delta = (d+d^\dagger)^2$. However, with this definition on $\mathbb{R}^m$ the action on functions is minus the common definition of Laplacian. In this work, to maintain the common notation in analysis, we chose this sign convention.

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