COHFTS AND THE TOPOLOGICAL RECURSION

1. Recap

Let $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$ be a global compact spectral curve such that

- x has only simple ramification points $\alpha_1, \dots, \alpha_r$
- $\omega_{0,1} = y\,dx$ is meromorphic on Σ, and holomorphic around the ramification points

Around each ramification point α_i, we have local coordinates ζ_i of the form

$$x - x(\alpha_i) = -\frac{\zeta_i^2}{2}. \tag{1}$$

Define a CohFT as follows.

- Constants:
$$C^i = \frac{dy(z)}{d\zeta_i(z)}\bigg|_{z=\alpha_i} \qquad i = 1, \dots, r. \tag{2}$$
- Meromorphic functions ξ^i and meromorphic differentials $d\xi^{k,i}$
$$\xi^i(z) = \int^z \frac{\omega_{0,2}(z_0, \cdot)}{d\zeta^i(z_0)}\bigg|_{z_0=\alpha_i}, \qquad d\xi^{k,i}(z) = d\left(\frac{d^k}{dx^k(z)}\xi^i(z)\right) \qquad k \in \mathbb{N},\ i = 1, \dots, r. \tag{3}$$
- TopFT on $V = \mathbb{C}\langle e_1, \dots, e_r\rangle$ with pairing $\eta(e_i, e_j) = \delta_{i,j}$ and amplitudes $w_{g,n} \colon V^{\otimes n} \to \mathbb{C}$
$$w_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \delta_{i_1,\dots,i_n} (C^i)^{-(2g-2+n)} \tag{4}$$
- Translation: $T(u) \in u^2 \cdot V[[u]]$, expressed on the basis as $T(u) = \sum_{i=1}^r T^i(u) e_i$:
$$T^i(u) = uC^i + \frac{1}{\sqrt{2\pi u}} \int_{\gamma_i} e^{\frac{x - x(\alpha_i)}{u}} \omega_{0,1} = uC^i - \sqrt{\frac{u}{2\pi}} \int_{\gamma_i} e^{\frac{x - x(\alpha_i)}{u}} dy \tag{5}$$
- Rotation: $R^{-1}(u) \in \mathrm{Id} + u \cdot \mathrm{End}(V)[[u]]$, expressed on the basis as $R^{-1}(u) e_i = \sum_{j=1}^r (R^{-1})_i^j(u) e_j$:
$$(R^{-1})_i^j(u) = -\sqrt{\frac{u}{2\pi}} \int_{\gamma_j} e^{\frac{x - x(\alpha_j)}{u}} d\xi^i = \frac{1}{\sqrt{2\pi u}} \int_{\gamma_j} e^{\frac{x - x(\alpha_j)}{u}} \xi^i dx \tag{6}$$
- CohFT: $\Omega_{g,n} \colon V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$
$$\Omega_{g,n} = (RTw)_{g,n} = \sum_{\substack{\Gamma \\ \text{stbl grph}}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\prod_v \mathrm{Cont}_v\right)\left(\prod_e \mathrm{Cont}_e\right)\left(\prod_\lambda \mathrm{Cont}_\lambda\right) \tag{7}$$

Theorem. Given $\mathcal{S}$, the descendant theory of Ω is computed by TR:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{i_1,\dots,i_n=1}^r \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{j=1}^n \sum_{k_j \geqslant 0} \psi_j^{k_j} d\xi^{k_j, i_j}(z_j) \tag{8}$$

CohFTs	TR
semisemplicity	only simple ram pnts
$\dim(V)$	number of simple ram pnts
TopFT	$\frac{dy}{d\zeta}$
Translation	$\omega_{0,1}$
Rotation	$d\xi$
Edge contribution	$\omega_{0,2}$

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2. EXAMPLES

2.1. **Mirzakhani curve.**

$$\mathcal{S}^{M} = \left(\mathbb{P}^1, x(z) = -\frac{z^2}{2},\ y(z) = \frac{\sin(2\pi z)}{2\pi},\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}\right) \tag{9}$$

Useful formula:

- Gaussian integral:

$$\int_{\mathbb{R}} e^{-a\frac{z^2}{2}}dz = \sqrt{\frac{\pi}{a}} \tag{10}$$

- Translation as κ-exp:

$$\sum_{m\geqslant 0}\frac{1}{m!}p_{m,*}\prod_{j=1}^{m}\sum_{k_j\geqslant 1}a_{k_j}\psi_{n+j}^{k_j+1} = \exp\left(\sum_{m\geqslant 1}b_m\kappa_m\right), \tag{11}$$

where the sequences $(a_k)_{k\geqslant 1}$ and $(b_m)_{m\geqslant 1}$ are related by the

$$1-\sum_{k\geqslant 1}a_k u^k = \exp\left(-\sum_{m\geqslant 1}b_m u^m\right) \tag{12}$$

2.2. **Lambert curve.**

$$\mathcal{S}^{L} = \left(\mathbb{P}^1, x(z) = \log(z)-z,\ y(z) = z,\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}\right) \tag{13}$$

Useful formulae:

- Hankel representation and Stirling approximation:

$$\frac{1}{\Gamma(t)} = \frac{i}{2\pi}\int_{C_H}(-w)^t e^{-w}dw \sim \frac{(-t)^{t+\frac{1}{2}}e^t}{\sqrt{2\pi}}\exp\left(\sum_{m\geqslant 1}\frac{B_{m+1}}{m(m+1)}t^{-m}\right) \tag{14}$$

where C_H goes from $+\infty$ along the positive real axis, around the origin counter clockwise and back to $+\infty$ along the positive real axis.

- Mumford's formula: setting $\Lambda(-1) = \sum_{k=0}^{g}(-1)^k\lambda_k$,

$$\Lambda(-1) = \exp\left(-\sum_{m\geqslant 1}\frac{B_{m+1}}{m(m+1)}\left(\kappa_m - \sum_{i=1}^{n}\psi_i^m + \delta_m\right)\right) \tag{15}$$

where $\delta_m = \frac{1}{2}j_*\sum_{k+\ell=m-1}(\psi')^k(\psi'')^\ell$ and $j\colon \partial\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ is the boundary map. We can rewrite it through the Givental action with

$$w_{g,n} = 1, \qquad T(u) = u\big(1-R^{-1}(u)\big) \qquad R^{-1}(u) = \exp\left(\sum_{m\geqslant 1}\frac{B_{m+1}}{m(m+1)}u^m\right) \tag{16}$$

2.3. **3-spin curve.**

$$\mathcal{S}^{\text{3-spin}} = \left(\mathbb{P}^1, x(z) = z^3-3\epsilon z,\ y(z) = z,\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}\right) \tag{17}$$

Useful formulae:

- Airy function and its asymptotic expansion:

$$\mathrm{Ai}(t) = \frac{1}{2\pi i}\int_{C_A}e^{\frac{w^3}{3}-tw}dw \sim \frac{e^{-\frac{2t^{3/2}}{3}}}{2\sqrt{\pi}}t^{-1/4}\sum_{k\geqslant 0}\frac{(6k)!}{(2k)!(3k)!}\left(-\frac{1}{576t^{3/2}}\right)^k \tag{18}$$

where C_A is the path starting at $e^{-\frac{\pi}{3}}\infty$ and ending at $e^{\frac{\pi}{3}}\infty$.

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