Supersymmetric Quantum Mechanics and Morse Theory

Alessandro Giacchetto
Supervisor: prof. Alessandro Tanzini

University of Trieste & SISSA

November 25, 2016

Presentation structure

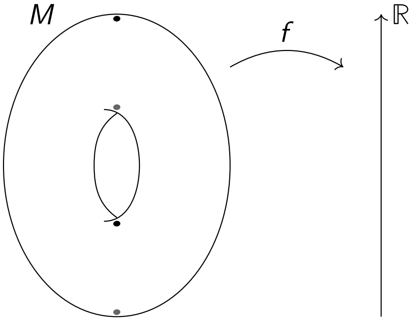
1. Morse Theory

2. Quantum Mechanics on Riemannian varieties
   - Quantum Mechanics and Supersymmetry
   - Hodge Theory

3. SUSY and Morse Theory
   - Weak Morse inequalities
   - Morse-Smale-Witten cochain complex

Morse function

Let M be a compact orientable smooth m-manifold, $f: M \to \mathbb{R}$ smooth. A point $p \in M$ is called critical for f if $df_p\colon T_pM \to \mathbb{R}$ is the null map.



Example: height function on the torus

Morse function

Fundamental idea

Critical points of f determine the topology of M

A smooth function $f\colon M \to \mathbb{R}$ is called a Morse function if for every critical point p the Hessian $d^2 f_p$ is non-degenerate. Define the Morse index in p as

$$\mu_p = \#\text{negative eigenvalues of } d^2 f_p$$

and the value

$$M_k = \#\text{critical points of Morse index } k.$$

Morse function

Fundamental idea

Critical points of f determine the topology of M

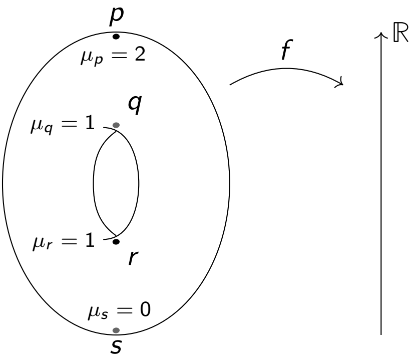
A smooth function $f\colon M \to \mathbb{R}$ is called a Morse function if for every critical point p the Hessian $d^2 f_p$ is non-degenerate. Define the Morse index in p as

$$\mu_p = \#\text{negative eigenvalues of } d^2 f_p$$

and the value

$$M_k = \#\text{critical points of Morse index } k.$$

Example: torus' height



$$M_0 = 1 \qquad M_1 = 2 \qquad M_2 = 1$$

Fundamental theorem

Principal result

From the sublevel sets

$$M^{\alpha} = \{\, p \in M \mid f(p) \leq \alpha \,\}$$

is possible to reconstruct the CW-complex structure of M

Fundamental theorem of Morse Theory

- Let $\alpha < \beta$ and suppose that $\{\, \alpha \leq f \leq \beta \,\}$ does not contain critical point of f. Then M^{α} is a deformation retract of M^{β}.
- Let p be a critical point with $f(p) = \alpha$ and $\epsilon > 0$ such that f does not have critical points in $(\alpha - \epsilon, \alpha + \epsilon)$ rather than p. Then $M^{\alpha+\epsilon}$ is homotopically equivalent to $M^{\alpha-\epsilon} \cup e^{\mu_p}$.

Fundamental theorem

Principal result

From the sublevel sets

$$M^\alpha = \{\, p \in M \mid f(p) \le \alpha \,\}$$

is possible to reconstruct the CW-complex structure of M

Fundamental theorem of Morse Theory

- Let $\alpha < \beta$ and suppose that $\{\, \alpha \le f \le \beta \,\}$ does not contain critical point of f. Then M^α is a deformation retract of M^β.
- Let p be a critical point with $f(p) = \alpha$ and $\epsilon > 0$ such that f does not have critical points in $(\alpha - \epsilon, \alpha + \epsilon)$ rather than p. Then $M^{\alpha+\epsilon}$ is homotopically equivalent to $M^{\alpha-\epsilon} \cup e^{\mu_p}$.

Fundamental theorem

Principal result

From the sublevel sets

$$M^\alpha = \{ p \in M \mid f(p) \leq \alpha \}$$

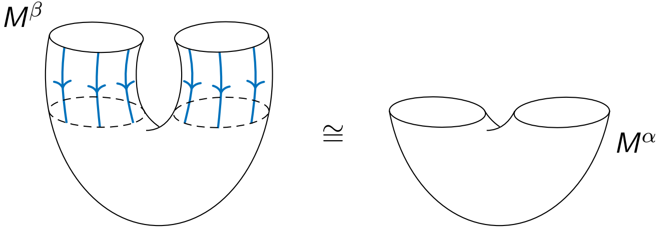
is possible to reconstruct the CW-complex structure of M

Fundamental theorem of Morse Theory

- Let $\alpha < \beta$ and suppose that $\{ \alpha \leq f \leq \beta \}$ does not contain critical point of f. Then M^α is a deformation retract of M^β.
- Let p be a critical point with $f(p) = \alpha$ and $\epsilon > 0$ such that f does not have critical points in $(\alpha - \epsilon, \alpha + \epsilon)$ rather than p. Then $M^{\alpha+\epsilon}$ is homotopically equivalent to $M^{\alpha-\epsilon} \cup e^{\mu_p}$.

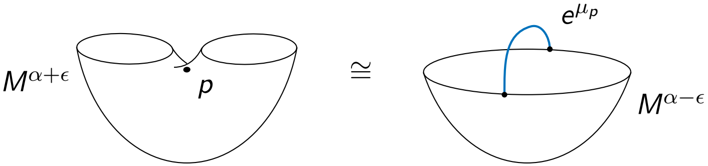
Example: torus' height

- Let $\alpha < \beta$ and suppose that $\{ \alpha \leq f \leq \beta \}$ does not contain critical point of f. Then M^α is a deformation retract of M^β.



Example: torus' height

- Let p be a critical point with $f(p) = \alpha$ and $\epsilon > 0$ such that f does not have critical points in $(\alpha - \epsilon, \alpha + \epsilon)$ rather than p. Then $M^{\alpha+\epsilon}$ is homotopically equivalent to $M^{\alpha-\epsilon} \cup e^{\mu_p}$.



Morse inequalities

Let β_k be the kth Betti number of M.

- Weak Morse inequalities:

$$\beta_k \leq M_k$$

- Strong Morse inequalities: for every $n = 0, \ldots, m$

$$\sum_{k=0}^{n} (-1)^{n-k} \beta_k \leq \sum_{k=0}^{n} (-1)^{n-k} M_k$$

- Morse index theorem:

$$\chi(M) = \sum_{k=0}^{m} (-1)^k M_k$$

Morse inequalities

Let β_k be the kth Betti number of M.

- Weak Morse inequalities:

$$\beta_k \leq M_k$$

- Strong Morse inequalities: for every $n = 0, \dots, m$

$$\sum_{k=0}^{n} (-1)^{n-k} \beta_k \leq \sum_{k=0}^{n} (-1)^{n-k} M_k$$

- Morse index theorem:

$$\chi(M) = \sum_{k=0}^{m} (-1)^k M_k$$

Morse inequalities

Let β_k be the kth Betti number of M.

- Weak Morse inequalities:

$$\beta_k \leq M_k$$

- Strong Morse inequalities: for every $n = 0, \dots, m$

$$\sum_{k=0}^{n} (-1)^{n-k} \beta_k \leq \sum_{k=0}^{n} (-1)^{n-k} M_k$$

- Morse index theorem:

$$\chi(M) = \sum_{k=0}^{m} (-1)^k M_k$$

Morse inequalities

Let β_k be the kth Betti number of M.

- Weak Morse inequalities:

$$\beta_k \leq M_k$$

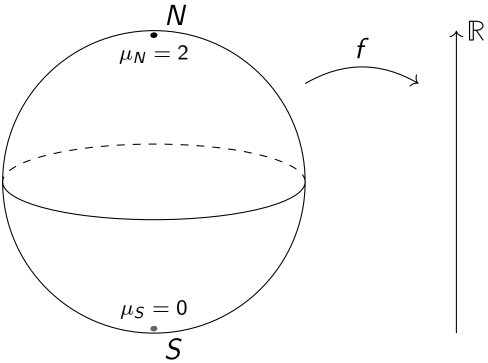
- Strong Morse inequalities: for every $n = 0, \dots, m$

$$\sum_{k=0}^{n} (-1)^{n-k} \beta_k \leq \sum_{k=0}^{n} (-1)^{n-k} M_k$$

- Morse index theorem:

$$\chi(M) = \sum_{k=0}^{m} (-1)^k M_k$$

Example: sphere's height

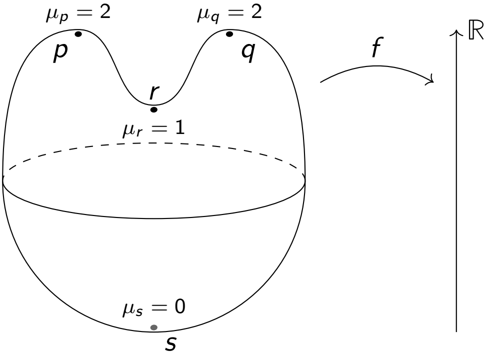


$$M_0 = 1 = \beta_0$$

$$M_1 = 0 = \beta_1$$

$$M_2 = 1 = \beta_2$$

Example: horned sphere's height



$$M_0 = 1 = \beta_0$$

$$M_1 = 1 > \beta_1$$

$$M_2 = 2 > \beta_2$$

$$-1 = \beta_1 - \beta_0 < M_1 - M_0 = 0$$

$$\chi(\mathbb{S}^2) = M_0 - M_1 + M_2$$

Fundamental postulates

In Quantum Mechanics the state of a particle is described by a complex-valued function of space-time variable: $\psi(x, t)$. The following conditions holds.

- $\psi(\cdot, t) \in L^2(\mathbb{R}^3)$, with $\|\psi(\cdot, t)\|_2 = 1$, so that $|\psi(\cdot, t)|^2$ can be interpreted as a probability distribution of the particle position in the frozen time t
- ψ solves the Schrödinger equation

$$i\hbar\, \partial_t \psi = \left(-\frac{\hbar^2}{2m} \Delta + V \right) \psi = H\psi$$

- Every observable (as the Hamiltonian H) is a linear self-adjoint operator of $L^2(\mathbb{R}^3)$. The possible outcomes of a measurement are precisely the eigenvalues of the given observable.

Generalisations

We can generalize the theory on a general Hilbert $\mathcal{H}$ space as follows.

- The states $\psi(t)$ are time-dependent elements of the Hilbert space $\mathcal{H}$, with unitary norm
- $\psi(t)$ evolves with the Schrödinger equation

$$i\hbar\,\partial_t\psi = H\psi, \qquad H\colon \mathcal{H} \to \mathcal{H} \text{ linear self-adjoint operator}$$

- Every observable is a linear self-adjoint operator of $\mathcal{H}$

Supersymmetry

A Quantum Mechanical system can have a further symmetry between bosons and fermions, called supersymmetry. In these theories every fermion has a bosonic counterpart, called superpartner.

The supersymmetry is described by two operators Q, $Q^\dagger$ obeying the algebra

$$[H, Q] = [H, Q^\dagger] = 0, \qquad \{Q, Q^\dagger\} = 2H, \qquad Q^2 = (Q^\dagger)^2 = 0.$$

In these theories, it is interesting to study the kernel of the Hamiltonian.

Supersymmetry broken

$$H\psi = 0 \begin{cases} \nearrow \text{ exists a non-trivial solution } \implies \text{ supersymmetry unbroken} \\ \searrow \text{ only trivial solutions } \implies \text{ supersymmetry broken} \end{cases}$$

Hilbert space structure of $\Omega^\bullet(M)$

Let (M, g) be an orientable compact smooth Riemannian m-manifold with atlas $\{(U, x^i)\}$. Define the Hodge star operator $*\colon \Omega^k(M) \to \Omega^{m-k}(M)$ as

$$*\omega = \frac{\sqrt{|g|}}{k!(m-k)!}\,\omega_{j_1\cdots j_k}\,\epsilon^{j_1\cdots j_k}{}_{j_{k+1}\cdots j_n}\,dx^{j_{k+1}} \wedge \cdots \wedge dx^{j_m}$$

This allows the definition of a scalar product on $\Omega^k(M)$ as

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta$$

which makes (the completition of) $\Omega^\bullet(M)$ a Hilbert space. We can define the adjoint of the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$:

$$\langle d\omega, \eta \rangle = \langle \omega, d^\dagger \eta \rangle \quad \Longrightarrow \quad d^\dagger = (-1)^{mk+1} * d *$$

Hodge theorem

We can construct the Laplace-de Rham operator

$$\Delta = -(d + d^\dagger)^2 = -(dd^\dagger + d^\dagger d),$$

which is a self-adjoint linear, negative definite operator on $\Omega^k(M)$.

A k-form ω is called harmonic if $\Delta\omega = 0$.

Hodge theorem

Every de Rahm cohomology class $[\omega] \in H^k_{\mathrm{dR}}(M)$ has a unique harmonic representative. In particular,

$$\beta_k = \dim\ker\big(\Delta\colon \Omega^k(M) \to \Omega^k(M)\big).$$

Supersymmetric Quantum Mechanics on $\Omega^\bullet(M)$

We have the SUSY QM on $\Omega^\bullet(M)$ given by the Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\Delta \qquad\qquad Q = \frac{\hbar}{\sqrt{m}}d$$

The Hilbert space can be divided into bosons and fermions

$$\Omega_{\text{bos}}(M) = \bigoplus_{k \text{ even}} \Omega^k(M)$$
$$\Omega_{\text{ferm}}(M) = \bigoplus_{k \text{ odd}} \Omega^k(M)$$

In particular, 1-forms as dx^i are fermions.

Witten deformation

Take a Morse function $f\colon M \to \mathbb{R}$. Main idea: Witten deformation. Define

$$d_s = e^{-fs} d\, e^{fs} = d + s\, \epsilon_{df}$$

The Hamiltonian becomes (with $\hbar = 1$, $m = 1/2$)

$$\begin{aligned} H_s &= -\Delta + s\big(\mathcal{L}_{\nabla f} + \mathcal{L}^{\dagger}_{\nabla f}\big) + s^2 \,\|df\|^2 \\ &= -\Delta + s \frac{\partial^2 f}{\partial x^j \partial x^l} [(a^j)^\dagger, a^l] + s^2\, \partial^j f \partial_j f, \end{aligned}$$

where

$$(a^j)^\dagger = \epsilon_{dx^j} \qquad a^l = i_{\partial_l}$$

are fermionic creation and annihilation operator respectively.

The theory is still supersymmetric. Our main goal for SUSY reasons is to understand the solutions of

$$H_s \omega = 0.$$

What we know?

- The operator d_s determine the same chain complex of d. In particular,

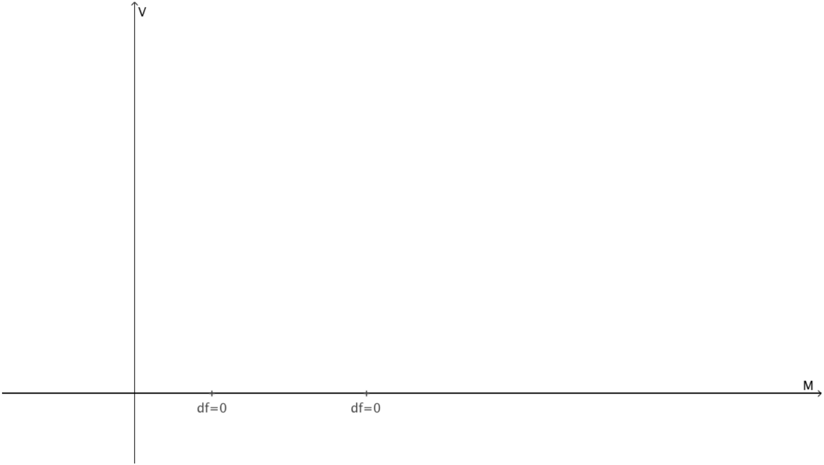
$$\beta_k = \dim\ker\big(H_s \colon \Omega^k(M) \to \Omega^k(M)\big).$$

- For $s \gg 0$, the vacua are determined by $s^2 \|df\|^2 = 0$, *i.e.* $df = 0$: the critical points of f.

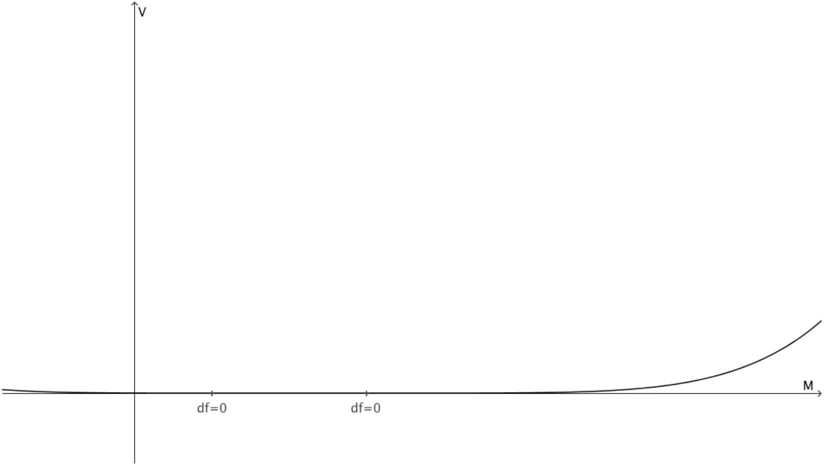
Idea: perturbation theory

Use perturbation theory to determine $\ker H_s$ around any critical point p.

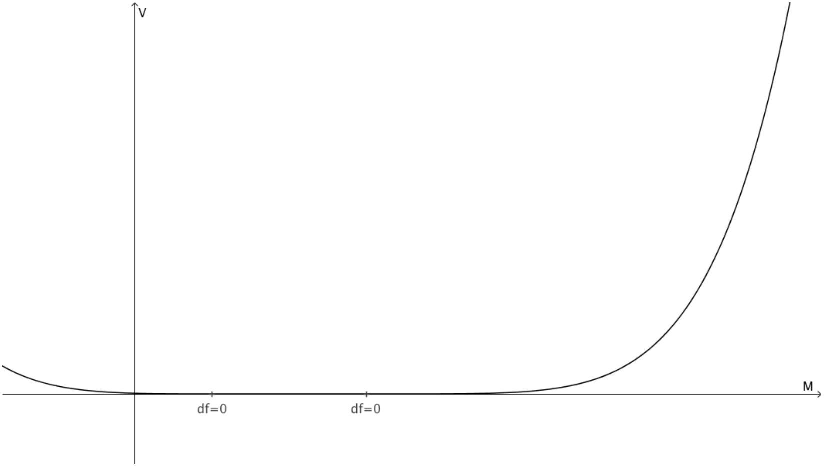
A picture of the potential as s increase.



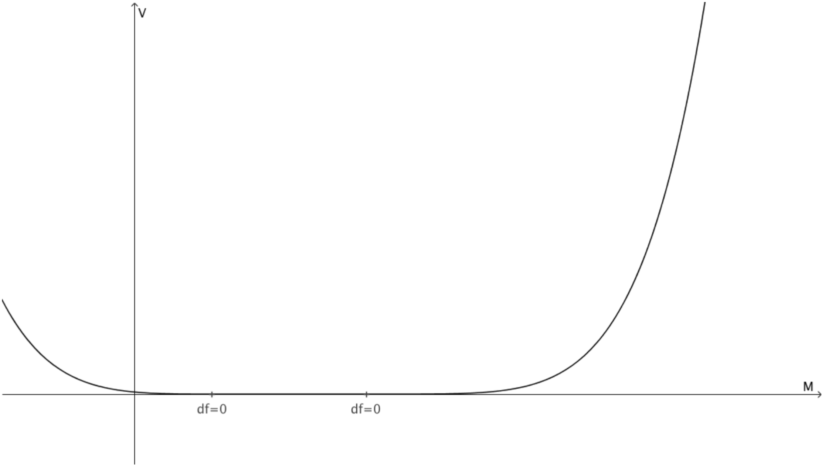
A picture of the potential as s increase.



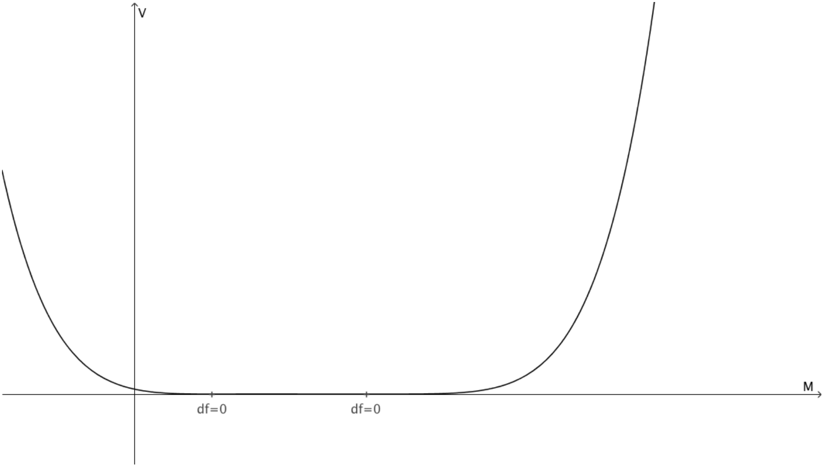
A picture of the potential as s increase.



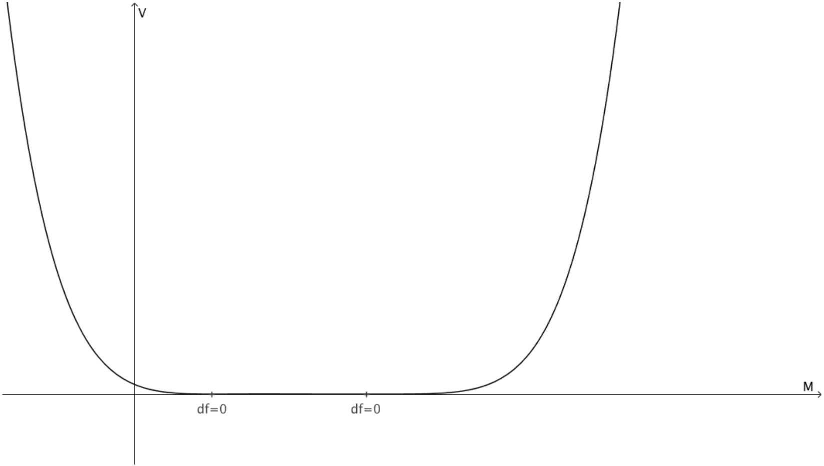
A picture of the potential as s increase.



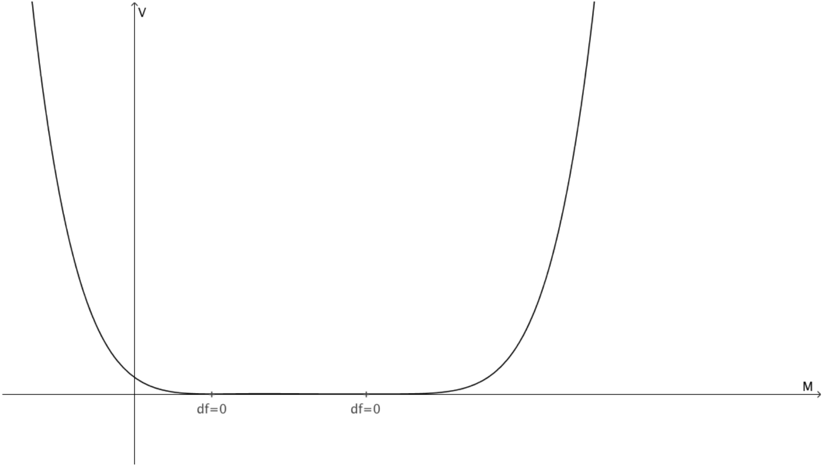
A picture of the potential as s increase.



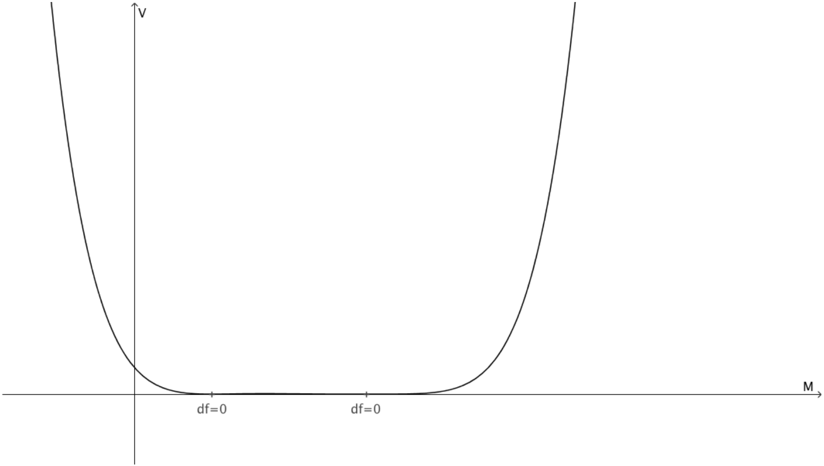
A picture of the potential as s increase.



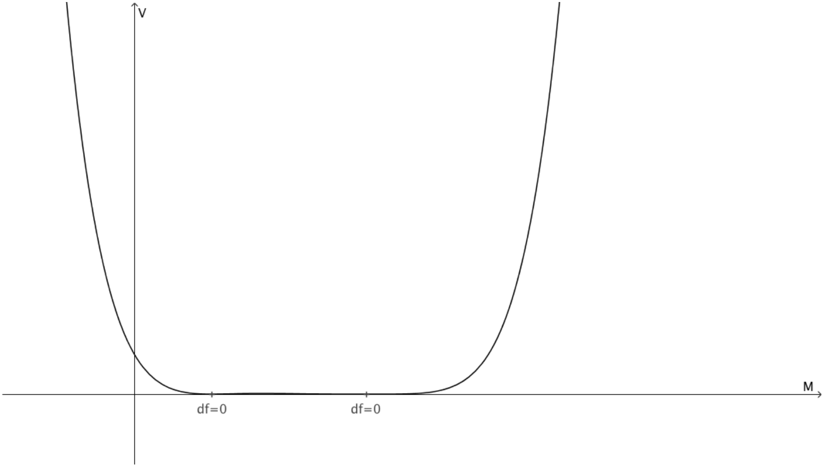
A picture of the potential as s increase.



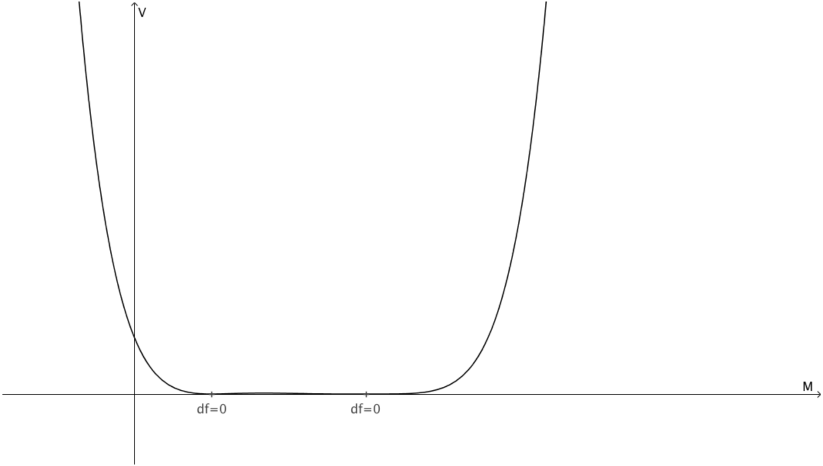
A picture of the potential as s increase.



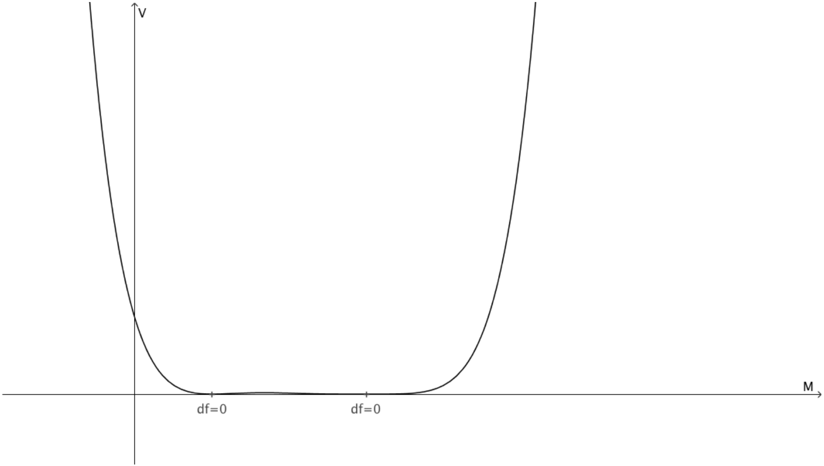
A picture of the potential as s increase.



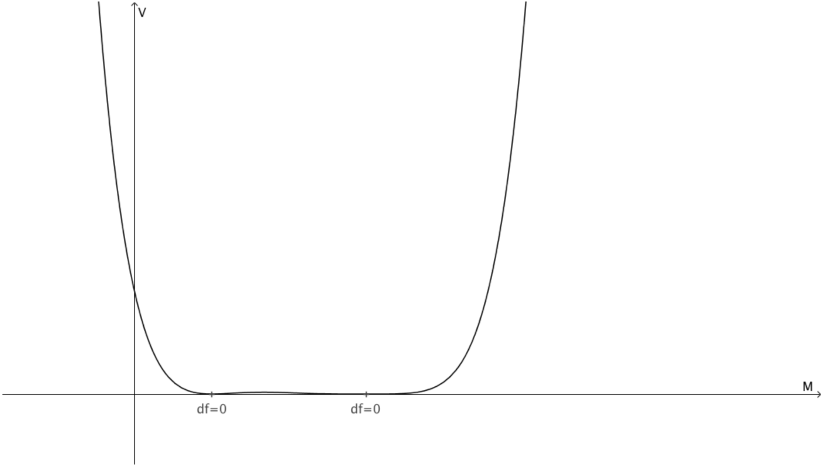
A picture of the potential as s increase.



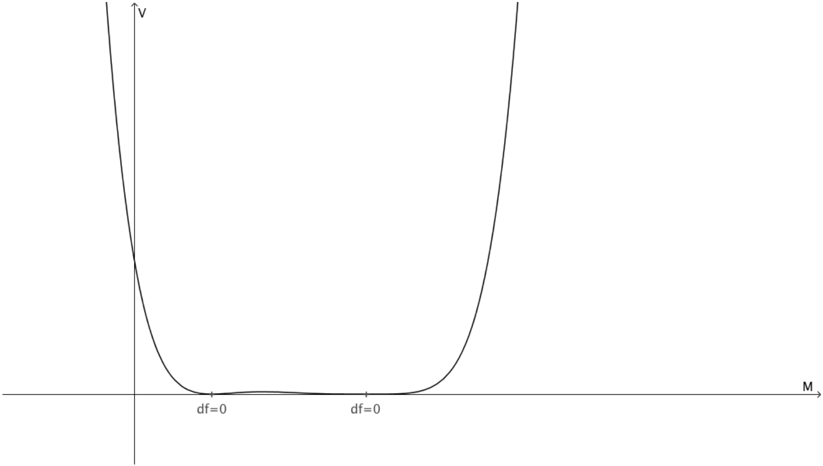
A picture of the potential as s increase.



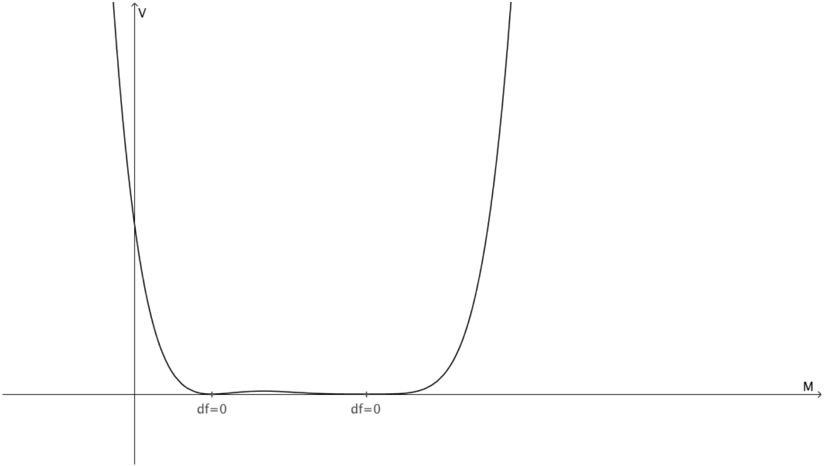
A picture of the potential as s increase.



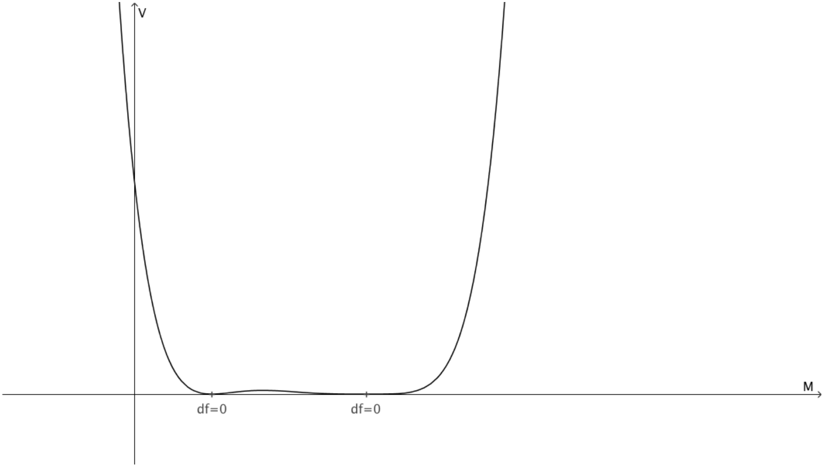
A picture of the potential as s increase.



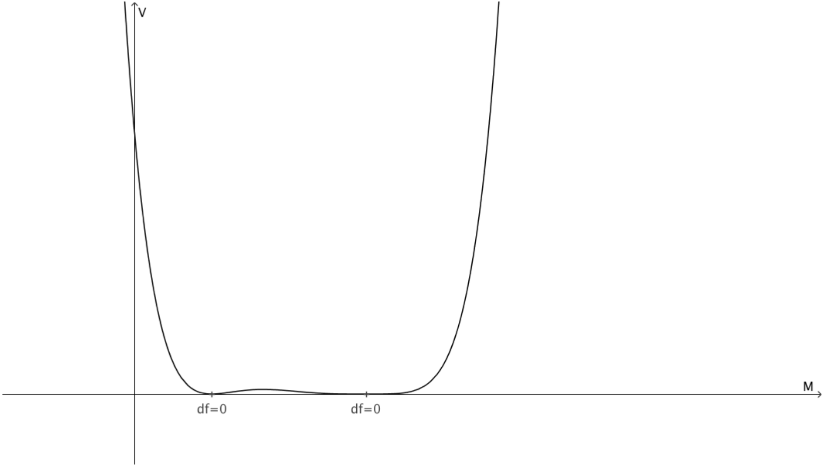
A picture of the potential as s increase.



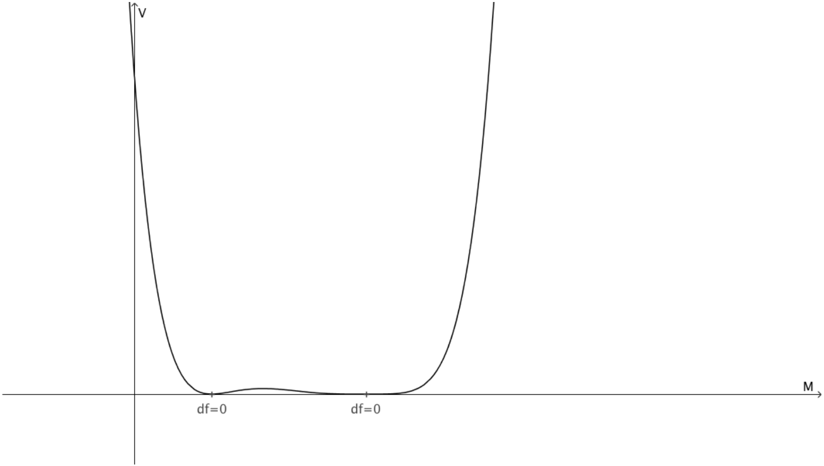
A picture of the potential as s increase.



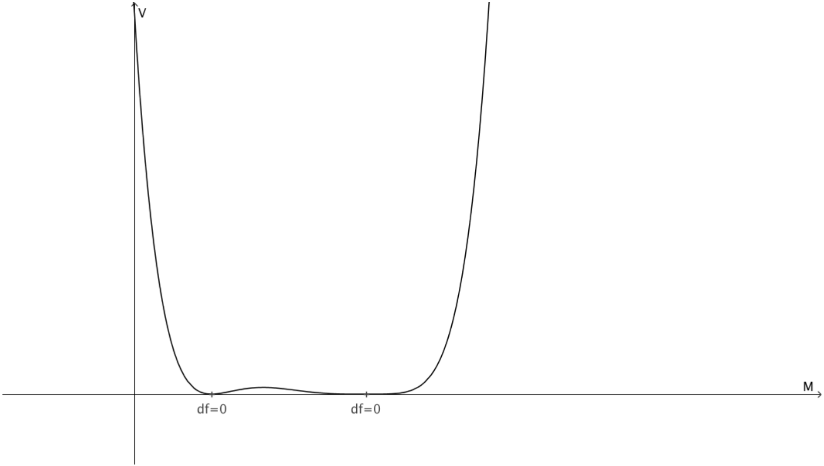
A picture of the potential as s increase.



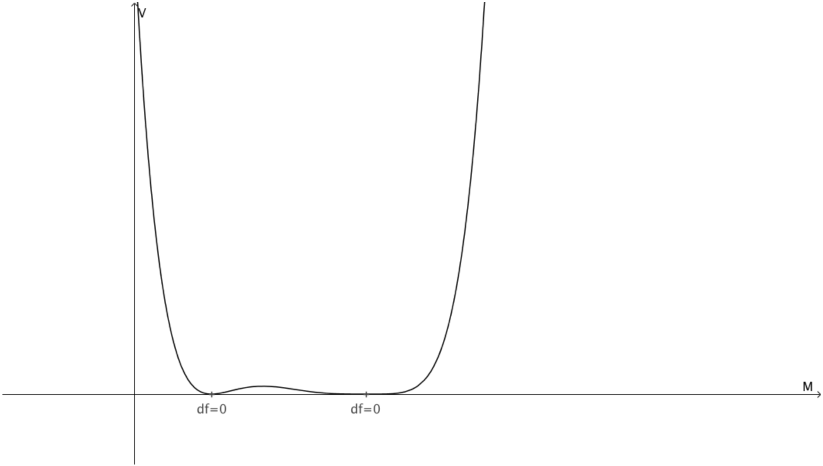
A picture of the potential as s increase.



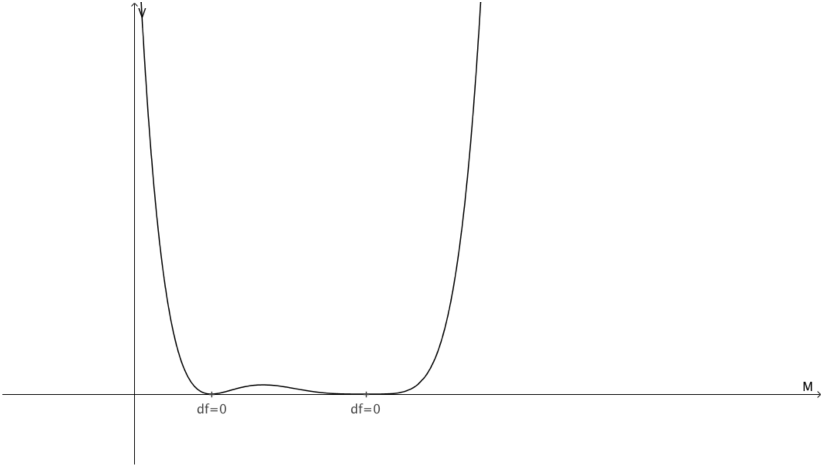
A picture of the potential as s increase.



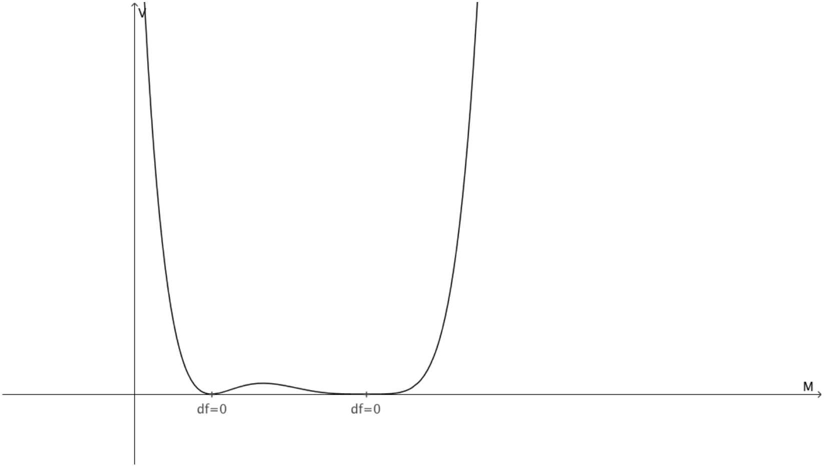
A picture of the potential as s increase.



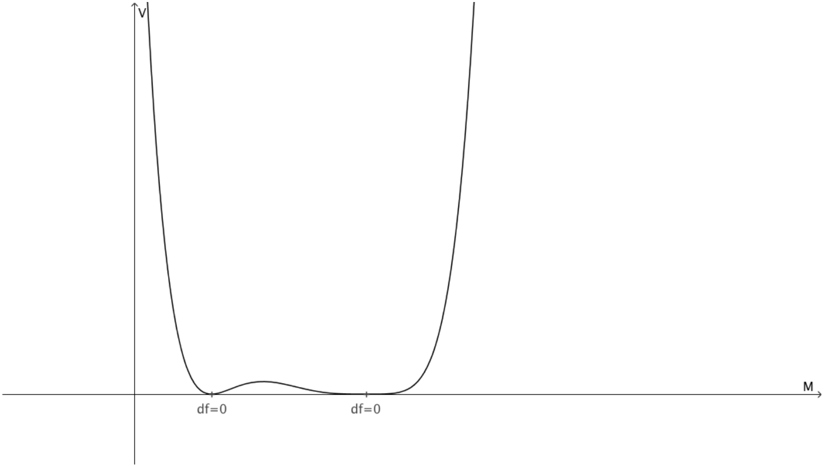
A picture of the potential as s increase.



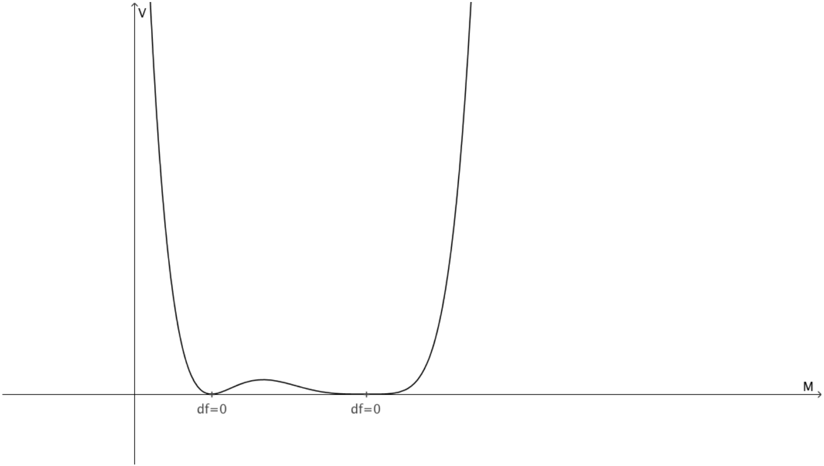
A picture of the potential as s increase.



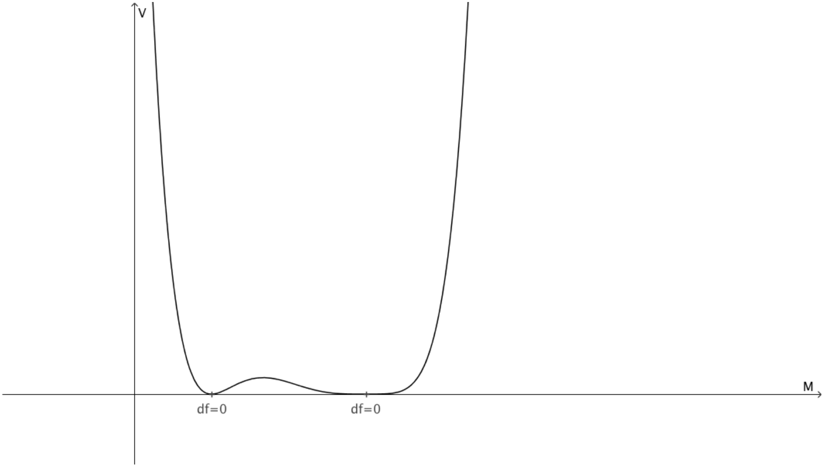
A picture of the potential as s increase.



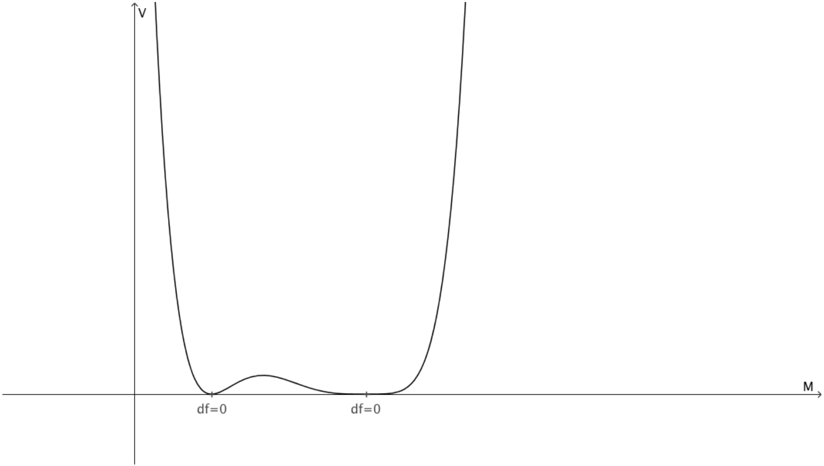
A picture of the potential as s increase.



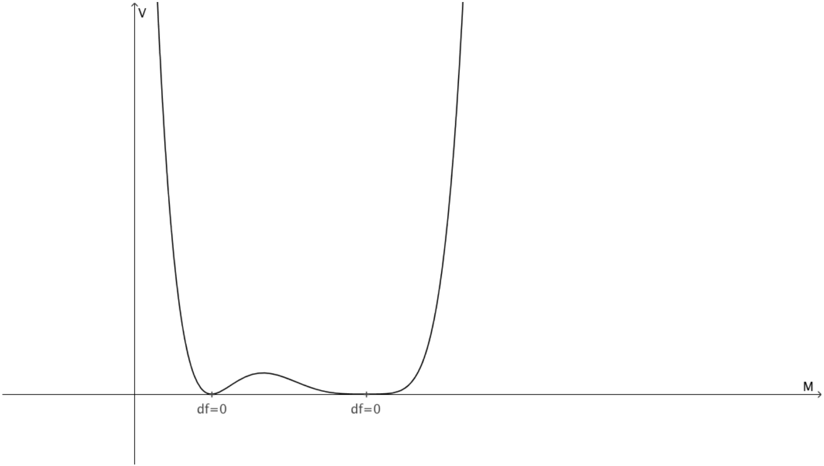
A picture of the potential as s increase.



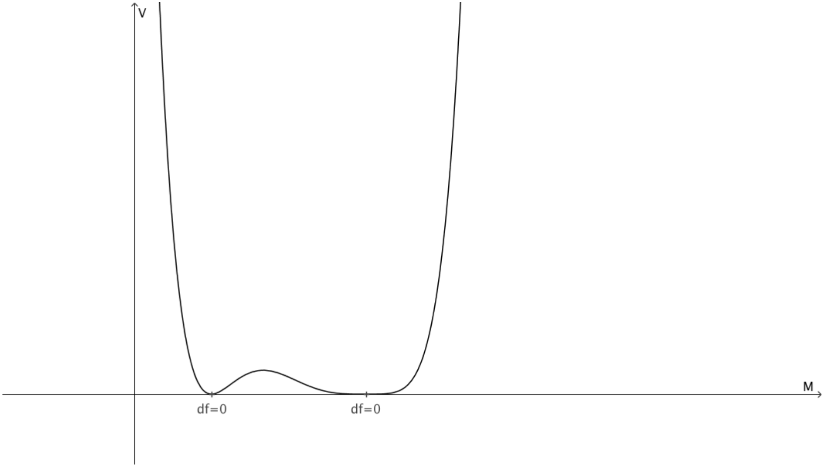
A picture of the potential as s increase.



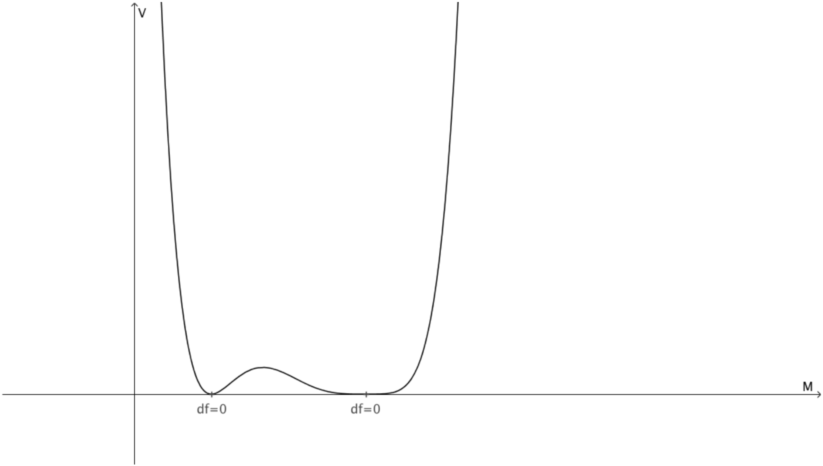
A picture of the potential as s increase.



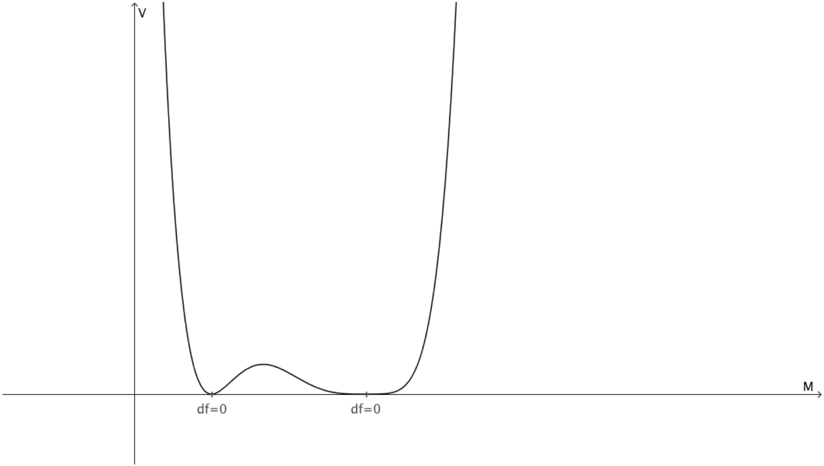
A picture of the potential as s increase.



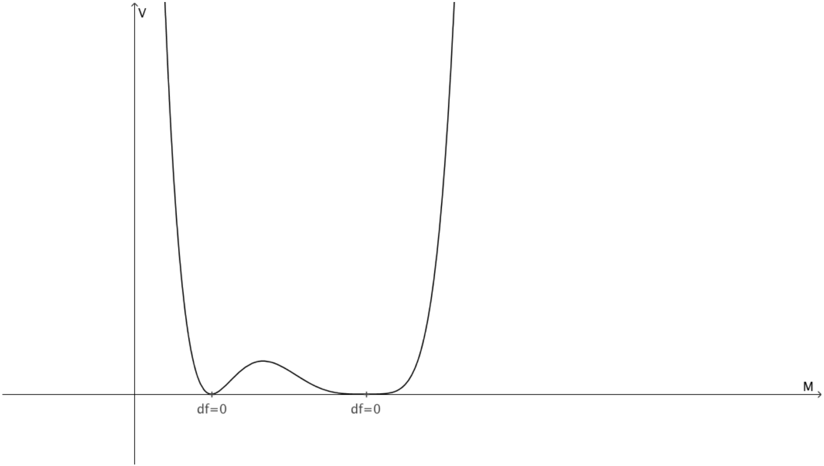
A picture of the potential as s increase.



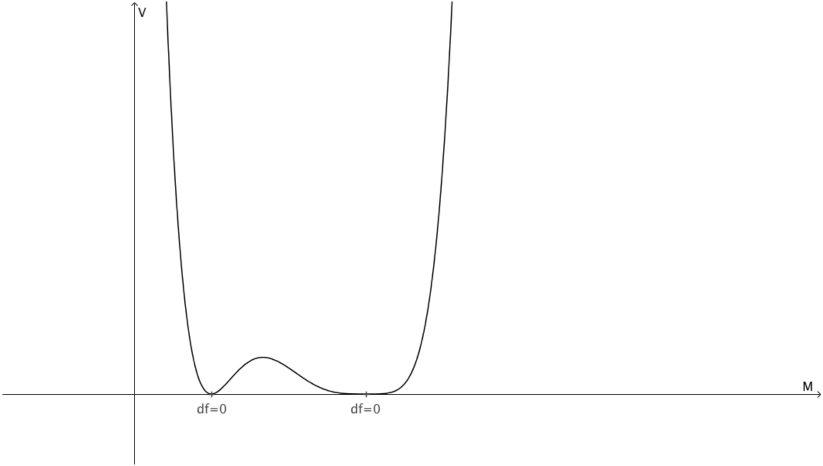
A picture of the potential as s increase.



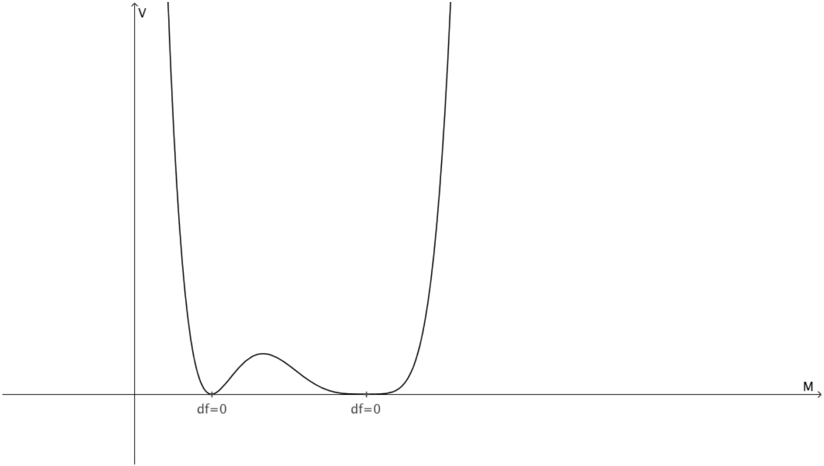
A picture of the potential as s increase.



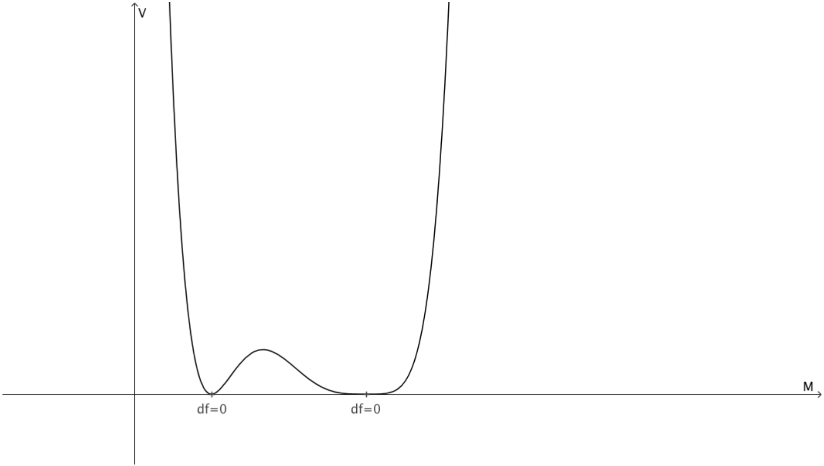
A picture of the potential as s increase.



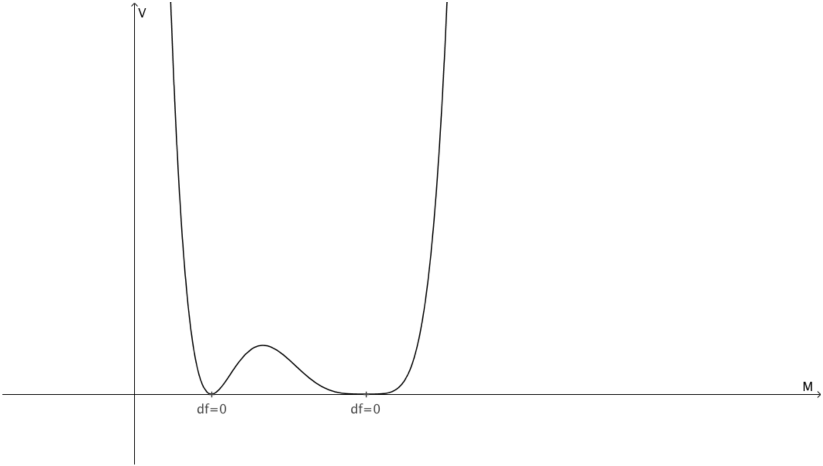
A picture of the potential as s increase.



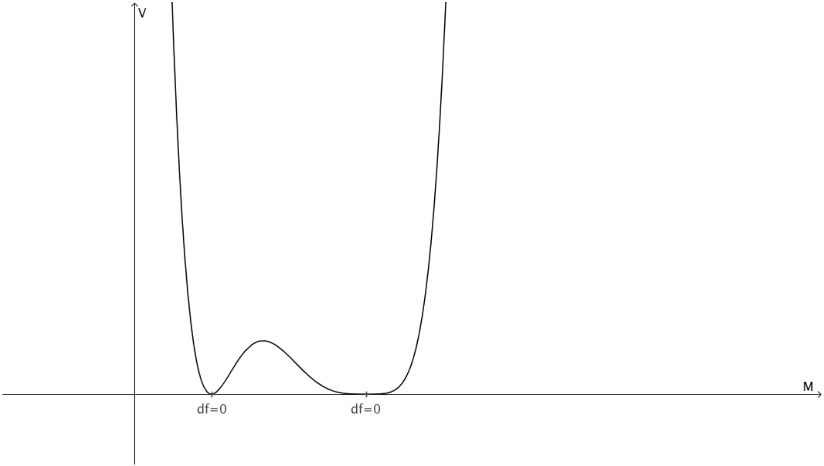
A picture of the potential as s increase.



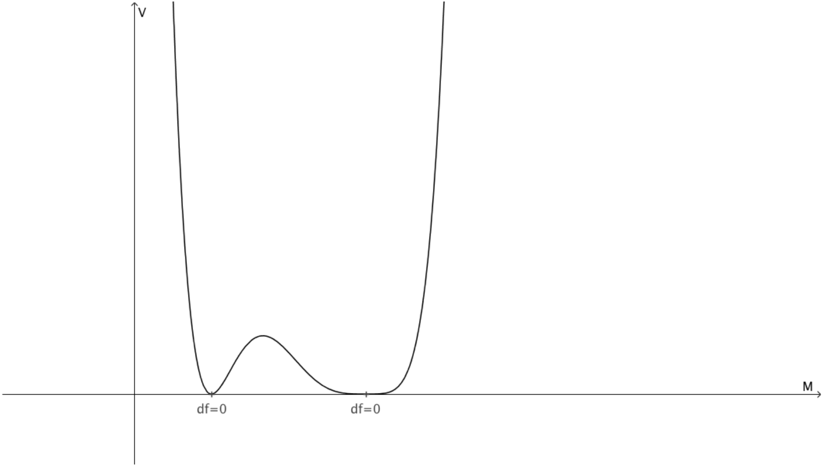
A picture of the potential as s increase.



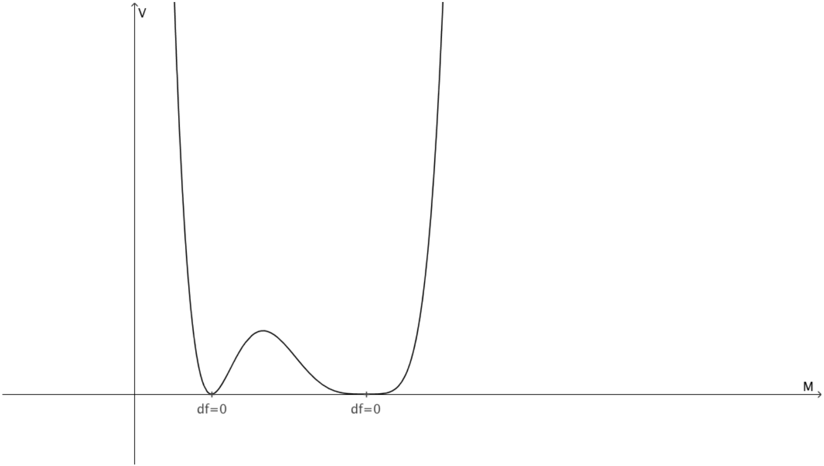
A picture of the potential as s increase.



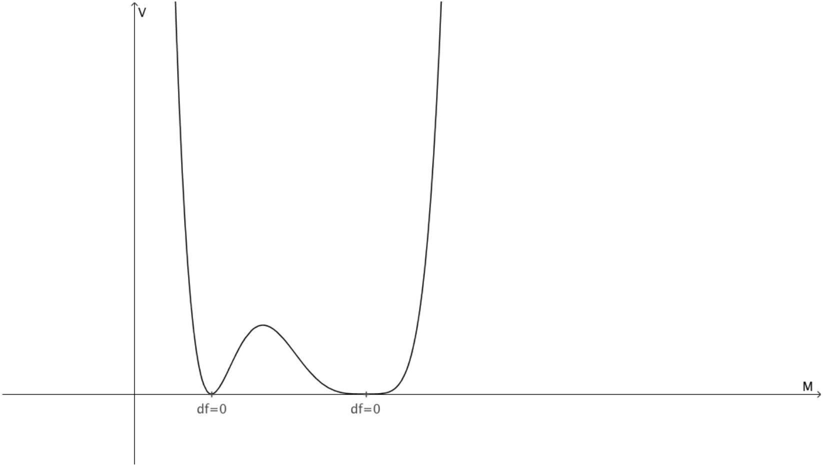
A picture of the potential as s increase.



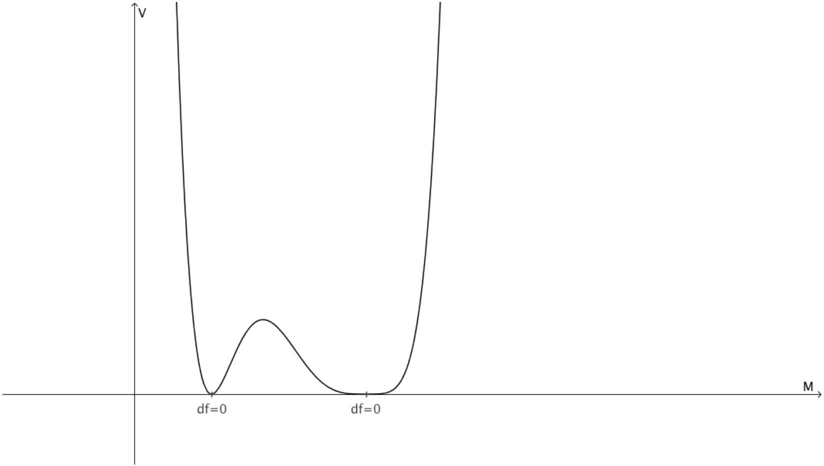
A picture of the potential as s increase.



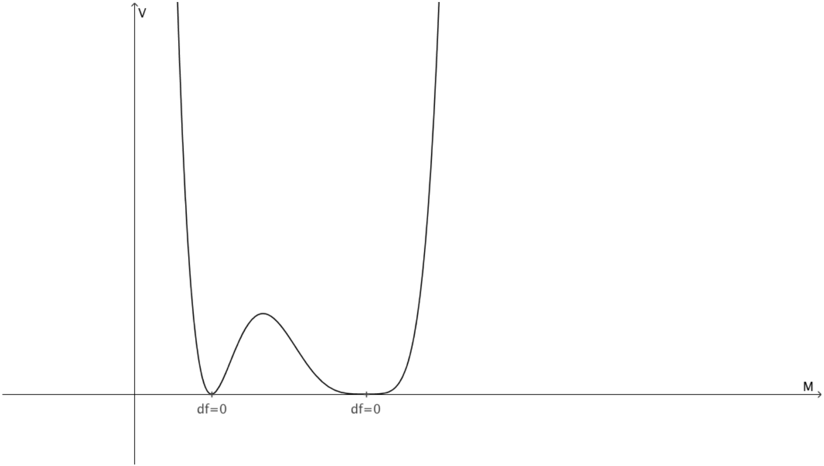
A picture of the potential as s increase.



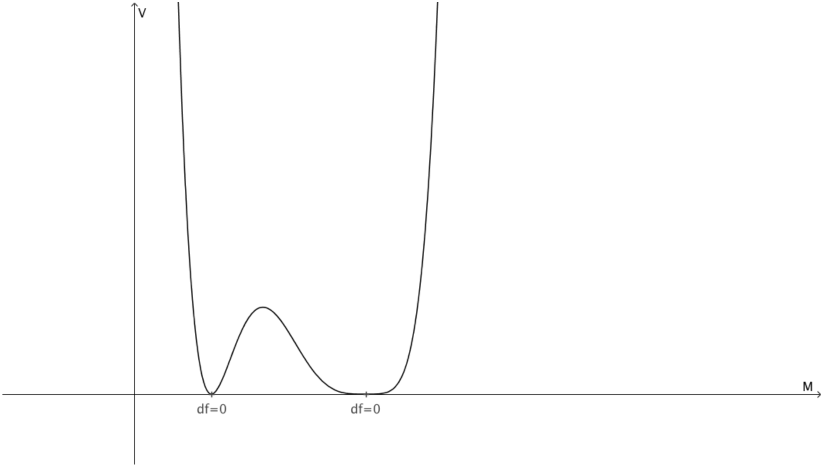
A picture of the potential as s increase.



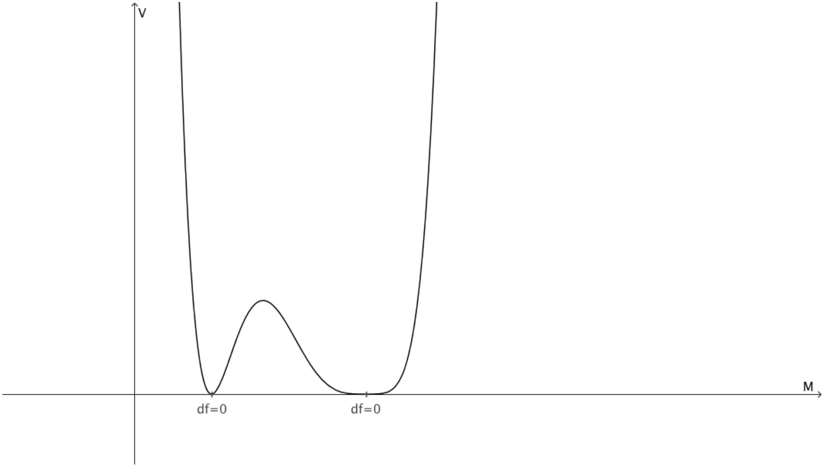
A picture of the potential as s increase.



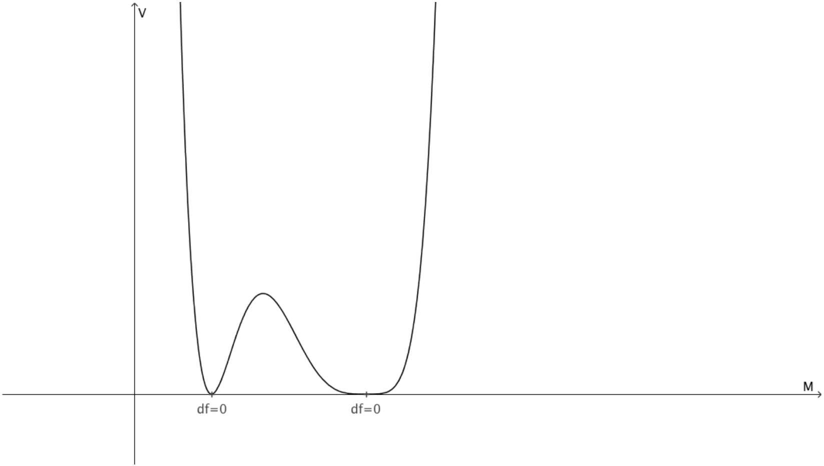
A picture of the potential as s increase.



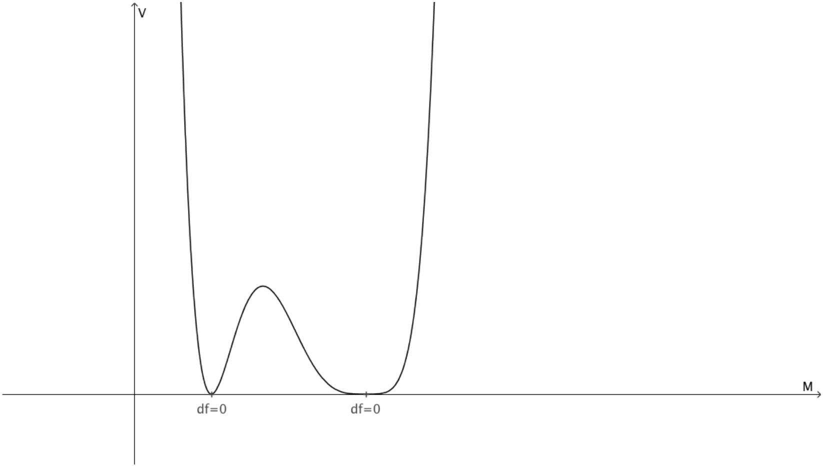
A picture of the potential as s increase.



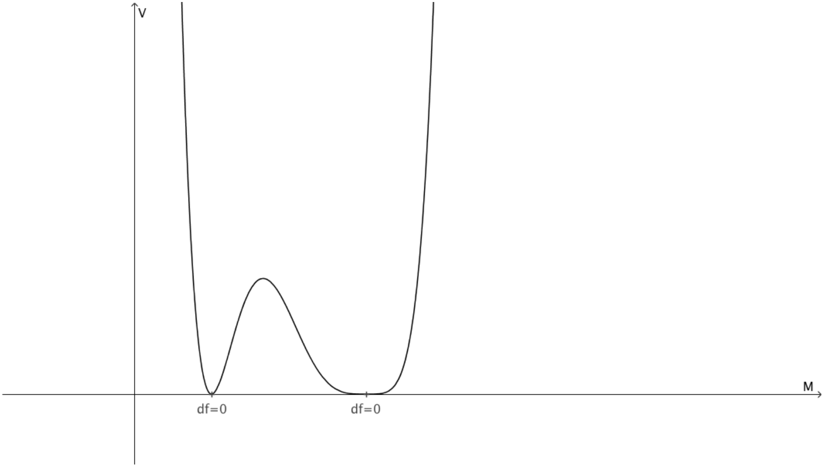
A picture of the potential as s increase.



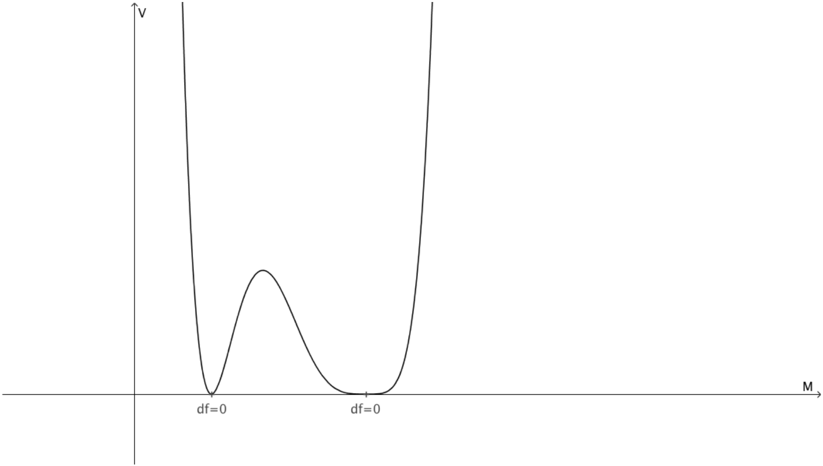
A picture of the potential as s increase.



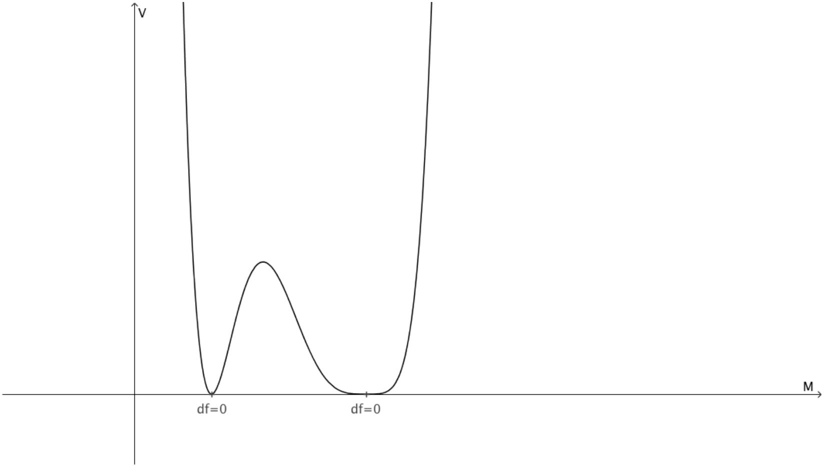
A picture of the potential as s increase.



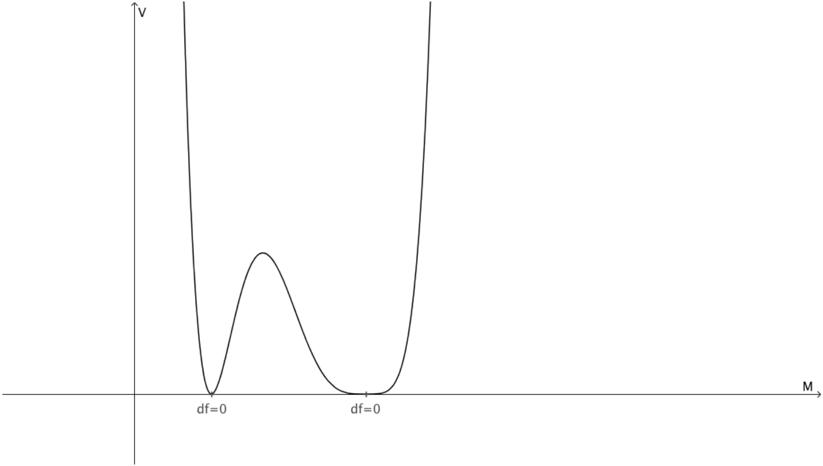
A picture of the potential as s increase.



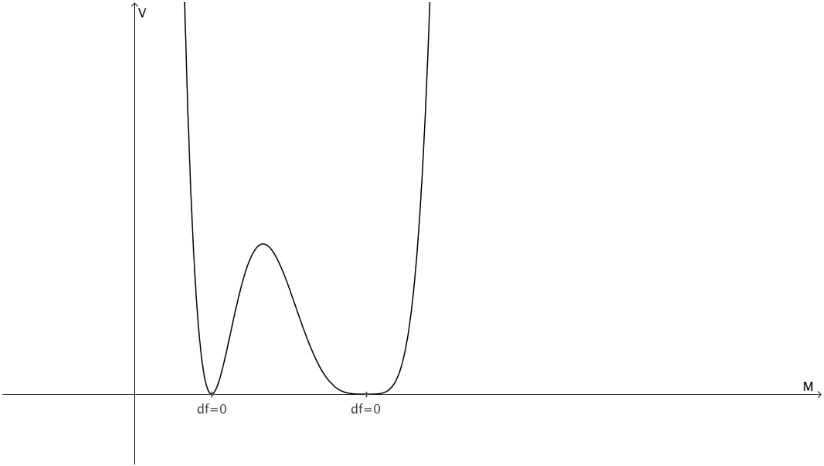
A picture of the potential as s increase.



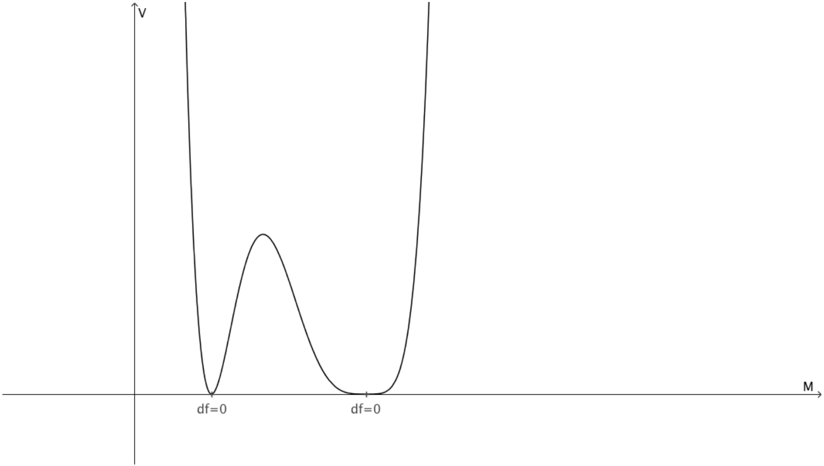
A picture of the potential as s increase.



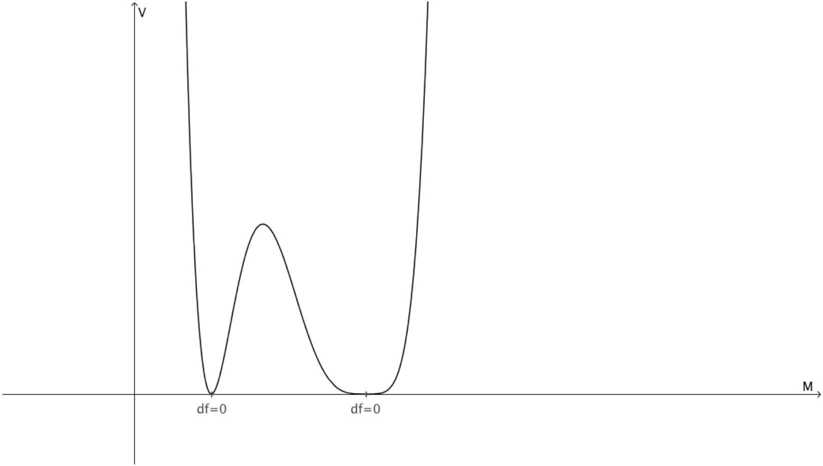
A picture of the potential as s increase.



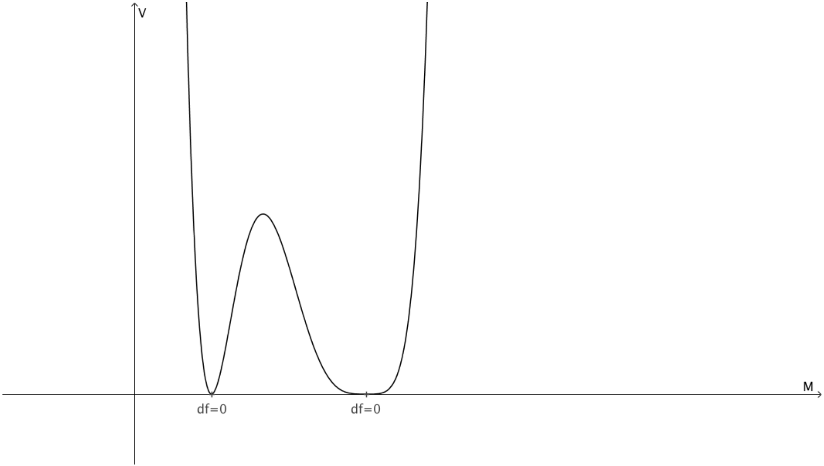
A picture of the potential as s increase.



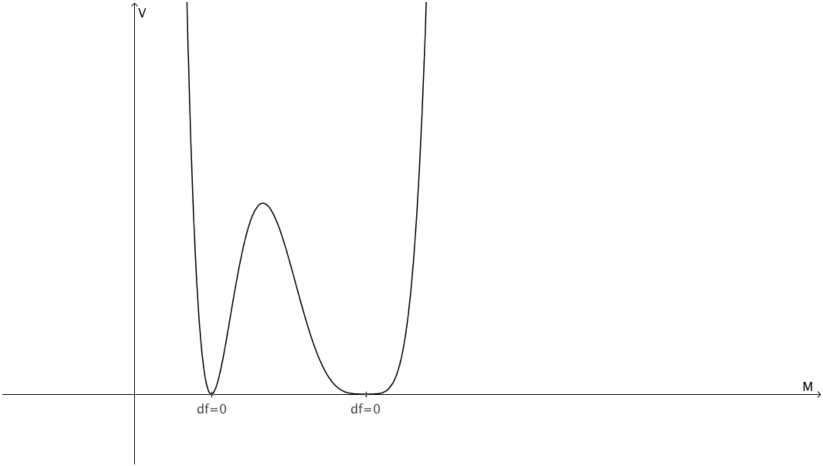
A picture of the potential as s increase.



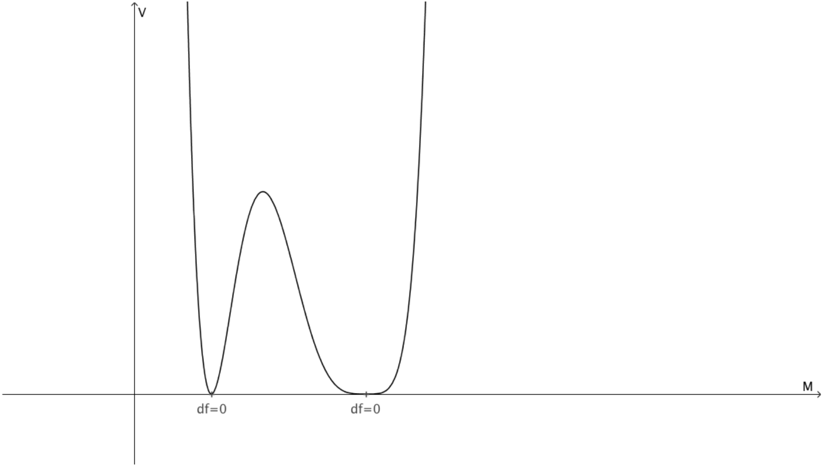
A picture of the potential as s increase.



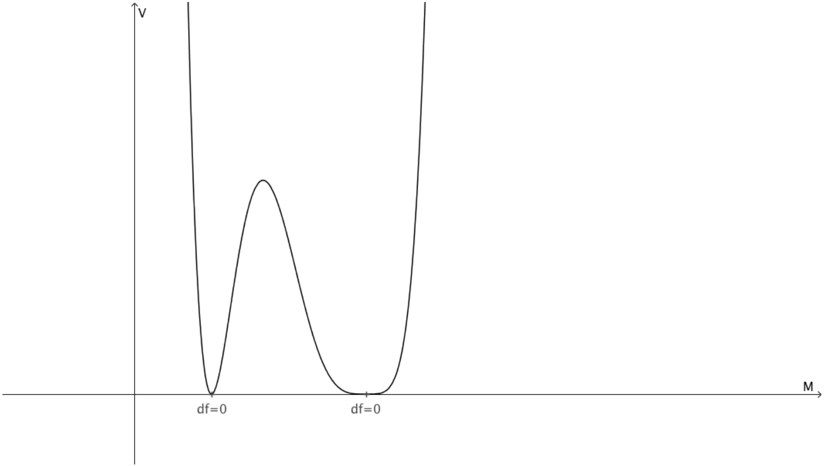
A picture of the potential as s increase.



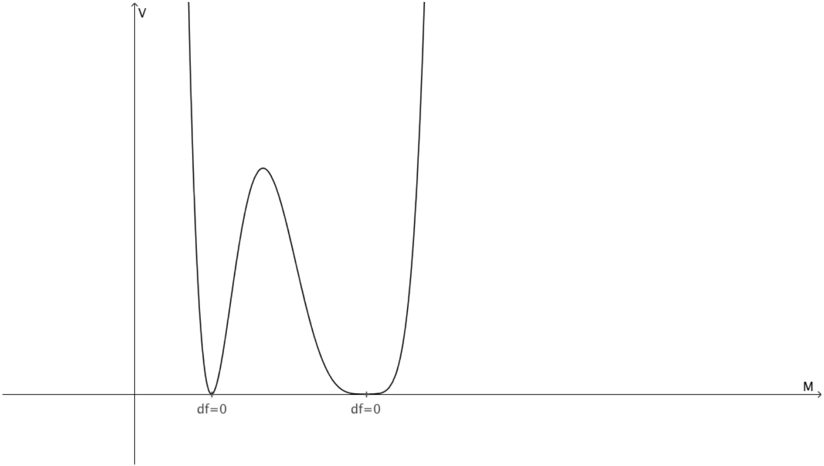
A picture of the potential as s increase.



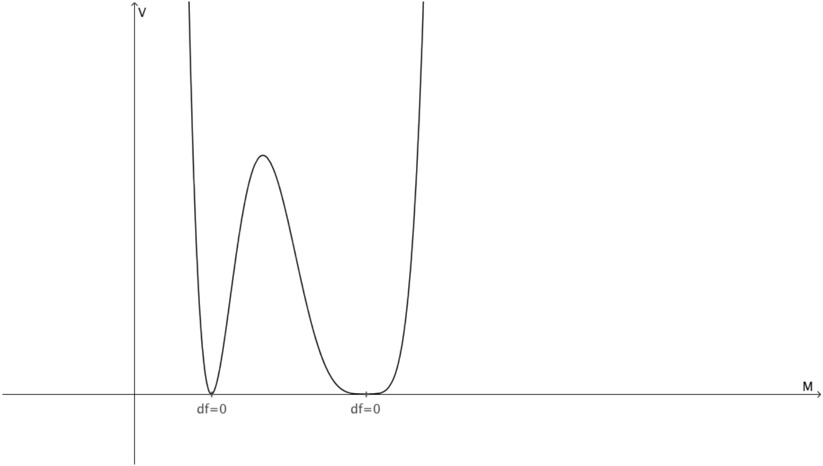
A picture of the potential as s increase.



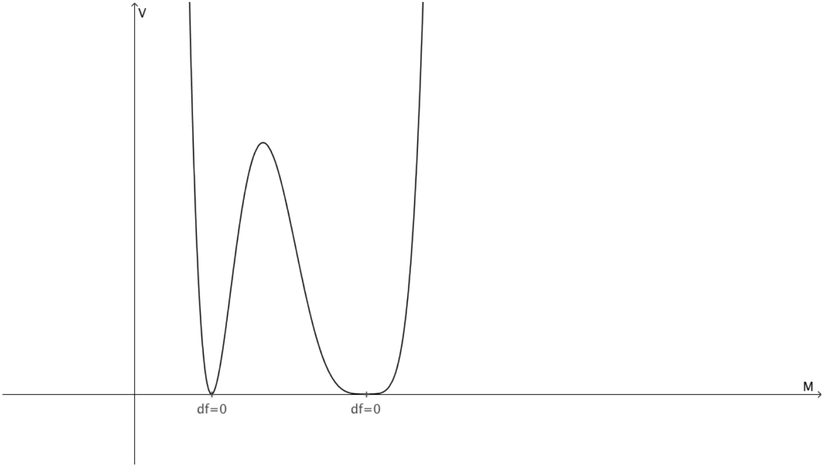
A picture of the potential as s increase.



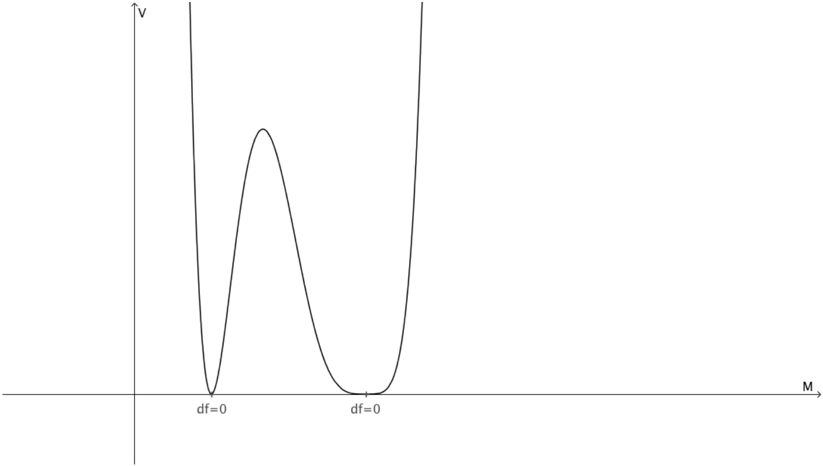
A picture of the potential as s increase.



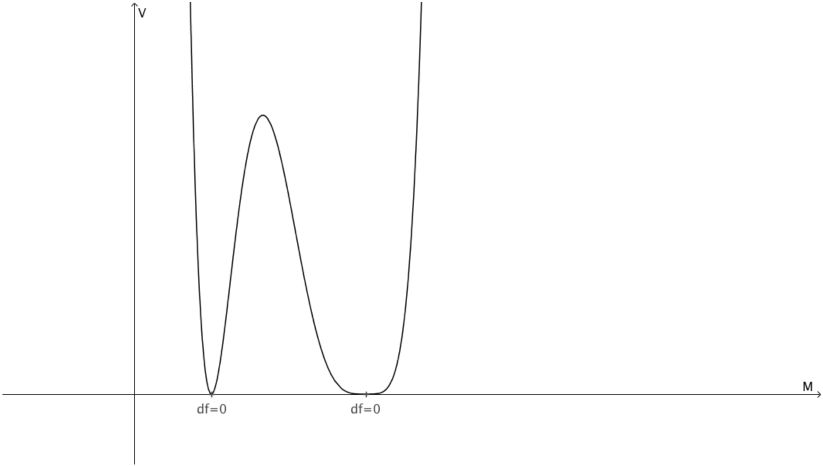
A picture of the potential as s increase.



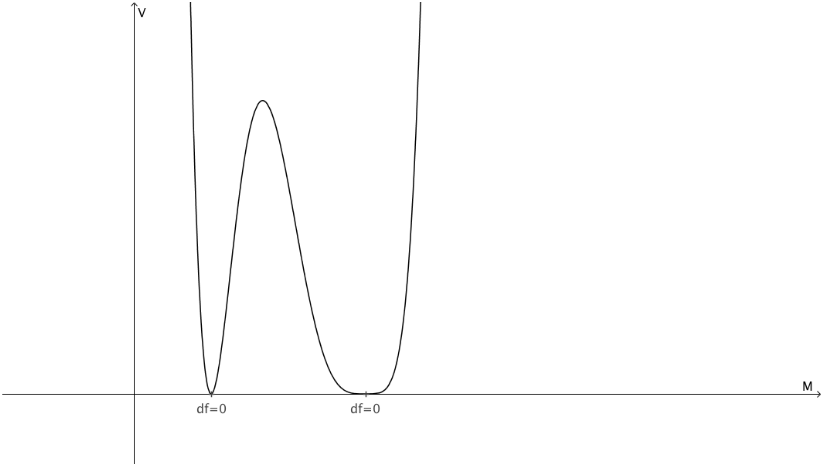
A picture of the potential as s increase.



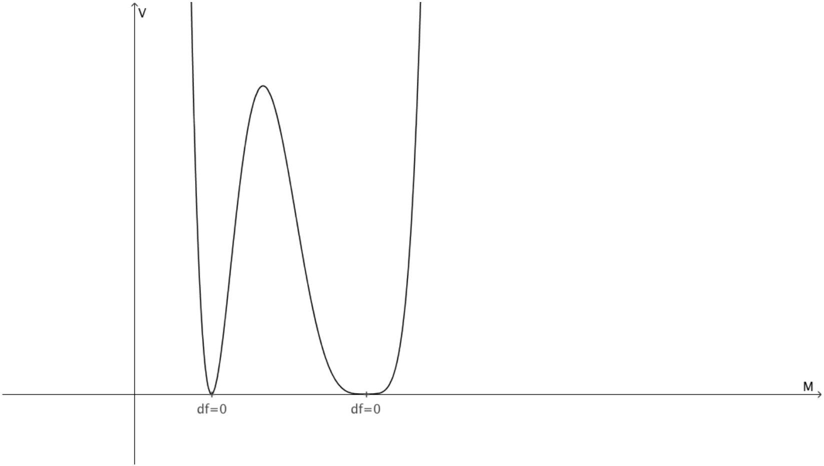
A picture of the potential as s increase.



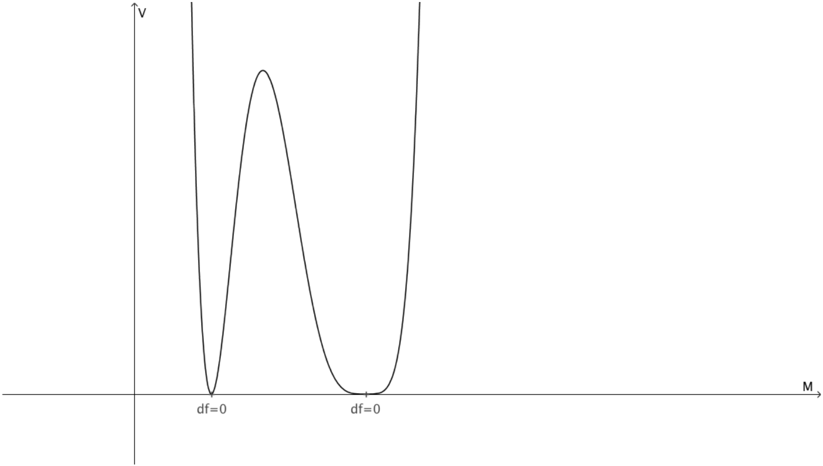
A picture of the potential as s increase.



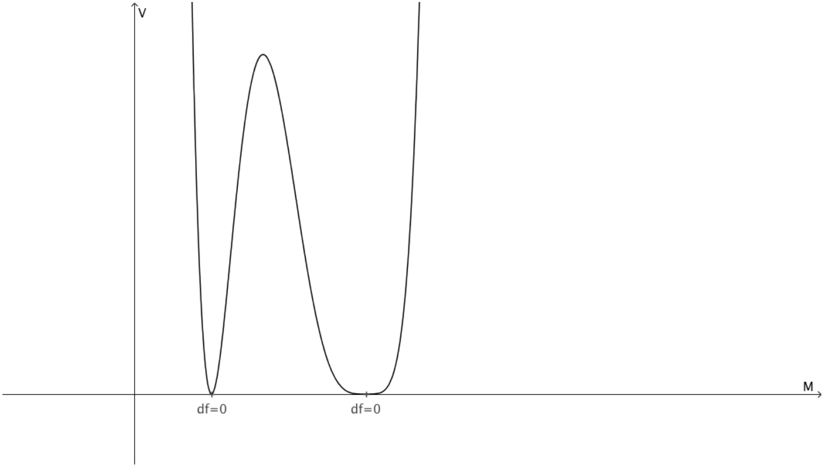
A picture of the potential as s increase.



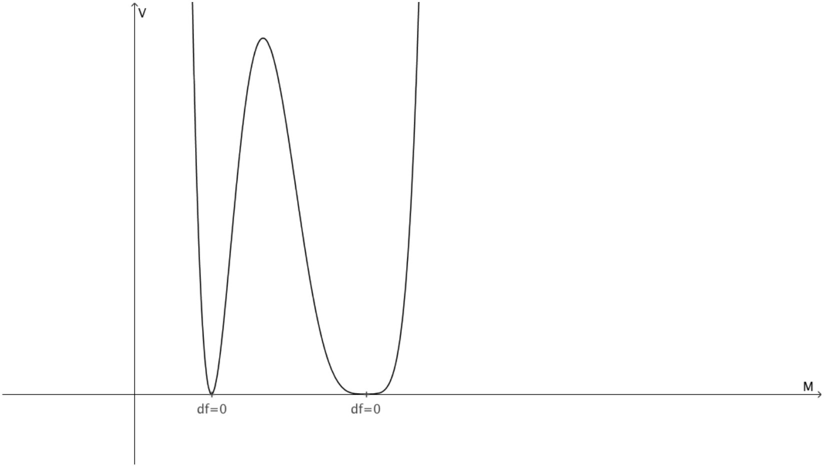
A picture of the potential as s increase.



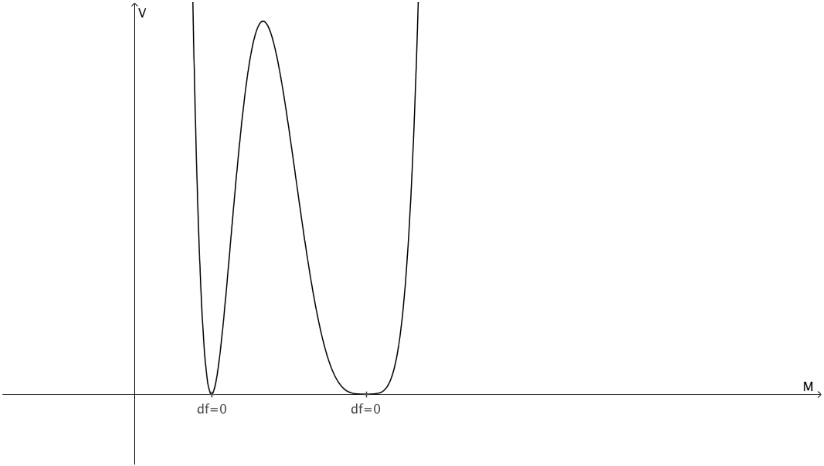
A picture of the potential as s increase.



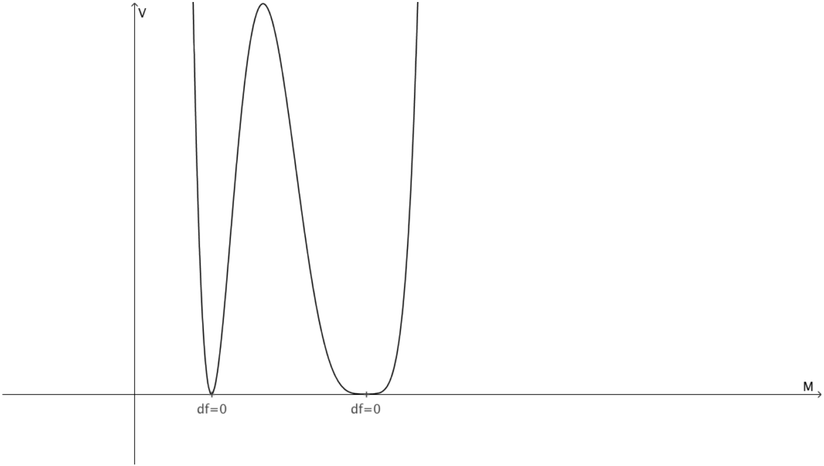
A picture of the potential as s increase.



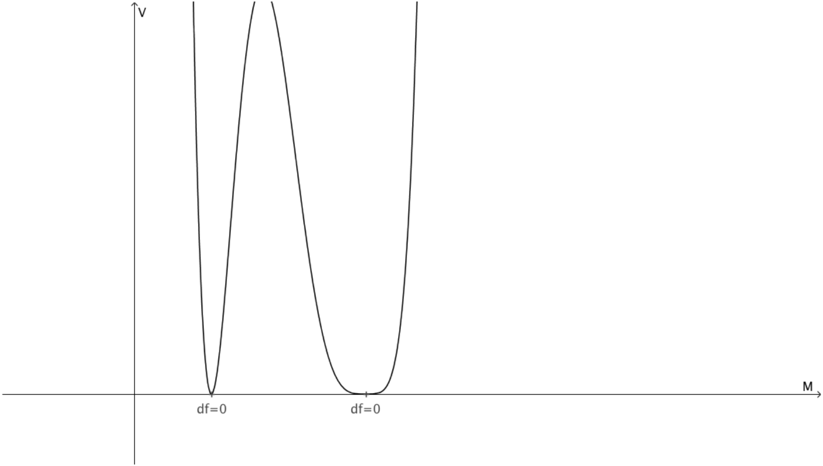
A picture of the potential as s increase.



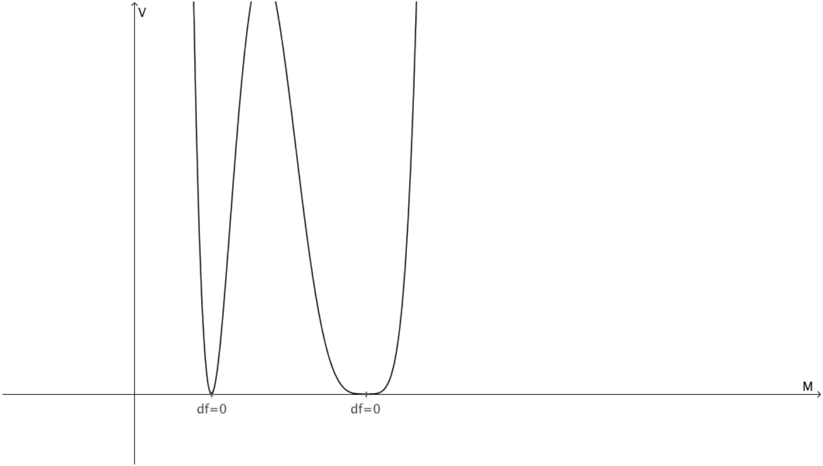
A picture of the potential as s increase.



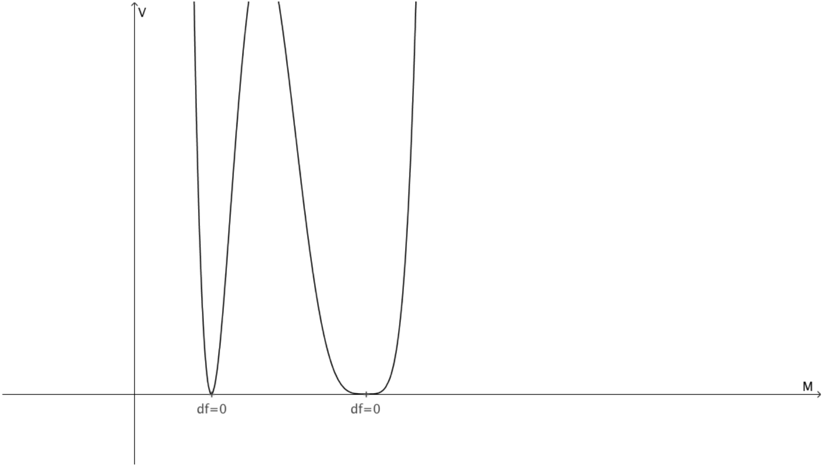
A picture of the potential as s increase.



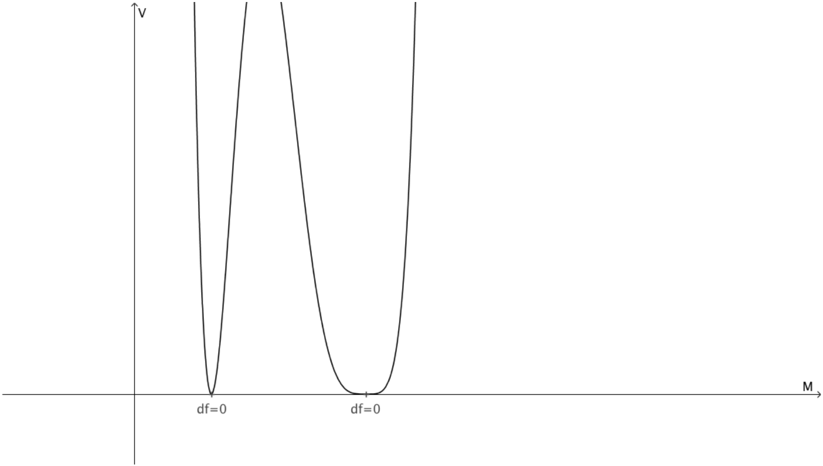
A picture of the potential as s increase.



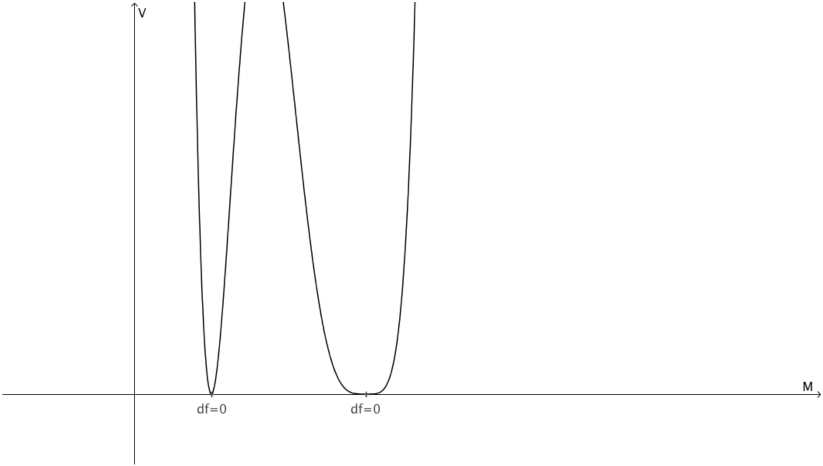
A picture of the potential as s increase.



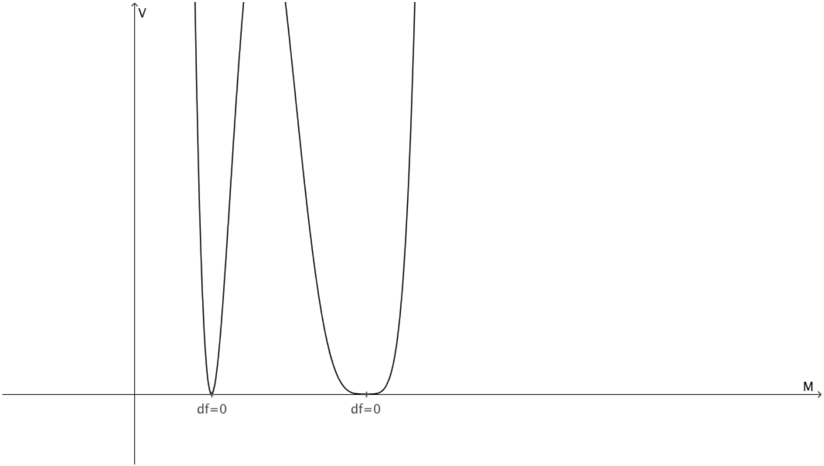
A picture of the potential as s increase.



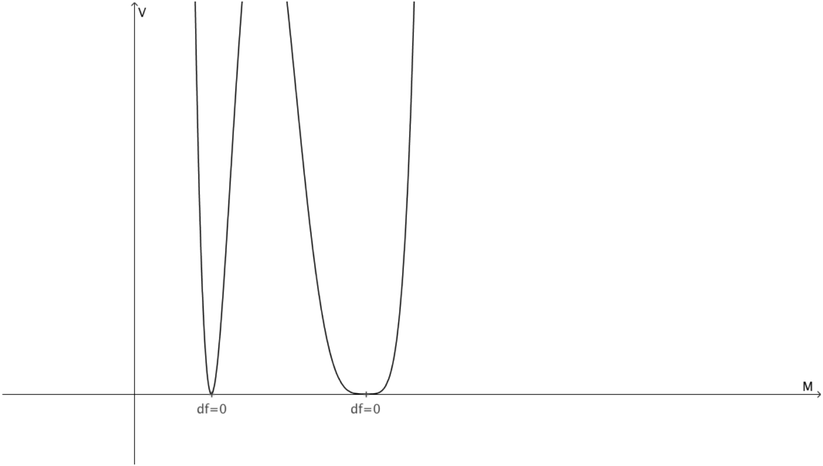
A picture of the potential as s increase.



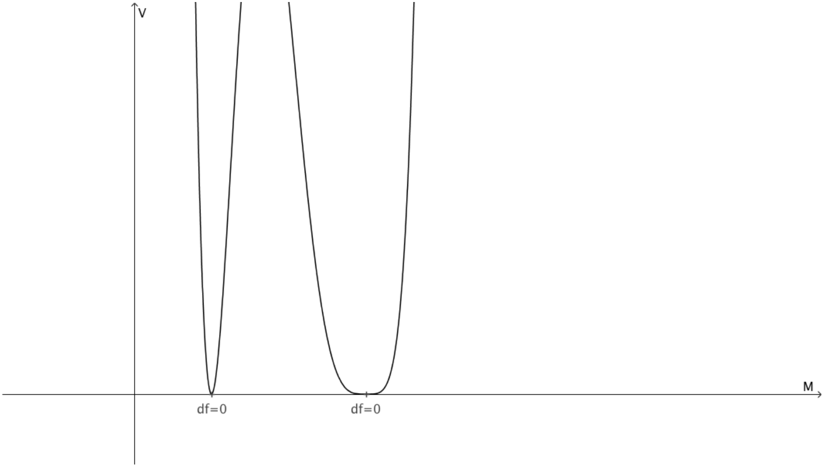
A picture of the potential as s increase.



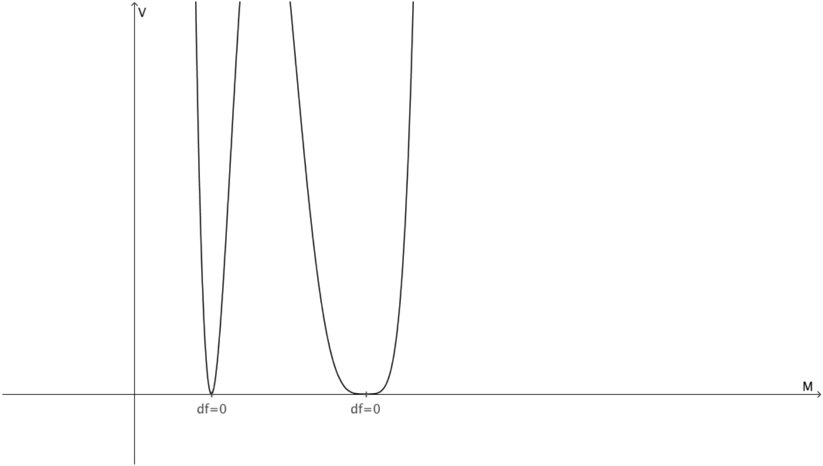
A picture of the potential as s increase.



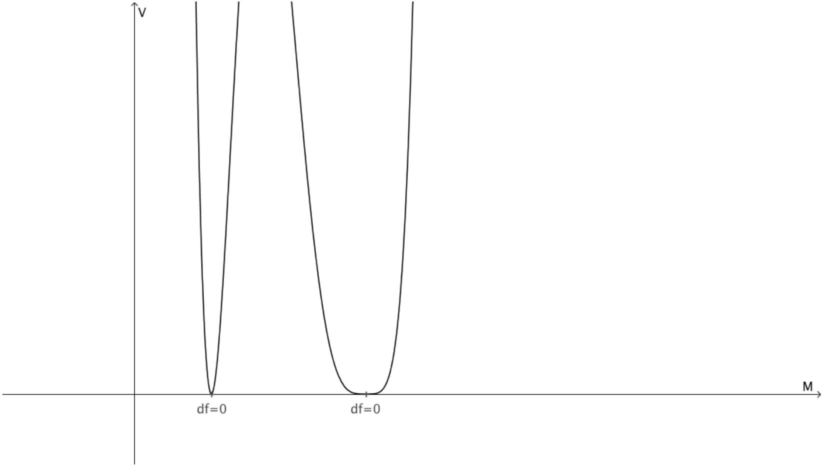
A picture of the potential as s increase.



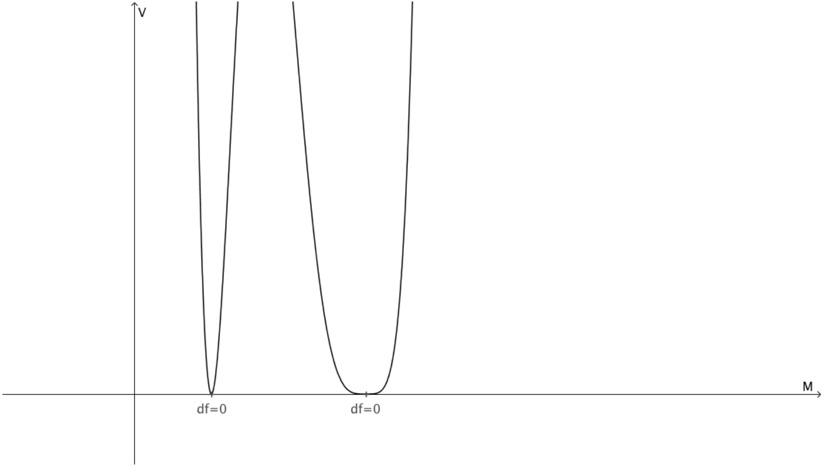
A picture of the potential as s increase.



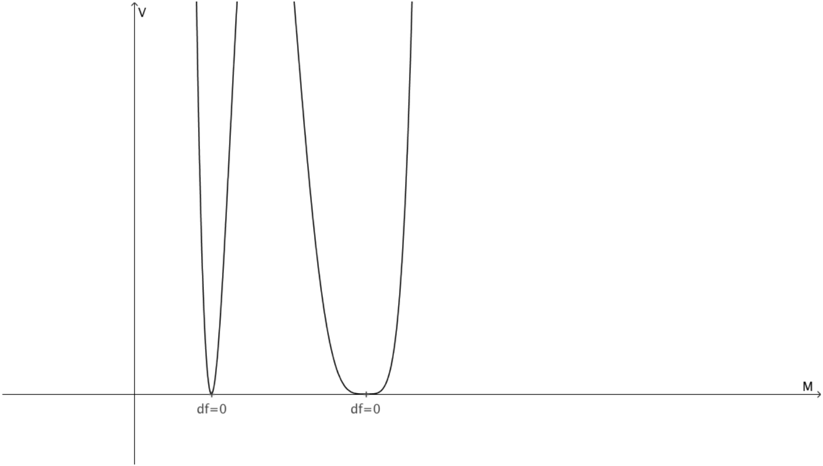
A picture of the potential as s increase.



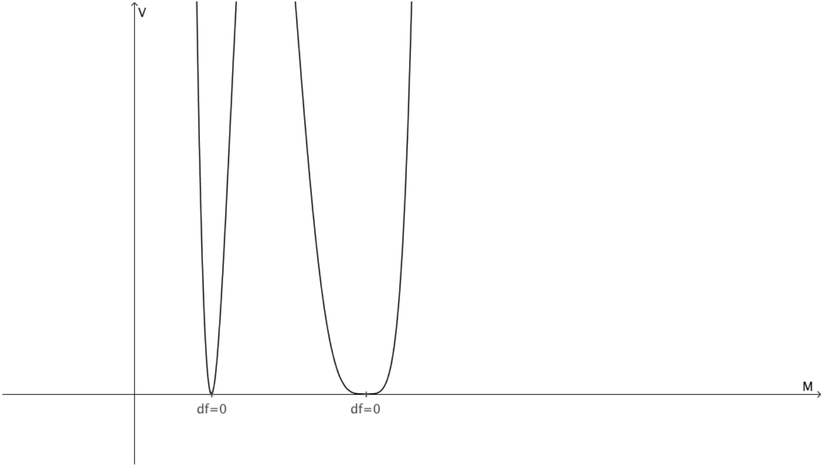
A picture of the potential as s increase.



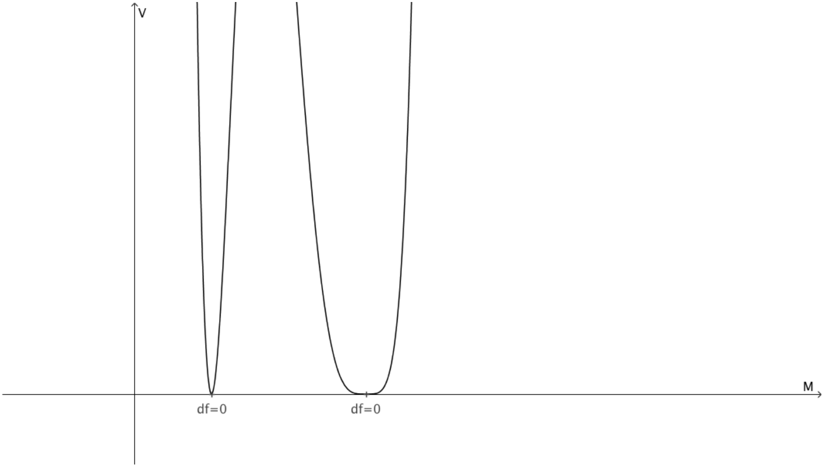
A picture of the potential as s increase.



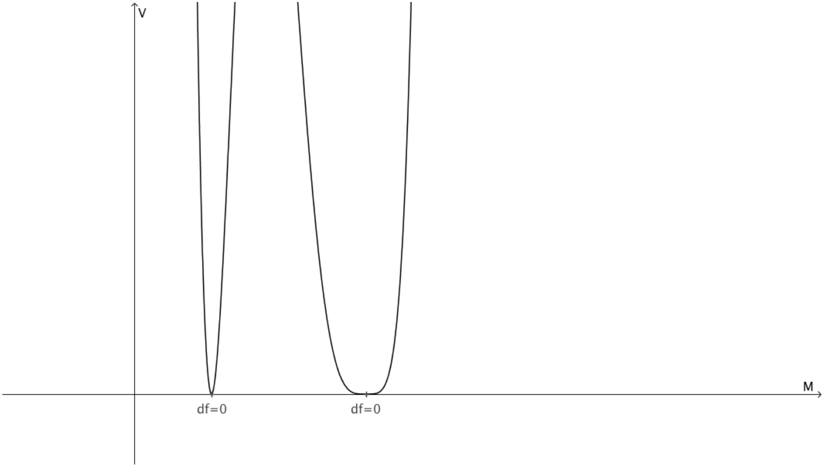
A picture of the potential as s increase.



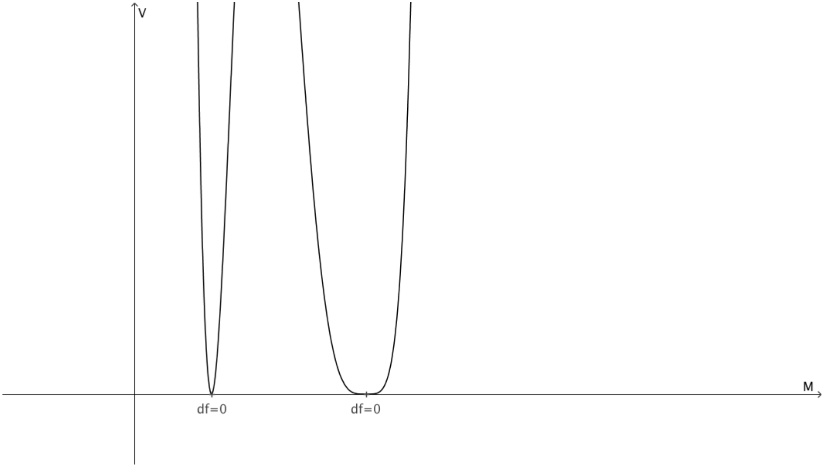
A picture of the potential as s increase.



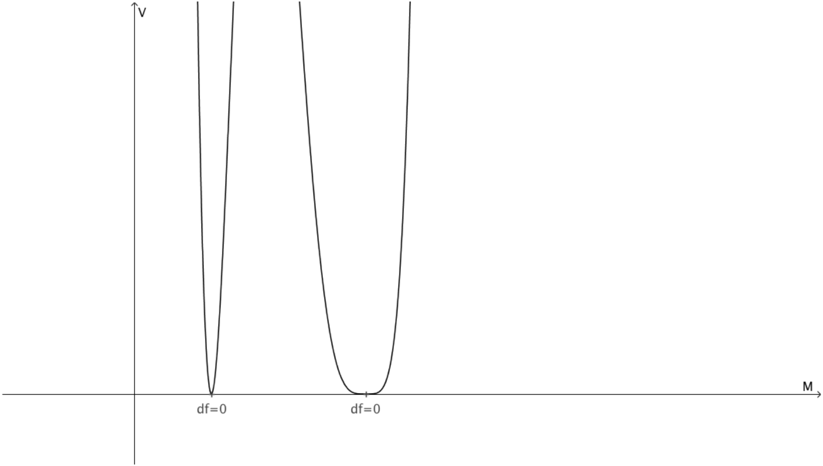
A picture of the potential as s increase.



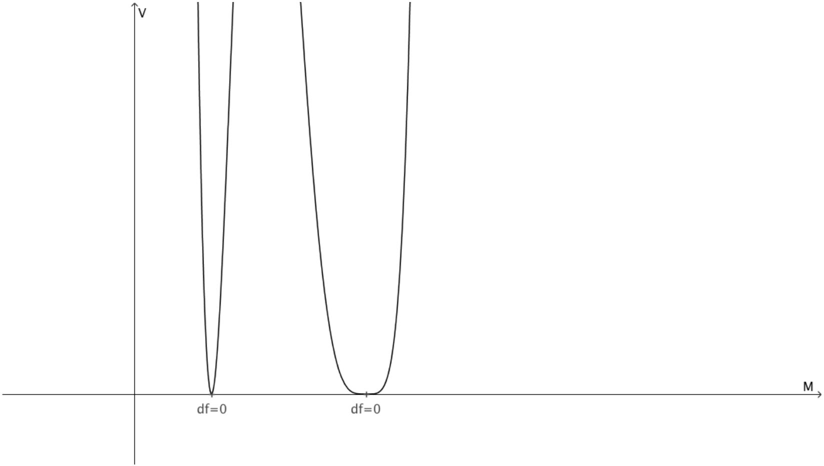
A picture of the potential as s increase.



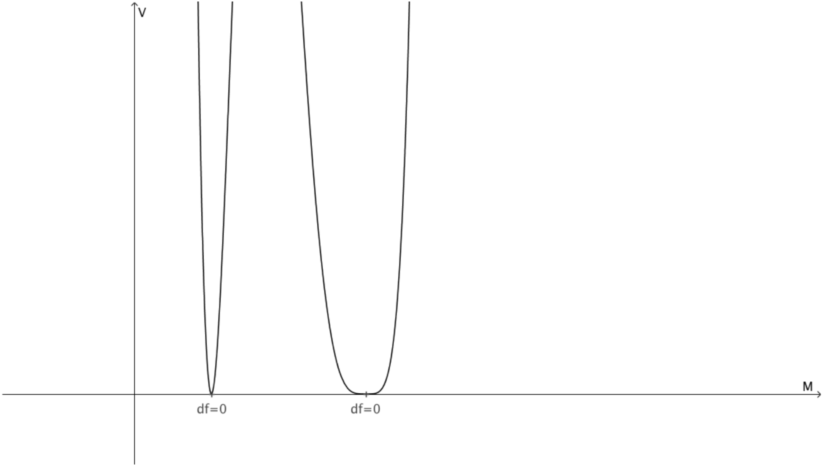
A picture of the potential as s increase.



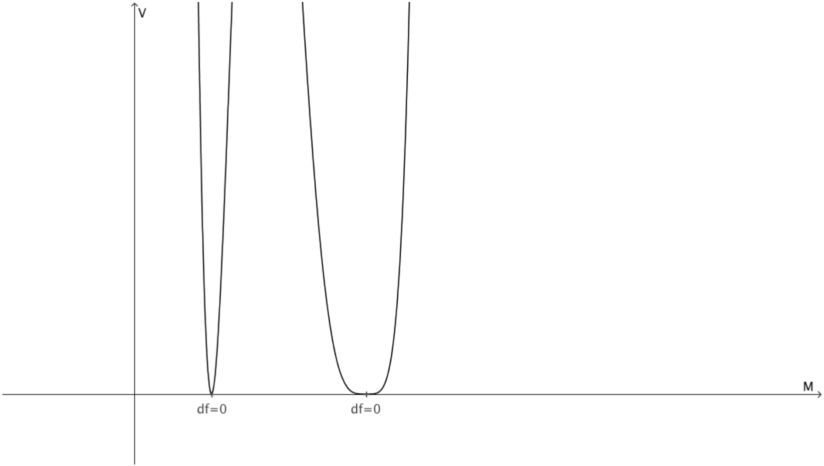
A picture of the potential as s increase.



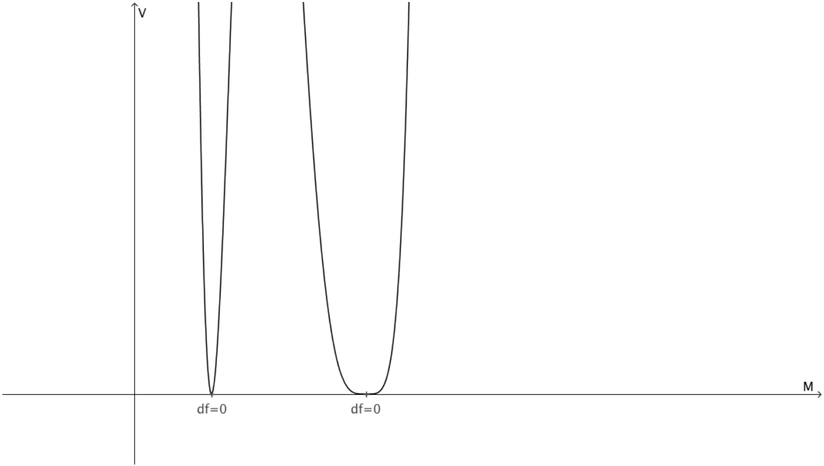
A picture of the potential as s increase.



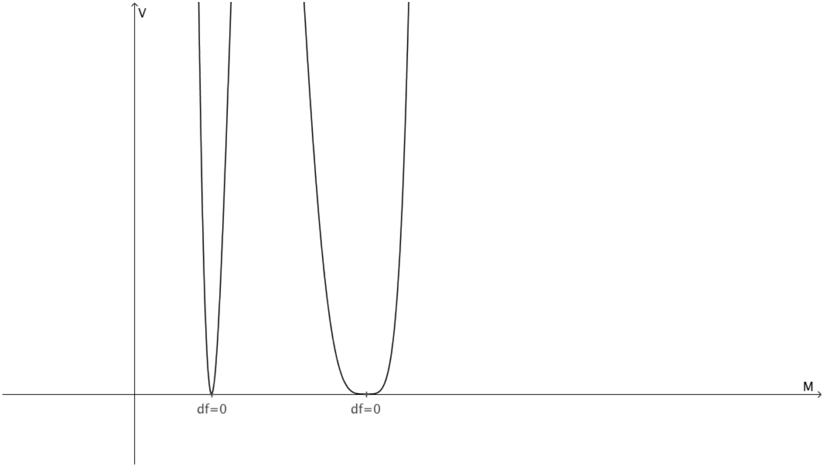
A picture of the potential as s increase.



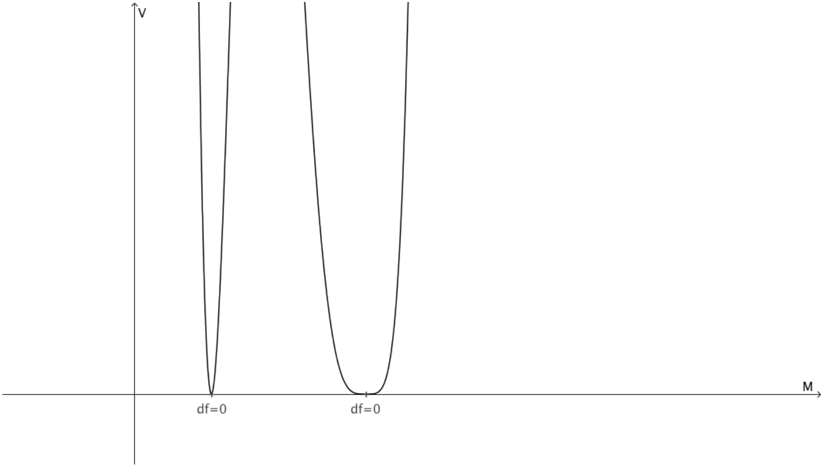
A picture of the potential as s increase.



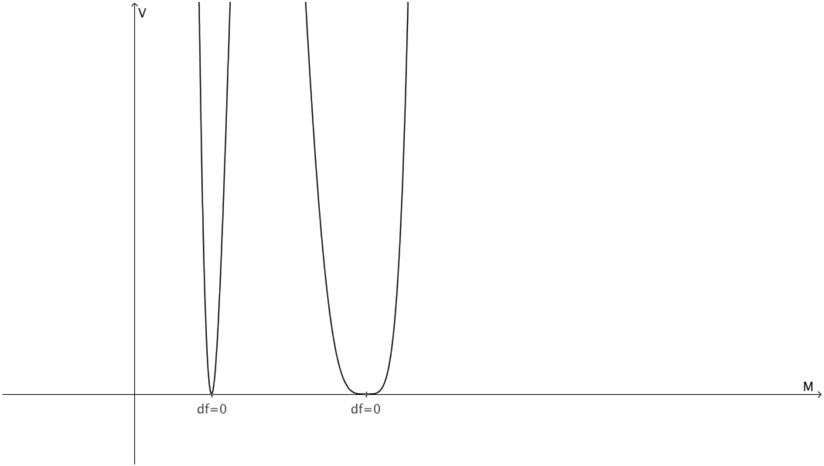
A picture of the potential as s increase.



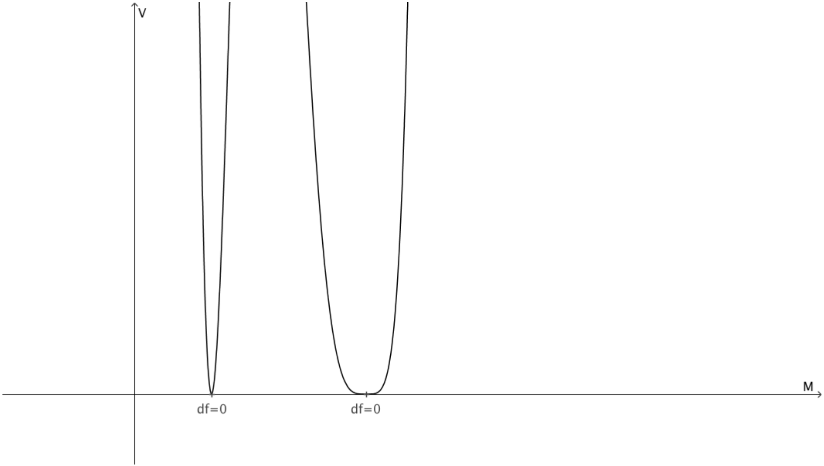
A picture of the potential as s increase.



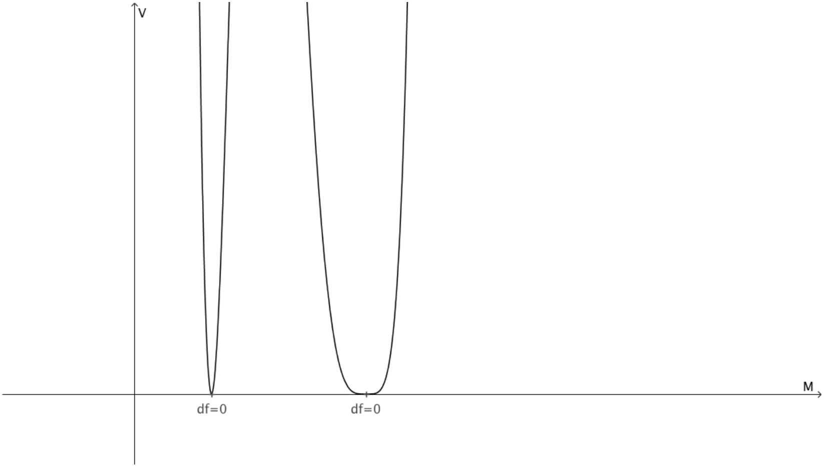
A picture of the potential as s increase.



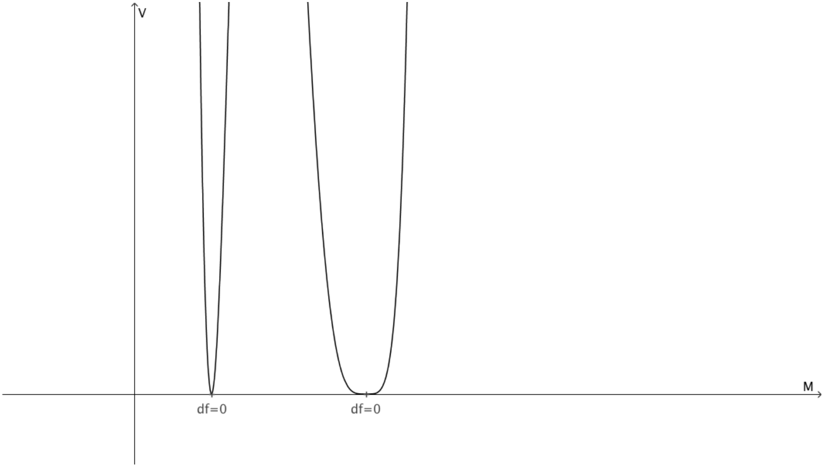
A picture of the potential as s increase.



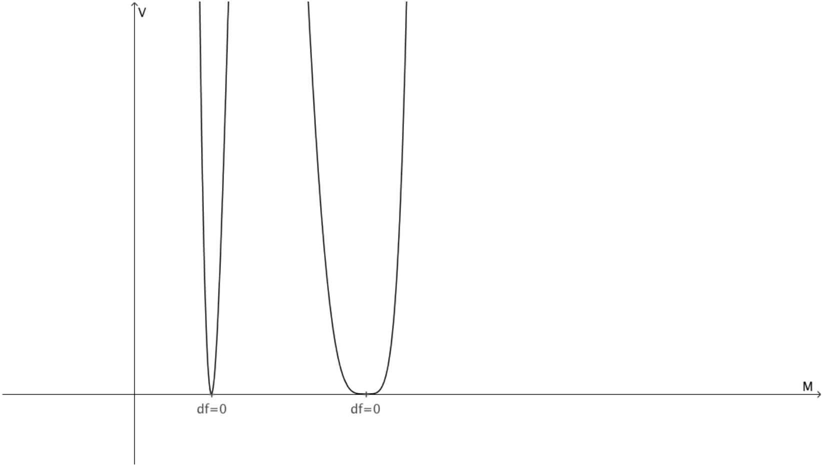
A picture of the potential as s increase.



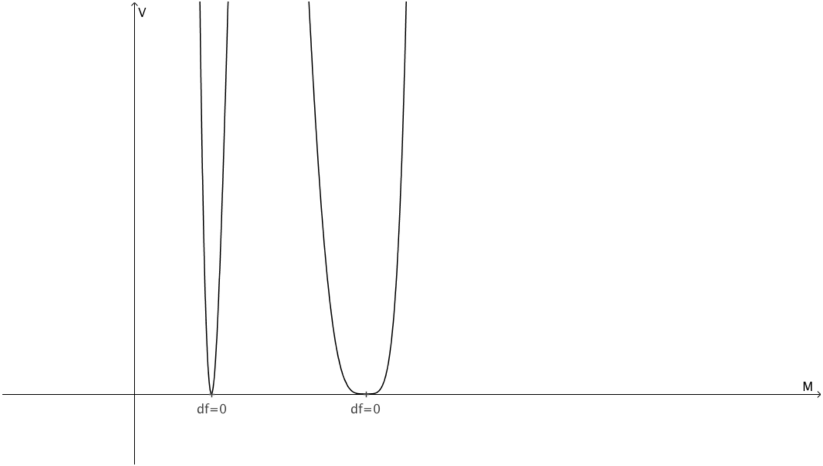
A picture of the potential as s increase.



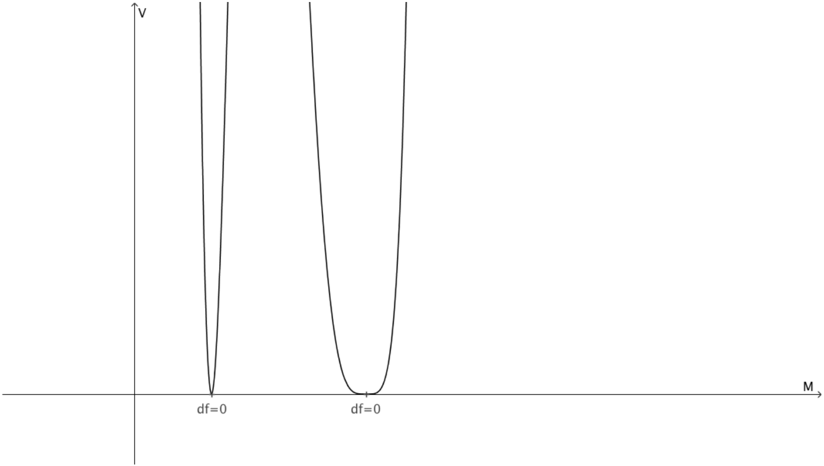
A picture of the potential as s increase.



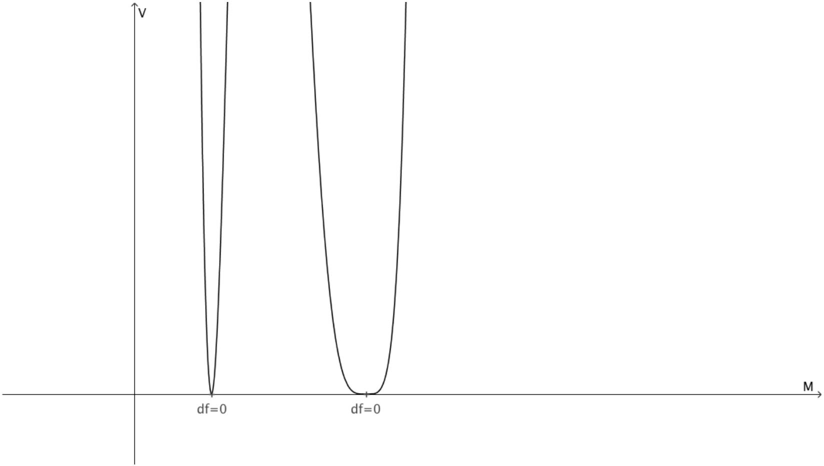
A picture of the potential as s increase.



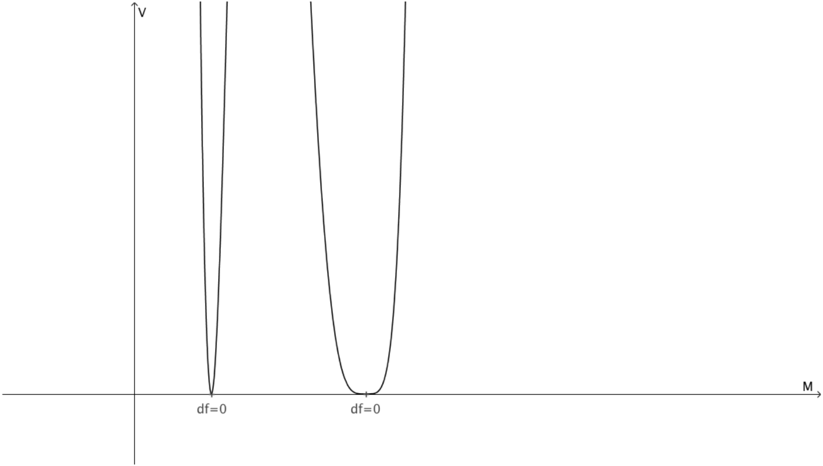
A picture of the potential as s increase.



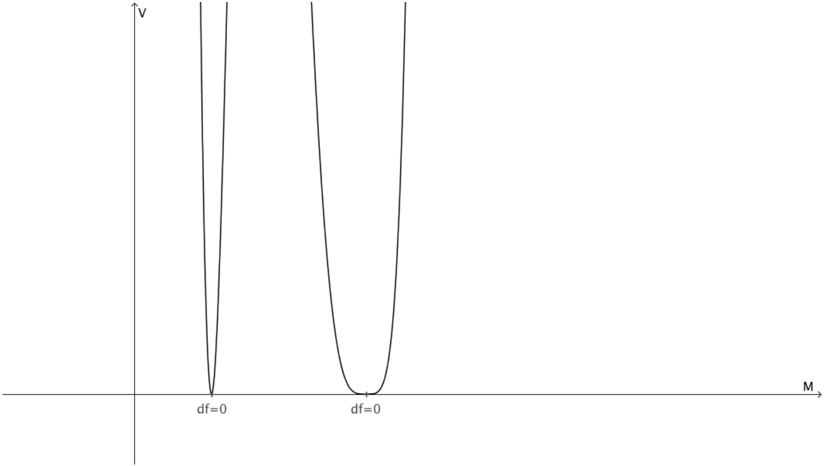
A picture of the potential as s increase.



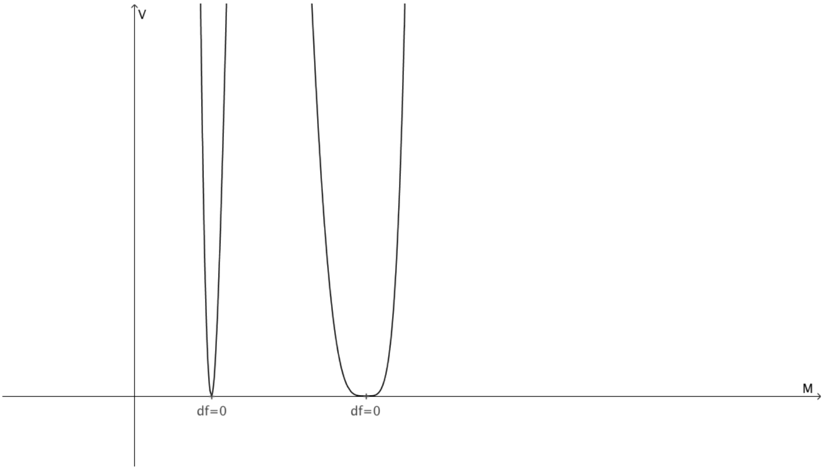
A picture of the potential as s increase.



A picture of the potential as s increase.



A picture of the potential as s increase.



With a good choice of coordinates around p,

$$H_s = s \sum_{i=1}^{m} \left(-\frac{\partial^2}{\partial x^i \partial x^i} + (\lambda^i x^i)^2 + \lambda^i \left[(a^i)^\dagger, a^i \right] \right) + O\big(\sqrt{s}\big),$$

where λ^i are the eigenvalues of the Hessian in p. The leading-order Hamiltonian is separated into Hamiltonians of the form

$$H' = -\frac{\partial^2}{\partial x^2} + \lambda^2 x^2 + \lambda \big[\epsilon_{dx}, i_\partial \big],$$

which is the sum of an harmonic oscillator Hamiltonian and the operator $\lambda \big[\epsilon_{dx}, i_\partial \big]$.

The eigenvalues at leading order are

$$s \sum_{i=1}^{m} \Big(|\lambda^i|(1 + 2N^{(i)}) \pm \lambda^i \Big), \qquad N^{(i)} = 0, 1, 2, \ldots$$

We have just one possibility to get zero: $N^{(i)} = 0$ and $+1$ if $\lambda^i < 0$, -1 if $\lambda^i > 0$. The choice corresponds to a k-form if and only if $\mu_p = k$.

Thus, the leading order Hamiltonian acting on $\Omega^k(M)$ has kernel dimension equal to M_k.

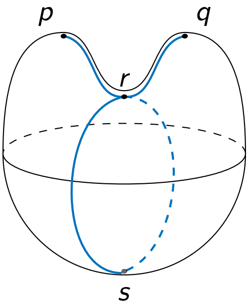
Since H_s do not necessarily annihilate such forms (it is just its leading coefficient that vanishes), we have established the weak Morse inequalities

$$\beta_k \leq M_k.$$

We have seen that

$$\text{perturbative analysis} \quad \Longrightarrow \quad \text{weak Morse inequalities.}$$

To learn something new we must perform a calculation which is sensitive to the existence on M of more than one critical point: we must allow for the possibility of "tunnelling" from one critical point to another. This non-perturbative effect can be calculated via semiclassical trajectories called instantons.



Expansion of the perturbative ground states

We already found a perturbative vacua ω_p for every critical point p, which is a μ_p-form. In the full theory, we expect that

$$Q\omega_p = \sum_{q \text{ critical pt}} \langle \omega_q, Q\omega_p \rangle \, \omega_q \; + \; \begin{array}{c}\text{expansion in}\\ \text{higher energetic states}\end{array}$$

The expansion coefficients (*i.e.* the tunnelling transition amplitudes from p to q) can be expressed as a path integral.

Tunnelling amplitudes

Computing the path integral, we get

$$\langle \omega_q, Q\omega_p \rangle = e^{-s\left(f(q)-f(p)\right)} \sum_{\substack{\gamma \\ \text{from } p \text{ to } q}} n_\gamma$$

where

- q is a critical point such that $\mu_q = \mu_p + 1$
- $\gamma\colon \mathbb{R} \to M$ are the steepest ascent paths:

$$\begin{cases} \dot{\gamma}^i = s\, g^{ij} \frac{\partial f}{\partial x^j} \\ \gamma(-\infty) = p,\ \gamma(+\infty) = q \end{cases}$$

- $n_\gamma = \pm 1$ is determined by means the orientation of M and the and by the steepest ascent γ

Action of Q

Properly normalizing the perturbative vacua, we find the action of Q on every ω_p

Q action on perturbative vacua

$$Q\omega_p = \sum_{\mu_q=\mu_p+1} \sum_{\substack{\gamma \\ \text{from } p \text{ to } q}} n_\gamma \, \omega_q .$$

Further, we it can be shown that

$$Q^2 = 0.$$

These facts suggest the following definition.

Morse-Smale-Witten cochain complex

Define the graded space of perturbative ground states

$$C^k = \bigoplus_{\mu_p = k} \mathbb{R}\left\langle \omega_p \right\rangle ,$$

we find the cochain complex with the coboundary operator given by Q

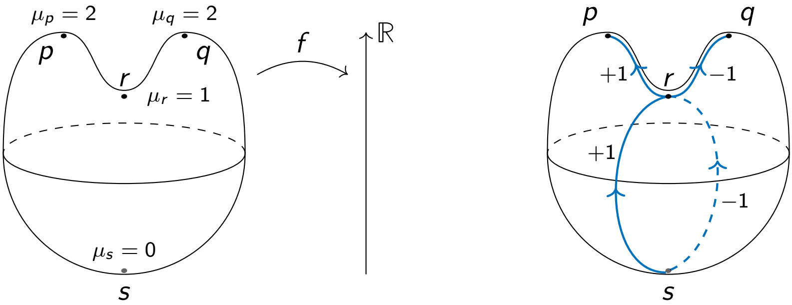
$$0 \longrightarrow C^0 \xrightarrow{Q} \cdots \xrightarrow{Q} C^m \longrightarrow 0$$

The pair $(C^\bullet, Q)$ is called the Morse-Smale-Witten cochain complex of f.
From

$$\beta_k = \dim\ker\left(H_s \colon \Omega^k(M) \to \Omega^k(M)\right)$$

and the fact that $H\omega = 0$ iff $Q\omega = Q^\dagger\omega = 0$, it can be shown that the cohomology of $(C^\bullet, Q)$ is nothing but the de Rham cohomology of M.

Example: horned sphere's height



$$0 \longrightarrow \mathbb{R}\langle \omega_s \rangle \xrightarrow{0} \mathbb{R}\langle \omega_r \rangle \xrightarrow{\omega_p - \omega_q} \mathbb{R}\langle \omega_p \rangle \oplus \mathbb{R}\langle \omega_q \rangle \longrightarrow 0$$

Example: horned sphere's height

$$0 \longrightarrow \mathbb{R}\langle\omega_s\rangle \xrightarrow{0} \mathbb{R}\langle\omega_r\rangle \xrightarrow{\omega_p-\omega_q} \mathbb{R}\langle\omega_p\rangle \oplus \mathbb{R}\langle\omega_q\rangle \longrightarrow 0$$

The cohomology will be

$$H^0\big((C^\bullet, Q)\big) = \frac{\mathbb{R}\langle\omega_s\rangle}{(0)} \cong \mathbb{R}$$

$$H^1\big((C^\bullet, Q)\big) = \frac{(0)}{(0)} \cong 0$$

$$H^2\big((C^\bullet, Q)\big) = \frac{\mathbb{R}\langle\omega_p\rangle \oplus \mathbb{R}\langle\omega_q\rangle}{\mathbb{R}\langle\omega_p - \omega_q\rangle} \cong \mathbb{R}$$

Thank you for the attention

Short bibliography

Hori, Kentaro et al. (2003). *Mirror Symmetry*. American Mathematical Society.

Milnor, John (1963). *Morse Theory*. Princeton University Press.

Rogers, Alice (2000). "The Topological particle and Morse theory". In: *Class. Quant. Grav.* 17.

Witten, Edward (1982). "Supersymmetry and Morse theory". In: *J. Differential Geom.* 17.4.

Euclidean action

The starting point is the Euclidean action associated to the previous model. In flat space, it is

$$S_E[\phi,\bar{\psi},\psi] = \int_{\mathbb{R}} dt \bigg(\frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j + \frac{1}{2} g_{ij} (\bar{\psi}^i \dot{\psi}^j - \dot{\bar{\psi}}^i \psi^j) + \\ + s \frac{\partial^2 f}{\partial x^i \partial x^j} \bar{\psi}^i \psi^j + \frac{1}{2} s^2 g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \bigg)$$

where $\phi \in C^\infty(\mathbb{R}, M)$ represent a boson and $\psi, \bar{\psi} \in \Gamma^\infty(\mathbb{R}, \phi^* TM \otimes \mathbb{C})$ its fermionic superpartner.

Expansion of the perturbative ground states

We already found a perturbative vacua ω_p for every critical point p, which is a μ_p-form. In the full theory, we expect that

$$Q\omega_p = \sum_{q \text{ critical pt}} \langle \omega_q, Q\omega_p \rangle \, \omega_q \; + \; \begin{array}{c}\text{expansion in} \\ \text{higher energetic states}\end{array}$$

The expansion coefficients (*i.e.* the tunnelling transition amplitudes from p to q) can be expressed as the path integral

$$\langle \omega_q, Q\omega_p \rangle = \frac{1}{f(q) - f(p)} \int \mathcal{D}\phi \mathcal{D}\bar{\psi} \mathcal{D}\psi \; \bar{\psi}^i \frac{\partial f}{\partial x^i} \, e^{-S_E[\phi, \bar{\psi}, \psi]}.$$

Ingredients

To compute

$$\int \mathcal{D}\phi \mathcal{D}\bar{\psi} \mathcal{D}\psi \; \bar{\psi}^i \frac{\partial f}{\partial x^i} \, e^{-S_E[\phi,\bar{\psi},\psi]}$$

with the saddle-point method we need:

- the minima of the bosonic action
- the bosonic and fermion determinants
- the zero modes

The answer will be

$$\sum_{\text{minima}} e^{-S_{\mathbf{b}}} \frac{\text{fermionic determinant'}}{\sqrt{\text{bosonic determinant'}}} \left(\int \text{zero modes} \right) \Bigg|_{\text{minima}}$$

Instantons

The minima of the bosonic action

$$S_{\mathrm{b}}[\phi] = \int_{\mathbb{R}} dt \left(\frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j + \frac{1}{2} s^2 g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)$$

$$= \frac{1}{2} \int_{\mathbb{R}} dt \left| \dot{\phi}^i - s\, g^{ij} \frac{\partial f}{\partial x^j} \right|^2 + s\big(f(q) - f(p)\big)$$

are given by the steepest ascent paths (instantons)

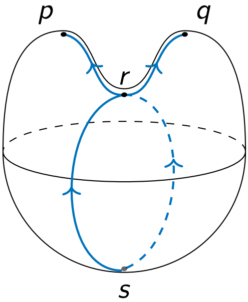
$$\dot{\gamma}^i = s\, g^{ij} \frac{\partial f}{\partial x^j}, \qquad \gamma(-\infty) = p,\ \gamma(+\infty) = q.$$

First ingredient

$$e^{-S_{\mathbf{b}}|_{\mathrm{minima}}} = e^{-s\left(f(q) - f(p)\right)}$$

Steepest ascent on the horned sphere

The steepest ascent for the height function on the horned sphere connecting critical points with relative Morse index $\Delta\mu = 1$.



Bosonic and fermionic determinants

The second variation of the bosonic action is given by

$$\mathcal{D}_- \delta\phi^i = \delta\dot{\phi}^i - s\, g^{ij} \frac{\partial^2 f}{\partial x^j \partial x^k} \delta\phi^k.$$

The fermionic action reads

$$\begin{aligned} S_{\mathrm{f}}[\bar{\psi}, \psi] &= \int_{\mathbb{R}} dt \left(\frac{1}{2} g_{ij} (\bar{\psi}^i \dot{\psi}^j - \dot{\bar{\psi}}^i \psi^j) + s \frac{\partial^2 f}{\partial x^i \partial x^j} \bar{\psi}^i \psi^j \right) \\ &= \int_{\mathbb{R}} dt\, g_{ij}\, \bar{\psi}^i \mathcal{D}_+ \psi^j = - \int_{\mathbb{R}} dt\, g_{ij}\, \mathcal{D}_- \bar{\psi}^i \psi^j \end{aligned}$$

where in general $\mathcal{D}_\pm \chi^i = \dot{\chi}^i \pm s\, g^{ij} \frac{\partial^2 f}{\partial x^j \partial x^k} \chi^k$.

Second ingredient

$$\left. \frac{\text{fermionic determinant}'}{\sqrt{\text{bosonic determinant}'}} \right|_{\text{minima}} = \left. \frac{\det' \mathcal{D}_-}{\sqrt{\det' |\mathcal{D}_-|^2}} \right|_\gamma = \pm 1$$

Zero modes

For the path integral

$$\int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \bar{\psi}^i \frac{\partial f}{\partial x^i}\, e^{-S_E[\phi,\bar{\psi},\psi]}$$

to be non-vanishing, the number of $\bar{\psi}$ zero modes (solutions of $\mathcal{D}_-\bar{\psi} = 0$) must be larger than the number of ψ zero modes (solutions of $\mathcal{D}_+\psi = 0$) by one, since there is a single insertion of $\bar{\psi}$. This is true if

$$\Delta\mu = \mu_q - \mu_p = 1.$$

Thus, only perturbative vacua with relative Morse index 1 contributes. For *generic* Morse function f, $\dim\ker\mathcal{D}_- = 1$ and in this case we find the

Third ingredient

$$\int \text{zero modes}\Big|_{\text{minima}} = f(q) - f(p)$$