Cardy-Frobenius algebras & 2d open-closed TQFTs

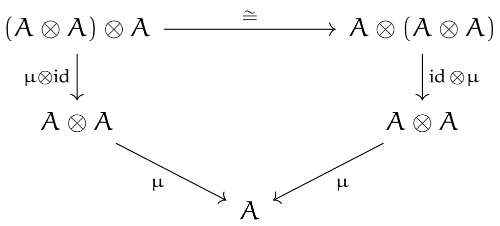
Alessandro Giacchetto

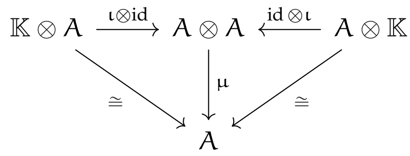
1. Cardy-Frobenius algebras

We will start introducing the main algebraic object of these notes, namely the notion of Cardy-Frobenius algebra, following [9]. In the following, we will denote by $\mathbb{K}$ a generic field.

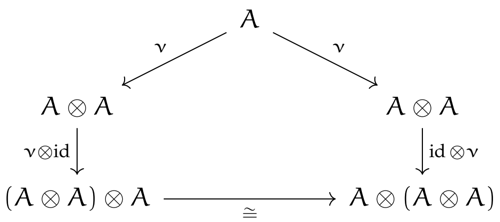
DEFINITION. A *Frobenius algebra* over $\mathbb{K}$ is a finite-dimensional $\mathbb{K}$-vector space A, equipped with the following structure.

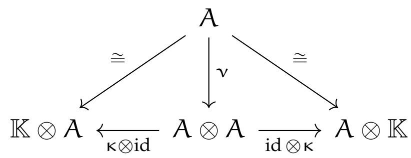
- An algebra structure, *i.e.* a product $\mu\colon A \otimes A \to A$ with a unit $\iota\colon \mathbb{K} \to A$, satisfying the associativity and the unity conditions.



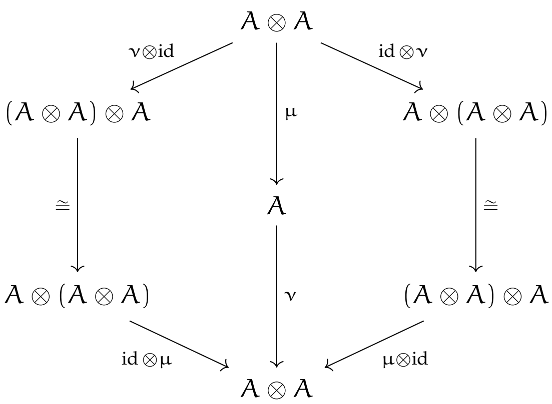


- A coalgebra structure, *i.e.* a coproduct $\nu\colon A \to A \otimes A$ with a counit $\kappa\colon A \to \mathbb{K}$, satisfying the coassociativity and the counity conditions.





- The following compatibility condition, called the *Frobenius relation*, holds.



We will denote it by $\mathcal{A} = (A, \mu, \iota, \nu, \kappa)$. A Frobenius algebra $\mathcal{A}$ is called

- symmetric if for all $a, b \in A$,

$$\kappa\big(\mu(a \otimes b)\big) = \kappa\big(\mu(b \otimes a)\big).$$

- commutative if for all $a, b \in A$,
$$\mu(a \otimes b) = \mu(b \otimes a).$$

A morphism between Frobenius algebras over $\mathbb{K}$, say $\mathcal{A}$ and $\mathcal{A}'$, is a linear map $f\colon A \to A'$ such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A' \otimes A' & \xrightarrow[\mu']{} & A' \end{array} \qquad \begin{array}{ccc} & & A \\ & \overset{\iota}{\nearrow} & \\ \mathbb{K} & & \downarrow{\scriptstyle f} \\ & \underset{\iota'}{\searrow} & \\ & & A' \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\nu} & A \otimes A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ A' & \xrightarrow[\nu']{} & A' \otimes A' \end{array} \qquad \begin{array}{ccc} A & & \\ & \overset{\kappa}{\searrow} & \\ {\scriptstyle f}\downarrow & & \mathbb{K} \\ & \underset{\kappa'}{\nearrow} & \\ A' & & \end{array}$$

DEFINITION. A *Cardy-Frobenius algebra* over $\mathbb{K}$ is the data $(\mathcal{A}, \mathcal{C}, \zeta, \zeta^*)$ of

- a symmetric Frobenius algebra $\mathcal{A} = (A, \mu_A, \iota_A, \nu_A, \kappa_A)$ over $\mathbb{K}$,
- a commutative Frobenius algebra $\mathcal{C} = (C, \mu_C, \iota_C, \nu_C, \kappa_C)$ over $\mathbb{K}$,
- two algebra morphisms $\zeta\colon C \to A$ and $\zeta^*\colon A \to C$, called *zipper* and *cozipper map* respectively, satisfying the following axioms.
  - Knowledge: for all $a \in A$ and $c \in C$
  $$\mu_A\big(\zeta(c) \otimes a\big) = \mu_A\big(a \otimes \zeta(c)\big)$$
  - Duality: for all $a \in A$ and $c \in C$
  $$\kappa_C\big(\mu_C\big(c \otimes \zeta^*(a)\big)\big) = \kappa_A\big(\mu_A\big(\zeta(c) \otimes a\big)\big)$$
  In other words, the maps ζ and ζ^* are dual with respect to the pairngs $\kappa_A \circ \mu_A$ and $\kappa_C \circ \mu_C$.
  - Cardy condition: for all $a \in A$,
  $$\mu_A\big(\tau\big(\nu_A(a)\big)\big) = \zeta\big(\zeta^*(a)\big),$$
  where $\tau\colon A \otimes A \to A \otimes A$ is the isomorphism $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$.

A morphism of Cardy-Frobenius algebras $(\mathcal{A}, \mathcal{C}, \zeta, \zeta^*)$ and $(\mathcal{A}', \mathcal{C}', \zeta', \zeta'^*)$ is a pair (f, g) of morphisms of Frobenius algebras $f\colon \mathcal{A} \to \mathcal{A}'$, $g\colon \mathcal{C} \to \mathcal{C}'$, such that the following diagrams commute.

$$\begin{array}{ccc} C & \xrightarrow{g} & C' \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \zeta'} \\ A & \xrightarrow[f]{} & A' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle \zeta^*}\downarrow & & \downarrow{\scriptstyle \zeta'^*} \\ C & \xrightarrow[g]{} & C' \end{array}$$

EXAMPLE. Consider $A = \mathrm{Mat}(n, \mathbb{K})$, the algebra of $n \times n$ matrices over $\mathbb{K}$, and $C = \mathbb{K}$, the base field. For $\alpha, \gamma \in \mathbb{K}^\times$, set

$$\kappa_A(a) = \alpha\,\mathrm{tr}(a), \qquad \zeta(x) = \mathrm{diag}(x, \dots, x),$$
$$\kappa_C(x) = \gamma x, \qquad \zeta^*(a) = \frac{\alpha}{\gamma}\,\mathrm{tr}(a).$$

The comultiplications and units for A a C are determined by the non-degeneracy of the pairings $\kappa \circ \mu$. Then the knowledge and duality condition hold, while the Cardy condition is equivalent to $\gamma = \alpha^2$.

DEFINITION. We denote by $\mathbf{CF\text{-}alg}_{\mathbb{K}}$ the symmetric monoidal category of Cardy-Frobenius algebras over $\mathbb{K}$.

2. Open-closed 2d cobordism and TQFTs

Let us introduce another interesting category which generalizes the 2d cobordism category. This will allow us to establish an equivalence of categories, relating the geometric construction of 2d open-closed TQFTs to the algebraic side of Cardy-Frobenius algebras.

Definition. An m-*atlas with corners* on a set M is a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$, labelled by an at most countable set of indices I, such that the following conditions hold.

- The sets U_α cover M.
- For any $\alpha \in I$, φ_α is a one-to-one map from U_α to an open domain in the space $\mathbb{R}^m_+ = [0, +\infty)^m$:
$$\varphi_\alpha \colon U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m_+.$$
- For any pair of intersecting sets $U_\alpha \cap U_\beta \neq \varnothing$, the domains $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^m_+$ and the one-to-one map
$$\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$
is the restriction of a diffeomorphism between open subsets of $\mathbb{R}^m$. These maps are called *transition functions*.

A pair (U, φ) is called a *chart*. A subset $U \subset M$ is defined to be open if its intersections with charts
$$\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{R}^m_+$$
are open for all $\alpha \in I$. This defines a topological structure on M.

Definition. A set M equipped with an m-atlas with corners is called a *manifold with corners* of dimension m if it is a Hausdorff, second countable topological space.

Definition. Let M and N be manifolds with corners of dimensions m and n respectively. A map $f\colon M \to N$ is said to be *smooth* if for each pair of charts (U, φ) and (V, ψ) on M and N respectively such that $f(U) \subset V$, the map
$$\psi \circ f \circ \varphi^{-1} \colon \varphi(U) \to \psi(V)$$
is the restriction of a smooth map between open subsets of $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$.

As pointed out in [8], the notion of manifold with corners is too general. The problem is that the boundary of a manifold with corners is not always a manifold with corners. Pretending to construct a cobordism theory, this should be solve somehow. To avoid the issue, as in the 2-disk with one corner, we have to introduce the concept of manifold with faces.

Definition. For an m-manifold with corners, fix a point $p \in M$ and a chart (U, φ) containing it. Define the value $c(p)$ as the number of zero coordinates in $\varphi(p) \in \mathbb{R}^m_+$. Then a connected face of M is the closure of a component of $\{p \in M \mid c(p) = 1\}$. A face is a free union of pairwise disjoint connected faces.
An m-*manifold with faces* M is a m-manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces.

Note that with this definition, every face of a manifold with faces is a manifold with faces as well.

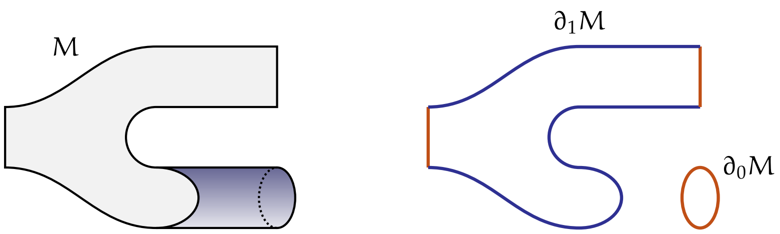
Definition. A *rooted* m-*manifold* $(M, \partial_0 M, \partial_1 M)$ is an m-manifold with faces M with a specified pair $(\partial_0 M, \partial_1 M)$ of faces of M such that the following two conditions hold.

- $\partial_0 M \cup \partial_1 M = \partial M$, where ∂M is the boundary of M as a topological manifold.
- $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$.

The face $\partial_0 M$ will be called the *root*.

Definition. A diffeomorphism $f\colon M \to N$ between two rooted manifolds is a diffeomorphism of the underlying manifolds with corners such that $f(\partial_i M) = \partial_i N$ for $i = 0, 1$.

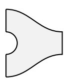
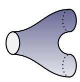
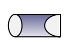
The diagram below shows a typical rooted 2-manifolds M and its faces decomposition (the root $\partial_0 M$ in red, $\partial_1 M$ in blue).

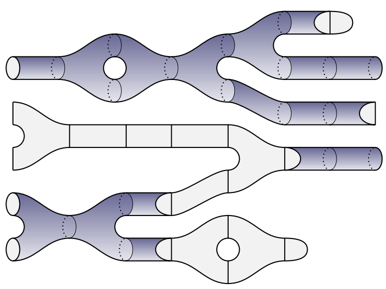


DEFINITION. The category of 2d *open-closed cobordisms* **OC-Cob**$_2$ is defined as follows.

- *Objects*. Its objects are finite disjoint union of labelled, compact, oriented, connected 1-manifolds with boundary. Specifically, denote by
$$I = [-1, 1] \times \{0\} \subset \mathbb{R}^2, \qquad \mathbb{S}^1 = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\},$$
where both manifolds have the standard orientation. For $k \in \mathbb{N}_0$, set
$$I_k = \big(3(k-1), 0\big) + I, \qquad \mathbb{S}^1_k = \big(3(k-1), 0\big) + \mathbb{S}^1.$$
Then the objects of **OC-Cob**$_2$ are elements $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, which corresponds to
$$\boldsymbol{\epsilon} = \left(\bigsqcup_{k \text{ s.t. } \epsilon_k = 0} I_k \right) \sqcup \left(\bigsqcup_{k \text{ s.t. } \epsilon_k = 1} \mathbb{S}^1_k \right).$$
Note that $(0) = I$ and $(1) = \mathbb{S}^1$.
- *Morphisms*. If $\boldsymbol{\epsilon}$, $\boldsymbol{\delta}$ are two objects, the morphisms $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$ are equivalence classes of compact, oriented, rooted 2-manifolds $(\Sigma, \partial_0\Sigma, \partial_1\Sigma)$, together with an orientation preserving diffeomorphism
$$\phi \colon \partial_0\Sigma \to \boldsymbol{\epsilon}^* \sqcup \boldsymbol{\delta}.$$
Here $\boldsymbol{\epsilon}^*$ indicates reversal of orientation. In the following, we will denote by $\partial_0^-\Sigma$ and $\partial_0^+\Sigma$ the boundary components corresponding to $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ respectively and we will call them *source* and *target*. We say that $(\Sigma, \partial_0\Sigma, \partial_1\Sigma, \phi)$ and $(\Sigma', \partial_0\Sigma', \partial_1\Sigma', \phi')$ are equivalent if there exists an orientation-preserving diffeomorphism $f \colon \Sigma \to \Sigma'$ of rooted 2-manifolds, such that the following diagram is commutative.
$$\begin{array}{ccc} \partial_0\Sigma & \xrightarrow{\;f|_{\partial_0\Sigma}\;} & \partial_0\Sigma' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & \boldsymbol{\epsilon}^* \sqcup \boldsymbol{\delta} & \end{array}$$
To each boundary component is given the labelling induced by its diffeomorphic image. Morphisms in this category will be called *open-closed cobordisms*.
- *Composition*. Composition of morphisms consists of gluing correspondingly labelled boundaries in an orientation-preserving manner. Specifically, if $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, $\boldsymbol{\epsilon}_3$ are three objects and $[\Sigma] \colon \boldsymbol{\epsilon}_1 \to \boldsymbol{\epsilon}_2$, $[\Sigma'] \colon \boldsymbol{\epsilon}_2 \to \boldsymbol{\epsilon}_3$ are morphisms, we have the diffeomorphisms
$$\phi \colon \partial_0^+\Sigma \to \boldsymbol{\epsilon}_2, \qquad \phi' \colon \partial_0^-\Sigma' \to \boldsymbol{\epsilon}_2^*.$$
Form a rooted 2-manifold $\Sigma_\phi \cup_{\phi'} \Sigma'$ by gluing together $\partial_0^+\Sigma$ and $\partial_0^-\Sigma'$ such that the parametrizations match. The resulting equivalence class of $\Sigma_\phi \cup_{\phi'} \Sigma'$ on the choice of representatives Σ and Σ'. This allows the definition of the composition of morphisms $[\Sigma'] \circ [\Sigma] = [\Sigma_\phi \sqcup_{\phi'} \Sigma']$.
- *Symmetric monoidal structure*. The disjoint union induces a symmetric monoidal structure on **OC-Cob**$_2$, for which the empty disjoint union is the unit object.

In the following, the source boundaries are drawn on the left, and the target ones on the right. The labelling goes from up to down. Further, representatives of morphisms in **OC-Cob**$_2$ will be simply called surfaces. Examples of morphisms in **OC-Cob**$_2$ are depicted below.

 $: (0, 0) \to (0)$,  $: (1) \to (1, 1)$,  $: (1) \to (0)$

 $: (1, 0, 0, 1, 1) \to (1, 0, 1)$

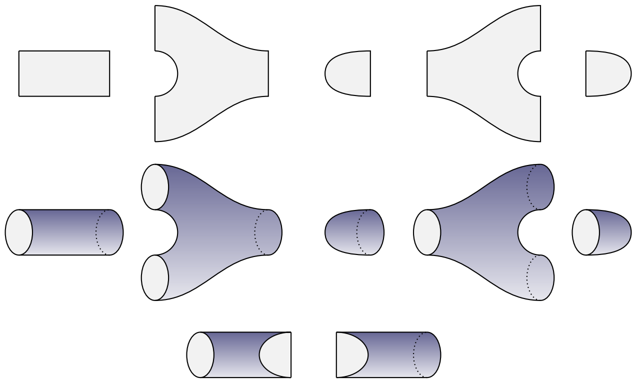


FIGURE 1. Elementary generators for representative of morphisms in **OC-Cob$_2$**.

Let us describe now the structure of **OC-Cob$_2$**. The idea is to show that every 2d open-closed cobordism is generated by a set of generators, corresponding to the identities, (co)multiplications, (co)units and the (co)zipper, and relations, corresponding to the Cardy-Frobenius axioms. Specifically,

THEOREM 1 ([9]). The objects in **OC-Cob$_2$** are generated by the interval and circle. The morphisms are generated by gluing copies of the twelve *elementary surfaces* in Fig. 1, subject to the sets of seventeen *elementary relations* of Fig. 2.

The object decomposition is clear. To prove the morphism decomposition, we proceed in two steps. Firstly, we prove that Fig. 1 provides the set of generators via Morse theory for surfaces with corners. Secondly, for the sufficiency of relations, the idea is to show that every representative Σ of a morphism is equivalent to a normal form NF(Σ). The normal form is defined from the number of negative/positive boundary components, the genus, the window number and the boundary permutation. Comparing to the closed case, these last two invariants have to be added to take into account the open sectors. Showing that a diffeomorphism from Σ to its normal form is obtained by applying a finite sequence of the above mentioned relations would prove the theorem.

2.1. Generator decomposition via Morse theory. The following generalization of Morse theory to manifold with corners was firstly introduced by D. Braess in [4].

DEFINITION. Consider an m-manifold with corners M. For every point $p \in M$, define the tangent space at p as

$$T_pM = \{\, v\colon C^\infty(M) \to \mathbb{R} \mid v \text{ is linear and } v(fg) = v(f)g + fv(g)\ \forall f, g \in C^\infty(M)\,\}.$$

We can define the *inwards pointing tangential cone* $C_pM \subset T_pM$ as the set of all tangent vectors $v \in T_pM$ for which there exists a smooth path $\gamma\colon [0,\epsilon] \to M$ for some $\epsilon > 0$ such that $\gamma(0) = p$ and the one-sided derivative is:

$$\lim_{t\to 0^+} \frac{\gamma(t)-\gamma(0)}{t} = v.$$

DEFINITION. Consider an m-manifold with corners M and a smooth function $f\colon M \to \mathbb{R}$. A point $p \in M$ is called *critical* if the restriction of d_pf to the inward pointing cone is not surjective, *i.e.*

$$d_pf(C_pM) \neq \mathbb{R}.$$

The value $f(p)$ is called a *critical value*. As d_pf is linear, cones are sent to cones. In particular, a critical point in the boundary is called *postive* if $d_pf(C_pM) = [0,+\infty)$, *negative* if $d_pf(C_pM) = (-\infty, 0]$ and *null* if $d_pf(C_pM) = \{0\}$.
A critical point $p \in M$ is called *non-degenerate* if the Hessian $\mathrm{Hess}_p(f)$ restricted to the kernel of d_pf has full rank:

$$\det \mathrm{Hess}_p(f)\big|_{\ker d_pf} \neq 0.$$

A function f such that all critical points are non-degenerate is called a *Morse function*.

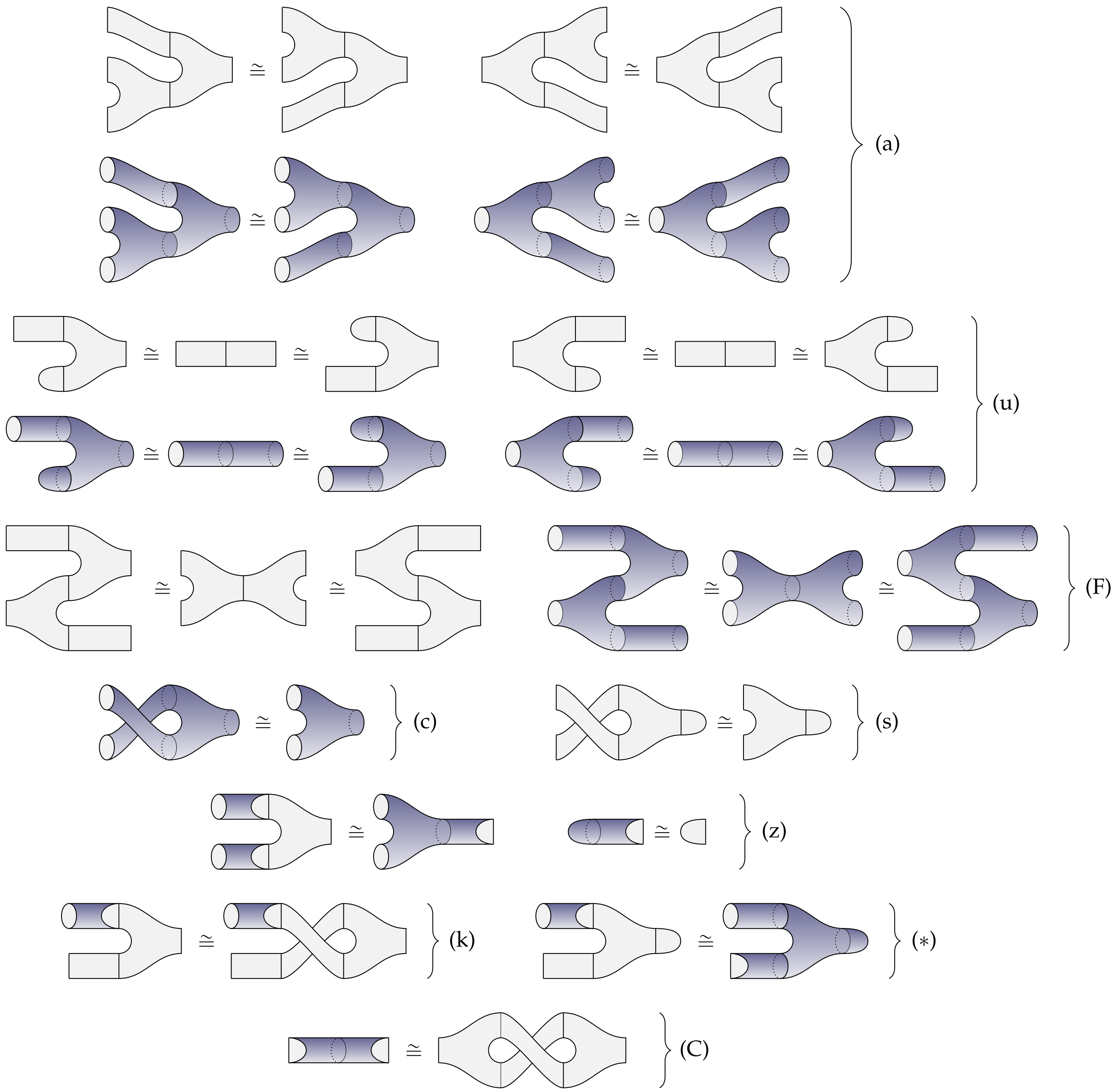


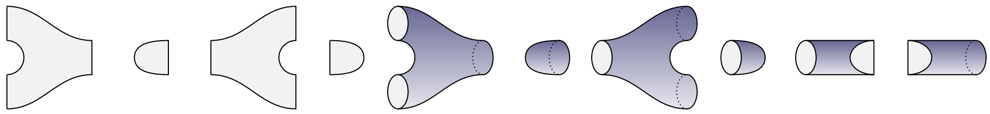
FIGURE 2. Elementary relations for representative of morphisms in **OC-Cob**$_2$.

DEFINITION. Let Σ be a representative of an open-closed cobordism $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$. A *special Morse function* for Σ is a Morse function $f\colon \Sigma \to \mathbb{R}$ satisfying the following conditions.

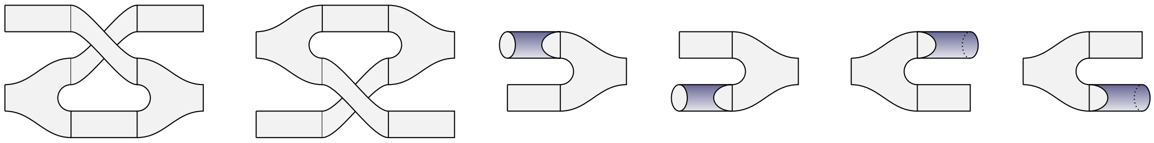
- It is normalized such that $f(\Sigma) \subset [0,1]$.
- It maps the source to 0 and the target to 1: $f(p) = 0$ if and only if $p \in \partial_0^- \Sigma$ and $f(p) = 1$ if and only if $p \in \partial_0^+ \Sigma$.
- Neither $\partial_0^- \Sigma$ nor $\partial_0^+ \Sigma$ contain critical points.
- The critical values of f are distinct.

By usual density techniques, it can be shown that every rooted surface admits a special Morse function.

THEOREM 2. Let Σ be a representative of a connected open-closed cobordism $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$, $f\colon \Sigma \to \mathbb{R}$ a special Morse function with precisely one critical point. Then M is equivalent to one of the following cobordisms



or the compositions



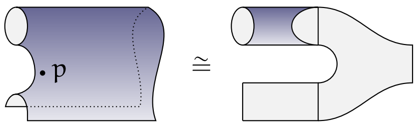
SKETCH OF THE PROOF. We have two cases: either the critical point $p \in \Sigma \setminus \partial\Sigma$ or $p \in \partial\Sigma$. In the first case, we can apply the ordinary Morse lemma: there exists a chart $\phi = (x_1, x_2)\colon U \to \mathbb{R}^2$ such that

$$f(p) = -\sum_{i=1}^{k} x_i^2(p) + \sum_{i=k+1}^{2} x_i^2(p).$$

Here k is the Morse index of f at p. Then we have three cases.

(1) $k = 2$. Then Σ is diffeomorphic to the closed counit surface.

(2) $k = 1$. If the cobordism is close-to-close, then Σ is diffeomorphic to the multiplication or comultiplication. If the boundary has open components, a case-by-case analysis shows that the only other possibilities are those of the second diagram of the statement.
One of the possible cases with the diffeomorphism to put it one of the forms of the statement is shown below.



The surface on the left is drawn in such a way that f is the height function with respect to the horizontal axis of the drawing plane.

(3) $k = 0$. Then Σ is diffeomorphic to the closed unit surface.

Let us consider now the case $p \in \partial\Sigma$. Then $p \in \partial_1\Sigma$ and it is not a corner, that is p is an internal critical point for the Morse function $f|_{\partial_1\Sigma}$. The index can be $k' = 0, 1$. For $k' = 1$ we have three possibilities.

(i) If p is a positive critical point, then Σ is diffeomorphic to the open counit.

(ii) If it is negative, then it is either the cozipper or the open multiplication.

(iii) If it is a null critical point, then non-degeneracy implies that $\mathrm{Hess}_p(f)$ is non-degenerate. Let k'' be number of negative eigenvalues of the Hessian. As $k' = 1$, it cannot be $k'' = 0$. So we have two cases: either $k'' = 2$, and we are in the same situation as in (i), or $k'' = 1$, and we are in (ii).

The case $k' = 0$ is completely symmetric to the former one, leading to the open unit, the zipper and the open multiplication. □

COROLLARY. Let $[\Sigma]\colon \boldsymbol{\epsilon} \to \boldsymbol{\delta}$ be any morphism. Then $[\Sigma] = [\Sigma_k] \circ \cdots \circ [\Sigma_1]$, where $[\Sigma_i]$ are elementary morphisms in Fig. 1.

2.2. Sufficiency of relations via normal form.

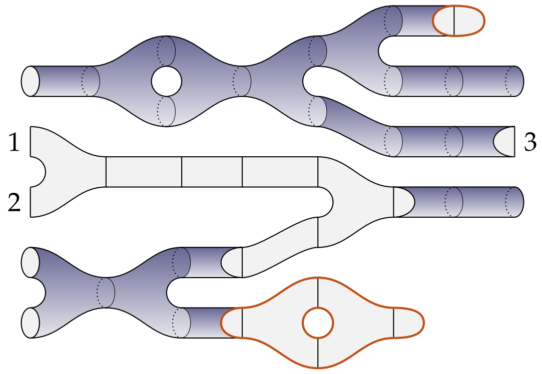
DEFINITION. Consider a morphism $[\Sigma]\colon \boldsymbol{\epsilon} \to \boldsymbol{\delta}$. We have the following invariants.

- *Genus*. Define $g_{[\Sigma]} \in \mathbb{N}_0$ as the genus of the underlying surface.
- *Window number*. Define $w_{[\Sigma]} \in \mathbb{N}_0$ as the number of boundary components of the face $\partial_1\Sigma$ which are diffeomorphic to a circle.
- *Boundary permutation*. Let k be the number of boundary components of the root $\partial_0\Sigma$ which are diffeomorphic to an interval, labelled from the source to the target in the natural way. Define $\beta_{[\Sigma]} \in S_k$ as follows. Consider a connected component X of $\partial\Sigma$ that contain a corner. The orientation of Σ induces an orientation on X. Then it is defined a cycle $(i_1 \cdots i_l)$, where the

$i_j \in \{1, \dots, k\}$ are the number of components of $\partial_0\Sigma$ contained in X. Then $\beta_{[\Sigma]}$ is the product of these cycles for all such components.

Note that all invariants are well-defines, *i.e.* they do not depend on the choice of a representative.

Consider for instance the morphism below.

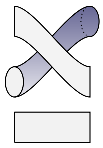


Then $g = 1$, $w = 3$ and $\beta = (1\ 2) \in S_3$. The "windows" are depicted in red.

This definition allow us to define the normal form associated an open-to-closed cobordism, *i.e.* a cobordism whose source is a disjoint union of intervals and the target is a disjoint union of circles. Before doing that, let us point out that, fixing an object $\boldsymbol{\epsilon}$ and a permutation $\sigma \in S_{|\boldsymbol{\epsilon}|}$, we can associate to them a morphism $[\sigma]\colon \boldsymbol{\epsilon} \to \sigma(\boldsymbol{\epsilon})$ in the natural way. An example is shown below.

$$\boldsymbol{\epsilon} = (0, 1, 0)$$

$$\sigma = (1\ 2) \in S_3$$

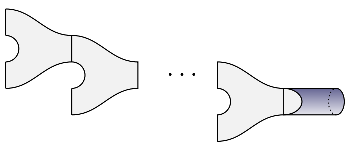


DEFINITION. Consider a representative $\bar{\Sigma}$ of a connected open-to-closed morphism $\mathbf{0} \to \mathbf{1}$ with genus g, window number w and boundary permutation β. Write the boundary permutation as a product of disjoint cycles $\beta = \beta_1 \cdots \beta_r$, with β_i of length ℓ_i. The normal form of $\bar{\Sigma}$ is defined as the composition

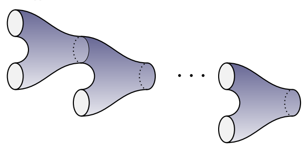
$$\mathrm{nf}(\bar{\Sigma}) = E_{|\mathbf{1}|} \circ D_g \circ C_w \circ B_r \circ \left(\bigsqcup_{i=1}^{r} A_{\ell_i} \right) \circ \sigma$$

of the following surfaces.

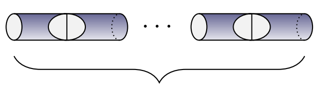
- A_ℓ consists of $\ell - 1$ open multiplications and a cozipper.

  $A_\ell =$ 

  Cycles of length one are represented by a single cozipper. In the case $r = 0$, the free union has to be replaced by the empty set.
- If $r \geqslant 1$, B_r consists of $r - 1$ closed multiplications.

  $B_r =$ 

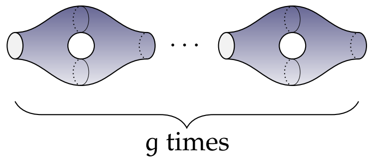
  If $r = 0$, set $B_0 =$ .
- Define C_w as

  $C_w =$ 

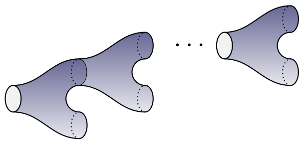
  w times

  if $w > 0$, the empty set otherwise.

- Similarly,

$D_g =$ 

if $g > 0$, the empty set otherwise.
- The surface E_n is given by $n-1$ closed comultiplications

$E_n =$ 

if $n > 0$ and the closed unit $E_0 =$  otherwise.
- Finally, the surface σ represents the following permutation. Let γ be the boundary permutation of the composition

$$E_{|\mathbf{1}|} \circ D_g \circ C_w \circ B_r \circ \Big(\bigsqcup_{i=1}^{r} A_{\ell_i} \Big).$$

Note that β and γ have the same cycle structure, since it is determined by the partition of $|\mathbf{0}| = \sum_{i=1}^{r} \ell_i$. Then there exists a permutation σ such that

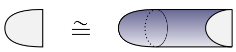
$$\beta = \sigma^{-1} \gamma \sigma.$$

The surface σ associated to $\mathbf{0}$ and the permutation σ is the missing term in the composition defining $\mathrm{nf}(\bar{\Sigma})$.

The main result of the section is to prove that the normal form of a surface is equivalent to the surface itself, providing an inductive proof which constructs a finite sequence of diffeomorphisms that puts an arbitrary open-closed cobordism into the normal form using only the elementary relations. Hence, we provide a constructive proof that the relations are sufficient to completely describe the category **OC-Cob**$_2$.

THEOREM 3. Let $\bar{\Sigma}$ be a representative of a connected morphism $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$. Then $\bar{\Sigma}$ is equivalent to its normal form $\mathrm{nf}(\bar{\Sigma})$ and the diffeomorphism is obtained by applying a finite number of times the elementary relations of Fig. 2.

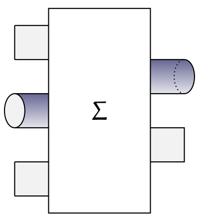
IDEA OF THE PROOF. The proof proceeds case-by-case, considering the possible configurations of the elementary generators and eliminating the configurations which do not compare in the normal form by applying the elementary relations. As an example, the open unit does not appear in the normal form. However, thanks to (z.2), we find



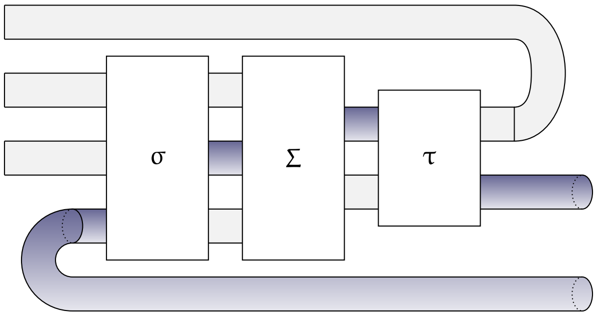
A complete proof can be found in [9]. □

Let us show now how to define the normal form for a general morphism. The idea is to move from a general cobordism to an open-to-closed one via permutations. Before proceeding with the statements, let us see an example.

Consider $[\Sigma]\colon (0,1,0) \to (1,0)$, which can be depicted schematically as below.



We can permute now the source and target boundaries, in such a way that all the components diffeomorphic to an interval are moved to the top. Then we can compose with pairings (composition of multiplication and counit) and copairings (composition of comultiplication and unit), in order to obtain an open-to-closed morphism.



LEMMA. Consider a representative Σ of a connected morphism $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$. Let $\mathbf{0}_{\boldsymbol{\epsilon}} = (0, \dots, 0)$, $\mathbf{1}_{\boldsymbol{\epsilon}} = (1, \dots, 1)$ such that $\mathbf{0}_{\boldsymbol{\epsilon}} \sqcup \mathbf{1}_{\boldsymbol{\epsilon}}$ is a permutation of $\boldsymbol{\epsilon}$:

$$\boldsymbol{\epsilon} = \sigma\big(\mathbf{0}_{\boldsymbol{\epsilon}} \sqcup \mathbf{1}_{\boldsymbol{\epsilon}}\big).$$

Similarly for $\boldsymbol{\delta}$: consider $\mathbf{0}_{\boldsymbol{\delta}} = (0, \dots, 0)$, $\mathbf{1}_{\boldsymbol{\delta}} = (1, \dots, 1)$ such that $\mathbf{0}_{\boldsymbol{\delta}} \sqcup \mathbf{1}_{\boldsymbol{\delta}}$ is a permutation of δ:

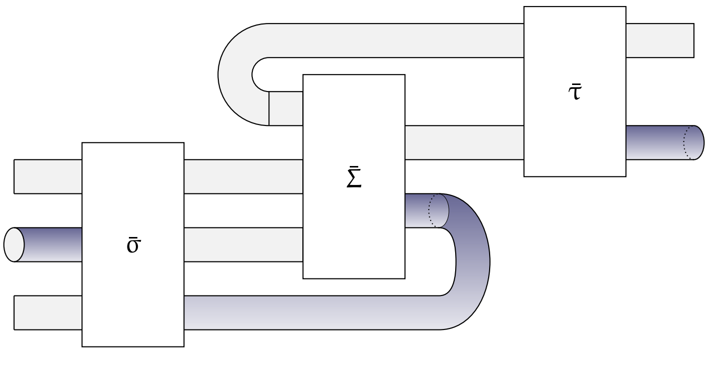
$$\boldsymbol{\delta} = \tau^{-1}\big(\mathbf{0}_{\boldsymbol{\delta}} \sqcup \mathbf{1}_{\boldsymbol{\delta}}.\big).$$

Then we define $\Lambda(\Sigma)$ to be the open-to-closed surface obtained from Σ by precomposing with σ, postcomposing with τ, gluing closed copairings on each circle in $\mathbf{1}_{\boldsymbol{\epsilon}}$, and gluing open pairings on each interval in $\mathbf{0}_{\boldsymbol{\delta}}$. The class of the resulting open-to-closed surface do not depend on the class of Σ.

We can also construct a sort of inverse map, but auxiliary structure to the open-to-closed morphism has to be specified, that is a decomposition of source and target boundaries and permutations of the first and second parts of these decomposition. For example, for $[\bar{\Sigma}]\colon (0,0,0) \to (1,1)$ with

$$(0,0,0) = (0) \sqcup (0,0) \qquad \bar{\sigma} = (2\ 3) \in S_3$$
$$(1,1) = (1) \sqcup (1) \qquad \bar{\tau} = (1\ 2) \in S_2$$

then we can construct a morphism in the following way.



LEMMA. Consider a representative $\bar{\Sigma}$ of a connected open-to-closed morphism $\mathbf{0} \to \mathbf{1}$ with the data of

- two decompositions of its source and target into free unions: $\mathbf{0} = \boldsymbol{\delta}_0 \sqcup \boldsymbol{\epsilon}_0$ and $\mathbf{1} = \boldsymbol{\delta}_1 \sqcup \boldsymbol{\epsilon}_1$,
- two elements of the symmetric groups $\bar{\sigma} \in S_{|\boldsymbol{\epsilon}_0|+|\boldsymbol{\epsilon}_1|}$ and $\bar{\tau} \in S_{|\boldsymbol{\delta}_0|+|\boldsymbol{\delta}_1|}$.

We define $\bar{\Lambda}(\bar{\Sigma})$ to be the surface from $\bar{\sigma}^{-1}(\boldsymbol{\epsilon}_0 \sqcup \boldsymbol{\epsilon}_1)$ to $\bar{\tau}(\boldsymbol{\delta}_0 \sqcup \boldsymbol{\delta}_1)$ given by gluing open copairings to the intervals in $\boldsymbol{\delta}_0$ and closed pairings to the circles in $\boldsymbol{\delta}_1$. The result of this gluing is then precomposed with $\bar{\sigma}$ and postcomposed with $\bar{\tau}$. The class of the resulting open-closed morphism do not depend on the class of $\bar{\Sigma}$.

Note that the element $\Lambda(\Sigma)$ comes with the above mentioned data. In particular, using the notation of the previous lemmata:

- the decomposition of its source and target into a free unions: $\mathbf{0} = \mathbf{0}_{\boldsymbol{\delta}} \sqcup \mathbf{0}_{\boldsymbol{\epsilon}}$ and $\mathbf{1} = \mathbf{1}_{\boldsymbol{\delta}} \sqcup \mathbf{1}_{\boldsymbol{\epsilon}}$,
- the two permutations $\bar{\sigma} = \sigma^{-1}$ and $\bar{\tau} = \tau^{-1}$.

Further, it is clear that $[\bar{\Lambda}(\Lambda(\Sigma))] = [\Sigma]$. Combining this with the normal form for open-to-closed morphisms, we obtain the following result.

COROLLARY. Consider a representative Σ of a connected morphism : $\boldsymbol{\epsilon} \to \boldsymbol{\delta}$. Define its *normal form* as

$$\mathrm{NF}(\Sigma) = \bar{\Lambda}\big(\mathrm{nf}(\Lambda(\Sigma))\big).$$

For disconnected surfaces, define it componentwise. Then the class of $\mathrm{NF}(\Sigma)$ do not depend on the class of Σ and in particular

$$\big[\mathrm{NF}(\Sigma)\big] = [\Sigma].$$

It is clear now that the elements in Fig. 1 generates $\mathbf{OC\text{-}Cob}_2$, as the normal form for open-to-closed morphisms and the permutations are obtained by gluing these pieces, and the theorem for normal form of open-to-closed cobordism proves the sufficiency of the relations in Fig. 2.

3. Open-closed TQFTs and Cardy-Frobenius algebras

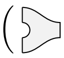
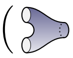
Let us denote by $\mathbf{Vect}_{\mathbb{K}}$ the category of finite-dimensional vector spaces over $\mathbb{K}$. The tensor product induces a symmetric monoidal structure on $\mathbf{Vect}_{\mathbb{K}}$, for which $\mathbb{K}$ is the unit object.

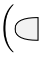
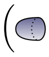
Following the axiomatization due to M.F. Atiyah [2], we give the

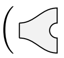
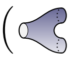
DEFINITION. A *two-dimensional open-closed topological quantum filed theory* (2d open-closed TQFT) over $\mathbb{K}$ is a symmetric monoidal functor $\mathcal{Z}\colon \mathbf{OC\text{-}Cob}_2 \to \mathbf{Vect}_{\mathbb{K}}$.

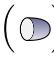
An important idea in the mathematical approach to quantum field theories is that manifolds and their gluings give rise to algebraic structures, where algebraic relations express diffeomorphisms between manifolds. The following theorem states the precise correspondence. The result corresponding to the closed sector has been folklore knowledge, and it seems to have been written in detail for the first time in [1].

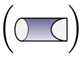
THEOREM 4 ([9]). A 2d open-closed TQFT $\mathcal{Z}$ determines a Cardy-Frobenius algebra structure on the pair (A, C), given by $\mathcal{Z}(I) = A$ and $\mathcal{Z}(\mathbb{S}^1) = C$, via the formulas

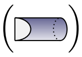
$\mathcal{Z}($  $) = \mu_A\colon A \otimes A \to A$ $\qquad$ $\mathcal{Z}($  $) = \mu_C\colon C \otimes C \to C$

$\mathcal{Z}($  $) = \iota_A\colon \mathbb{K} \to A$ $\qquad$ $\mathcal{Z}($  $) = \iota_C\colon \mathbb{K} \to C$

$\mathcal{Z}($  $) = \nu_A\colon A \to A \otimes A$ $\qquad$ $\mathcal{Z}($  $) = \nu_C\colon C \to C \otimes C$

$\mathcal{Z}($  $) = \kappa_A\colon A \to \mathbb{K}$ $\qquad$ $\mathcal{Z}($  $) = \kappa_C\colon C \to \mathbb{K}$

$\mathcal{Z}($  $) = \zeta\colon C \to A$

$\mathcal{Z}($  $) = \zeta^*\colon A \to C$

Conversely, for any Cardy-Frobenius algebra, these formulas determine a unique 2d open-closed TQFT.

PROOF. Consider a 2d open-closed TQFT $\mathcal{Z}$. We have to check that $A = \mathcal{Z}(I)$ form a symmetric Frobenius algebra, $C = \mathcal{Z}(\mathbb{S}^1)$ form a commutative Frobenius algebra and the knowledge, duality and Cardy conditions hold. These relations are all shown in Fig. 2.

It is clear that μ_A and μ_C are associative. The same holds for the coassociativity of ν_A and ν_C. The (co)associativity laws are given by the diffeomorphisms (a) in Fig. 2. Similarly, ι_A and ι_C are units, while κ_A and κ_C are counits by (u). The Frobenius relation is given by (F). The commutative of μ_C comes from the fact that we always find an automorphism of a pair of pants which exchanges the two boundaries one one side, that is (c). The same does not hold for μ_A, though it is symmetric, see (s). Furthermore, the map ζ is an algebra morphism, as shown in (z), and the knowledge, duality and Cardy conditions hold: (k), $(*)$ and (C). As a consequence, we have a Cardy-Frobenius algebra structure.

The converse follows from the classification theorem of 2d open-closed cobordisms, namely Theorem 1: the value $\mathcal{Z}(\boldsymbol{\epsilon})$ is determined by the values on I and $\mathbb{S}^1$, while the morphism are determine by the values on the generators, subjected to the relations. These are precisely the Cardy-Frobenius axioms. $\square$

We can also say something more about the above result. In particular, 2d open-closed TQFTs form a symmetric monoidal category, and the above result extends to an equivalence of categories.

Let us introduce the following notation. Consider a 2d open-closed TQFT $\mathcal{Z}$ and set $A = \mathcal{Z}(I)$, $C = \mathcal{Z}(\mathbb{S}^1)$. For $\boldsymbol{\epsilon} \in \{0,1\}^n$, set $(A,C)^{\otimes \boldsymbol{\epsilon}}$ for the tensor product of A's and C's, with order given by the components of $\boldsymbol{\epsilon}$. Here 0 correspond to A and 1 corresponds to C.

DEFINITION. A morphism $(f,g)\colon \mathcal{Z} \to \mathcal{Z}'$ between 2d open-closed TQFTs over $\mathbb{K}$ is a collection of maps

$$(f,g)^{\otimes \boldsymbol{\epsilon}}\colon (A,C)^{\otimes \boldsymbol{\epsilon}} \to (A',C')^{\otimes \boldsymbol{\epsilon}}$$

such that the following diagrams commute for every morphism $[\Sigma]\colon \boldsymbol{\epsilon} \to \boldsymbol{\delta}$.

$$\begin{array}{ccc} (A,C)^{\otimes \boldsymbol{\epsilon}} & \xrightarrow{(f,g)^{\otimes \boldsymbol{\epsilon}}} & (A',C')^{\otimes \boldsymbol{\epsilon}} \\ {\scriptstyle \mathcal{Z}[\Sigma]}\downarrow & & \downarrow{\scriptstyle \mathcal{Z}'[\Sigma]} \\ (A,C)^{\otimes \boldsymbol{\delta}} & \xrightarrow[(f,g)^{\otimes \boldsymbol{\delta}}]{} & (A',C')^{\otimes \boldsymbol{\delta}} \end{array}$$

We can also introduce the tensor product of 2d open-closed TQFTs over $\mathbb{K}$, say $\mathcal{Z}$ and $\mathcal{Z}'$, as

$$(\mathcal{Z} \otimes \mathcal{Z}')(\boldsymbol{\epsilon}) = \mathcal{Z}(\boldsymbol{\epsilon}) \otimes \mathcal{Z}'(\boldsymbol{\epsilon}).$$

Thanks to the isomorphism $(A,C)^{\otimes \boldsymbol{\epsilon}} \otimes (A',C')^{\otimes \boldsymbol{\epsilon}} \cong (A \otimes A', C \otimes C')^{\otimes \boldsymbol{\epsilon}}$, it makes sense to define

$$(\mathcal{Z} \otimes \mathcal{Z}')[\Sigma] = \mathcal{Z}[\Sigma] \otimes \mathcal{Z}'[\Sigma].$$

With these notion of morphisms and tensor product, 2d open-closed TQFTs over $\mathbb{K}$ form a symmetric monoidal category, denoted by **2d OC-TQFT**$_{\mathbb{K}}$.

THEOREM 5. There is an equivalence of categories $\mathcal{F}\colon$ **CF-alg**$_{\mathbb{K}} \to$ **2d OC-TQFT**$_{\mathbb{K}}$.

PROOF. We have already seen the correspondence of objects. For the morphisms correspondence, it is clear that given a morphism of 2d open-closed TQFTs $(f,g)\colon \mathcal{Z} \to \mathcal{Z}'$, then

$$(f,g)^{\otimes(0)}\colon A \to A', \qquad (f,g)^{\otimes(1)}\colon C \to C'$$

is a morphism of Cardy-Frobenius algebras. On the other hand, from a morphism of Cardy-Frobenius algebras $(f,g)\colon (A,C) \to (A',C')$ we can construct all morphisms $(f,g)^{\otimes \boldsymbol{\epsilon}}$ via tensor products.

Further, it is clear that those assignments respect the tensor product. □

REMARK. The above results can be generalized in various directions. On one hand, the original work by A. Lauda and H. Pfeiffer presented the results for 2d open-closed TQFTs with values in a symmetric monoidal category $\mathscr{C}$, *i.e.* a symmetric monoidal functor

$$\mathcal{Z}\colon \textbf{OC-Cob}_2 \to \mathscr{C}.$$

The algebraic counterpart is played by Cardy-Frobenius algebras over $\mathscr{C}$. In the same article, they presented a result for colored TQFTs, in which the boundary $\partial_1\Sigma$ can be colored in different ways and all morphisms respect the coloring.

On the other hand one can explore the notion of deformations of Cardy-Frobenius algebras in the sense of [6]. This has be done by S.M. Natanzon in [10], with the concepts of Cardy-Frobenius manifold and extended cohomological field theory.

A classification has been carried out in the semisimple supersymmetric case. In particular, every semisimple super Cardy-Frobenius algebra is the direct sum of super Cardy-Frobenius algebras of three simple types. More precisely:

DEFINITION. A super Cardy-Frobenius algebra (A, C, ζ, ζ^*) is called semisimple if A and C are semisimple in the category of $\mathbb{Z}_2-$graded algebras.

THEOREM 6 ([7]). Let (A, C, ζ, ζ^*) be a semisimple super Cardy-Frobenius algebra over an algebraically closed field $\mathbb{K}$. Then it is the direct sum of the following elementary algebras.

- $A = 0$ and $C = \mathbb{K}$, with $\kappa_C(x) = \alpha x$ for some $\alpha \in \mathbb{K}^\times$.

- $A = \mathrm{Mat}(n|m, \mathbb{K})$ and $C = \mathbb{K}$, with

$$\kappa_C(a) = \alpha\, \mathrm{str}(a) \qquad \zeta(x) = \mathrm{diag}(x, \dots, x)$$

$$\kappa_A(x) = \alpha^2 x \qquad \zeta^*(a) = \frac{1}{\alpha}\, \mathrm{str}(a).$$

  for some $\alpha \in \mathbb{K}^\times$.
- $A = \mathrm{Mat}(n, \mathbb{K})[\xi]$, with $\xi^2 = 1$ and grading given by

$$A = \underbrace{\mathrm{Mat}(n, \mathbb{K})}_{=A_0} \oplus \underbrace{\mathrm{Mat}(n, \mathbb{K}) \cdot \xi}_{=A_1},$$

  and $C = \mathbb{K}$, with

$$\kappa_A(a + b\xi) = \alpha\, \mathrm{tr}(b) \qquad \zeta(x) = \mathrm{diag}(x, \dots, x)$$

$$\kappa_C(x) = \frac{\alpha^2}{2} x \qquad \zeta^*(a + b\xi) = \frac{2}{\alpha}\, \mathrm{tr}(b).$$

  for some $\alpha \in \mathbb{K}^\times$.

4. Matrix factorization

A geometric example of Cardy-Frobenius structure comes from matrix factorization. This structure arises in the context of Homological Mirror Symmetry, as the B-category associated to a Landau-Ginzburg model [3]. In the following, $\mathbb{K}$ will be an algebraically closed field.

DEFINITION. Let R be a commutative ring, $w \in R$. we define the category of matrix factorizations on w, denoted by $\mathbf{MF}(w)$, as follows. The objects, called *matrix factorizations* on w, are periodic complexes

$$M^0 \xrightarrow{d_M^0} M^1 \xrightarrow{d_M^1} M^0$$

where M^0, M^1 are free R-modules, $d_M^0 d_M^1 = w \cdot \mathrm{id}$, $d_M^1 d_M^0 = w \cdot \mathrm{id}$. Alternatively, we can consider the data

$$M = M^0 \oplus M^1, \qquad d_M = \begin{pmatrix} 0 & d_M^1 \\ d_M^0 & 0 \end{pmatrix} \quad \text{s.t.} \quad d_M^2 = w \cdot \mathrm{id}\,.$$

A morphism of matrix factorizations M, N on w is a pair of R-linear morphisms $f^i \colon M^i \to N^i$, such that the following diagram commute.

$$\begin{array}{ccccc} M^0 & \xrightarrow{d_M^0} & M^1 & \xrightarrow{d_M^1} & M^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ N^0 & \xrightarrow[d_N^0]{} & N^1 & \xrightarrow[d_N^1]{} & N^0 \end{array}$$

Now, a homotopy between two matrix factorizations on w, say $f, g \colon M \to N$, is a pair of R-linear maps $h^i \colon M^i \to N^{i-1}$, where the indices are intended mod 2, such that the following diagram commute.

$$\begin{array}{ccccc} M^0 & \xrightarrow{d_M^0} & M^1 & \xrightarrow{d_M^1} & M^0 \\ \downarrow & \swarrow{\scriptstyle h^1} & \downarrow & \swarrow{\scriptstyle h^0} & \downarrow \\ N^0 & \xrightarrow[d_N^0]{} & N^1 & \xrightarrow[d_N^1]{} & N^0 \end{array}$$

Here the vertical arrows are the maps $f^i - g^i$. Null-homotopic morphisms constitute an ideal in the category $\mathbf{MF}(w)$. Let $\mathbf{HMF}(w)$ be the quotient category by this ideal. It has the same objects as $\mathbf{MF}(w)$, but fewer morphisms. More precisely:

DEFINITION. The *homotopy matrix factorization category* associated to $w \in R$ is the category $\mathbf{HMF}(w)$ whose objects are matrix factorizations on w, and the morphisms are

$$\mathrm{Mor}_{\mathbf{HMF}(w)}(M, N) = {\mathrm{Mor}_{\mathbf{MF}(w)}(M, N)}\big/{\{\text{null-homotopic morphisms}\}}.$$

REMARK. Note that if M, N are matrix factorizations, then $\mathrm{Hom}_R(M, N)$ is a $\mathbb{Z}_2$ cochain complex, with

$$\mathrm{Hom}_R(M,N) = \underbrace{\big(\mathrm{Hom}_R(M^0,N^0)\oplus \mathrm{Hom}_R(M^1,N^1)\big)}_{\mathrm{Hom}_R^{\mathrm{ev}}(M,N)} \oplus \underbrace{\big(\mathrm{Hom}_R(M^0,N^1)\oplus \mathrm{Hom}_R(M^0,N^1)\big)}_{\mathrm{Hom}_R^{\mathrm{odd}}(M,N)}$$

and differential

$$d_{M,N} f = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

In particular, it makes sense to speak about the cohomology of $(\mathrm{Hom}_R(M,N), d_{M,N})$ and we have

$$\mathrm{Mor}_{\mathbf{HMF}(w)}(M,N) = H^0\big(\mathrm{Hom}_R(M,N), d_{M,N}\big).$$

EXAMPLE. Consider $R = \mathbb{K}[x]$, $w = x^n$ and for every $k \in \{1, \dots, n-1\}$, the matrix factorization

$$M_k = \mathbb{K}[x] \oplus \mathbb{K}[x], \qquad d_k = \begin{pmatrix} 0 & x^{n-k} \\ x^k & 0 \end{pmatrix}.$$

It is clear that $d_k^2 = x^n \cdot \mathrm{id}$ for every k. Let us compute now $\mathrm{End}_{\mathbf{MF}(x^n)}(M_k)$. Fix an even R-linear morphisms $f\colon M_k \to M_k$, that is $f = \mathrm{diag}(p, q)$ with $p, q \in \mathbb{K}[x]$. Imposing the commutativity with the differential, we find

$$0 = d_k \circ f - f \circ d_k = \begin{pmatrix} 0 & -x^k(p-q) \\ x^{n-k}(p-q) & 0 \end{pmatrix},$$

which implies $p = q$. In particular, we have the algebra isomorphism

$$\mathrm{End}_{\mathbf{MF}(x^n)}(M_k) \cong \mathbb{K}[x] \oplus \mathbb{K}[x].$$

let us compute now $\mathrm{End}_{\mathbf{HMF}(x^n)}(M_k)$. Consider a null-homotopic morphism $f \in \mathrm{End}_{\mathbf{MF}(x^n)}(M_k)$. That is, there exists an odd R-linear map

$$h = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{K}[x]$, such that $f = d_k \circ h + h \circ d_k$. In particular, setting $f = \mathrm{diag}(p, p)$, we find

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} x^{n-k}\alpha + x^k\beta & 0 \\ 0 & x^{n-k}\alpha + x^k\beta \end{pmatrix}.$$

As a consequence, $p \in (x^k, x^{n-k})$. In particular, we have the algebra isomorphism

$$\mathrm{End}_{\mathbf{HMF}(x^n)}(M_k) \cong \mathbb{K}[x]\big/(x^k, x^{n-k}) \oplus \mathbb{K}[x]\big/(x^k, x^{n-k}).$$

Consider now a matrix factorization (M, d_M) on a polynomial $w \in \mathbb{K}[x_1, \dots, x_n]$. We can define a super Cardy-Frobenius algebra structure as follows. Let $I_w = (\partial_1 w, \dots, \partial_n w)$ be the Jacobian ideal. Set

$$A = \mathrm{End}_{\mathbf{HMF}(w)}(M) \qquad C = \mathbb{K}[x_1, \dots, x_n]\big/I_w$$

$$\kappa_A[a] = \mathrm{Res}_{\mathbb{K}[x]/\mathbb{K}}\left[\frac{\mathrm{str}(a \cdot \partial_1 d_M \cdots \partial_n d_M)dx}{\partial_1 w, \dots, \partial_n w}\right] \qquad \kappa_C[p] = \mathrm{Res}_{\mathbb{K}[x]/\mathbb{K}}\left[\frac{p\, dx}{\partial_1 w, \dots, \partial_n w}\right]$$

$$\zeta[p] = \big[p \cdot \mathrm{id}_M\big] \qquad \zeta^*[a] = (-1)^{\binom{n-1}{2}}\, \mathrm{str}\big(a \cdot \partial_1 d_M \cdots \partial_n d_M\big) \mod I_W.$$

Here the $\mathbb{Z}_2$-grading on A is given by

$$A = \mathrm{End}_{\mathbf{HMF}(w)}(M^0) \oplus \mathrm{End}_{\mathbf{HMF}(w)}(M^1),$$

while the grading on the Milnor ring C is trivial (it is purely even). Further, after a choice of basis, we can express d_M as a matrix with entries in $\mathbb{K}[x_1, \dots, x_n]$, so that it is clear what $\partial_i d_M$ means. In particular, it can be shown that the supertrace of the morphism $a \cdot \partial_1 d_M \cdots \partial_n d_M$ do not depend on the homotopy class of a, so that κ_A and ζ^* are well-defined. Further, by differentiating $d_M^2 = w \cdot \mathrm{id}$, we obtain that $\partial_i w \cdot \mathrm{id}$ is null-homotopic, so that ζ is well-defined too. Finally, from the definition of residue it can be shown that if $p \in I_w$, then

$$\mathrm{Res}_{\mathbb{K}[x]/\mathbb{K}}\left[\frac{p\, dx}{\partial_1 w, \dots, \partial_n w}\right] = 0,$$

so that also κ_C is well-defined.

THEOREM 7 ([5, 11]). The above $A, C, \kappa_A, \kappa_C, \zeta, \zeta^*$ form a super Cardy-Frobenius algebra.

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Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn
E-mail address: agiacche@mpim-bonn.mpg.de