# Detached maps and the TR kernel

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MAX-PLANCK-GESELLSCHAFT

IMPRS Moduli Spaces

# Maps: discretised Riemann surfaces

### Definition

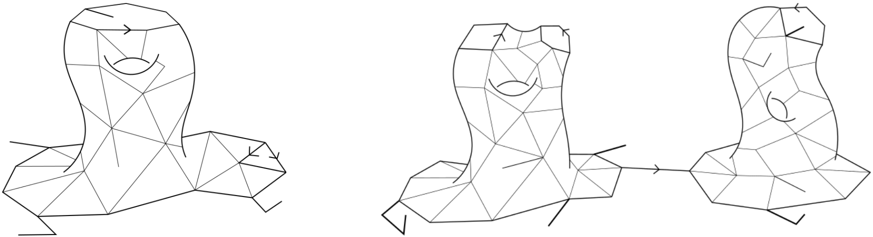
A **map** of topology $(g,n)$ is an equivalence class of a connected graph $\Gamma$ properly embedded into a topological, connected, oriented, closed surface $\Sigma$ of genus $g$, such that

- $\Sigma \setminus \Gamma \cong \mathbb{D}^2 \sqcup \cdots \sqcup \mathbb{D}^2$, called *faces*;
- it has $n$ marked faces (*boundaries*), labelled by $1,\ldots,n$, with a marked edge;
- unmarked faces are at least triangles and have finite length $d$.

Two such embeddings $\Gamma \hookrightarrow \Sigma$ and $\Gamma' \hookrightarrow \Sigma'$ are isomorphic iff there exists an orientation-preserving homomorphism $\phi\colon \Sigma \to \Sigma'$ with

- $\phi|_\Gamma$ is a graph isomorphism between $\Gamma$ and $\Gamma'$;
- restriction of $\phi$ to marked edges is the identity.

# Maps: discretised Riemann surfaces



Set $\mathbb{M}_{g,n}(v) = \{\,$maps of topology $(g,n)$ with $v$ vertices$\,\}$, which is finite.

**Goal**

Count the number of maps of a fixed topology and $\#$vertices.

# Generating series

Define the generating series

$$W_{g,n}(x_1,\dots,x_n) = \sum_{v=1}^{\infty} t^v \sum_{m\in\mathbb{M}_{g,n}(v)} \frac{1}{|\operatorname{Aut}(m)|} \frac{\prod_{k\geq 3} t_k^{N_k(m)}}{x_1^{1+\ell_1(m)}\cdots x_n^{1+\ell_n(m)}} + \\ + \frac{t}{x_1}\delta_{g,0}\delta_{n,1},$$

where

$$N_3(m) = \#\triangle, \qquad N_4(m) = \#\square, \qquad \dots$$

$\ell_i(m) =$ length of the $i$th boundary.

**Question**

Can we find a recursive formula for $W_{g,n}$?

# Tutte's equation...

Removing the marked edge on the first boundary, we obtain Tutte's equation

$$\mathcal{L}_{x_1} W_{g,n}(x_1, x_L) = \sum_{i=2}^{n} \partial_i \left( \frac{W_{g,n-1}(x_1, x_{L_i}) - W_{g,n-1}(x_i, x_{L_i})}{x_1 - x_i} \right) +$$

$$+ W_{g-1,n+1}(x_1, x_1, x_L) + \sum_{\substack{h'+h''=g \\ J' \sqcup J'' = L}}' W_{h',|J'|+1}(x_1, x_{J'}) W_{h'',|J''|+1}(x_1, x_{J''}),$$

where

$$\mathcal{L} f(x) = [V'(x) f(x)]_{-x} - 2 W_{0,1}(x) f(x), \qquad V(x) = \frac{x^2}{2} - \sum_{k \geq 3} \frac{t_k}{k} x^k.$$

**Problem**

The equation is not recursive in $W_{g,n}$: we need to invert $\mathcal{L}_{x_1}$.

# ... and topological recursion

**Theorem**

*With the change of variable* $x = \alpha + \gamma\left(z + \frac{1}{z}\right)$*, Tutte's equation is equivalent to the TR formula for initial data*

$$\begin{cases} \Sigma = \mathbb{C}P^1, \\ x(z) \\ \omega_{0,1}(z) = W_{0,1}(x)dx \\ \omega_{0,2}(z_1, z_2) = W_{0,2}(x_1, x_2)dx_1 dx_2 + \frac{dx_1 dx_2}{(x_1 - x_2)^2} \end{cases}$$

*and* $\omega_{g,n}(z_1, \ldots, z_n) = W_{g,n}(x_1, \ldots, x_n)dx_1 \cdots dx_n$ *for stable* $(g, n)$*. The parameters* $\alpha$ *and* $\gamma$ *are determined in terms of the weights* $t, t_3, t_4, \ldots$*.*

# Problem is solved, but...

From Tutte's recursion

$$\begin{array}{c} \text{removing the marked edge} \\ \text{without changing topology} \end{array} \qquad \longleftrightarrow \qquad \mathcal{L}_{x_1}$$

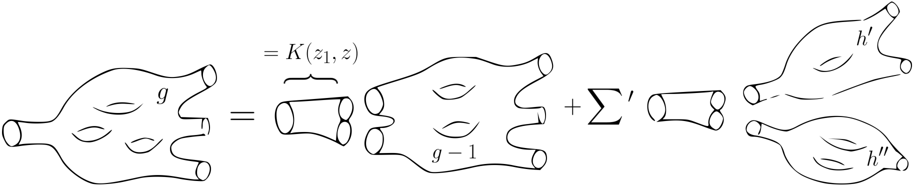
and from TR

$$\mathcal{L}_{x_1}^{-1} \qquad \longleftrightarrow \qquad \operatorname*{Res}_{\text{brnch pts}} K(z_1, z)$$

**Question**

Does the kernel $K(z_1, z)$ has a geometric interpretation in terms of "removing the marked edge until we change topology"? Is it the generating series of some particular type of maps?

# Detached maps




### Goal

Describe the TR kernel as the generating series of some detached maps (or degenerate pair of pants, or pair of shorts), obtained by applying Tutte's recursion until we change topology.

Thanks for the attention