Topological recursion for Masur–Veech volumes

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IMPRS Moduli Spaces

Some definitions

- A bordered surface Σ of type (g, n) is a smooth, compact, oriented, connected stable surface of genus $g \geqslant 0$ and $n > 0$ labelled boundary components $\partial_1\Sigma, \ldots, \partial_n\Sigma$.
- The Teichmüller space

$$\mathcal{T}_\Sigma \coloneqq \left\{ \begin{array}{c} \text{hyperbolic metrics on } \Sigma \\ \text{s.t. } \partial_i\Sigma \text{ are geodesic} \end{array} \right\} \Big/ \sim$$

  fibers over $\mathbb{R}_+^n$ via the perimeter map, and we denote the fiber over $L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n$ by $\mathcal{T}_\Sigma(L)$.
- Example: $\mathcal{T}_P \cong \mathbb{R}_+^3$.

Some definitions

- The pure mapping class group $\mathrm{Mod}_{\Sigma}^{\partial}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ that restrict to the identity on $\partial\Sigma$. It acts on $\mathcal{T}_{\Sigma}(L)$.
- The quotient space $\mathcal{M}_{g,n}(L)$ is a smooth orbifold (moduli space of bordered Riemann surfaces).
- The space $\mathcal{M}_{g,n}(L)$ is endowed with the Weil–Petersson measure μ_{WP}, and we define the Weil–Petersson volumes

$$V_{g,n}^{\mathrm{WP}}(L) := \int_{\mathcal{M}_{g,n}(L)} d\mu_{\mathrm{WP}}.$$

Some definitions

- The moduli space $\mathfrak{M}_{g,n}$ of smooth complex curves of genus g with n labelled punctures.
- The moduli space $Q\mathfrak{M}_{g,n}$ of pairs (C, q), where C is a smooth curve of genus g with n marked points, and q a meromorphic quadratic differential on C with n simple poles at the marked points and no other poles.
- The space $Q\mathfrak{M}_{g,n}$ has an integral piecewise linear structure which allows to define a measure by lattice point counting: the Masur–Veech measure μ_{MV}.
- Define the Masur–Veech volumes as

$$MV_{g,n} := \mu^1_{\mathrm{MV}}(Q^1\mathfrak{M}_{g,n}).$$

First values of Masur–Veech volumes $\frac{MV_{g,n}}{\pi^{6g-6+2n}}$:

$g \backslash n$	0	1	2	3	4
0	—	—	—	1	$\frac{1}{2}$
1	—	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{11}{96}$	$\frac{21}{64}$
2	$\frac{1}{64}$	$\frac{29}{2560}$	$\frac{337}{9216}$	$\frac{319}{2048}$	$\frac{10109}{12288}$
3	$\frac{345}{28672}$	$\frac{20555}{1327104}$	$\frac{77633}{884736}$	$\frac{1038595}{1769472}$	$\frac{16011391}{3538944}$
4	$\frac{2106241}{66060288}$	$\frac{1103729}{18874368}$	$\frac{160909109}{339738624}$	$\frac{14674841399}{3397386240}$	$\frac{99177888029}{2264924160}$

Geometric recursion

- Andersen, Borot, Orantin in 2017, inspired by fundamental results of Mirzakhani
- Input: initial data (A, B, C, D), where

$$A, B, C \in \mathrm{Maps}(\mathcal{T}_P, \mathbb{R}) \cong \mathrm{Maps}(\mathbb{R}^3_+, \mathbb{R}), \qquad D_T \in \mathrm{Maps}(\mathcal{T}_T, \mathbb{R}).$$

- Output: a distinguished element $\Omega_\Sigma \in \mathrm{Maps}(\mathcal{T}_\Sigma, \mathbb{R})^{\mathrm{Mod}^\partial_\Sigma}$, computed recursively in the Euler characteristic.

$$(A, B, C, D) \quad \xrightarrow{\mathrm{GR}} \quad (\Omega_\Sigma)_\Sigma$$

The GR formula

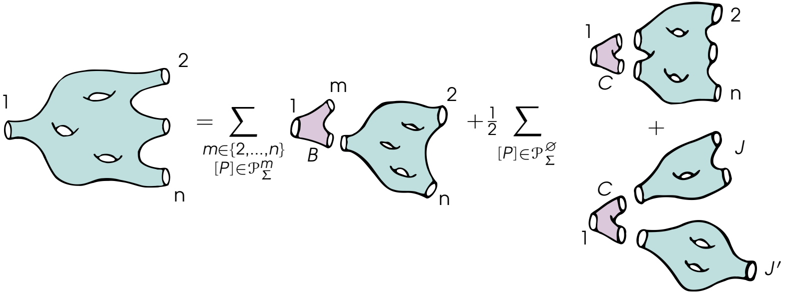
Input. Initial data (A, B, C, D).

Output. Set

$$\Omega_P \coloneqq A, \qquad \Omega_T \coloneqq D_T.$$

and for a bordered surface Σ s.t. $2g - 2 + n > 1$, set recursively

$$\Omega_\Sigma(\sigma) \coloneqq \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^m} B(\vec{\ell}_\sigma(\partial P))\, \Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \frac{1}{2} \sum_{[P]\in\mathcal{P}_\Sigma^\varnothing} C(\vec{\ell}_\sigma(\partial P))\, \Omega_{\Sigma-P}(\sigma|_{\Sigma-P}).$$



The GR formula

$$\Omega_\Sigma(\sigma) = \sum_{m=2}^{n} \sum_{[P]\in\mathcal{P}_\Sigma^m} B(\vec{\ell}_\sigma(\partial P))\,\Omega_{\Sigma-P}(\sigma|_{\Sigma-P}) + \frac{1}{2}\sum_{[P]\in\mathcal{P}_\Sigma^\varnothing} C(\vec{\ell}_\sigma(\partial P))\,\Omega_{\Sigma-P}(\sigma|_{\Sigma-P}).$$

Theorem (Andersen, Borot, Orantin '17)

If (A, B, C, D) are admissible initial data, then

- the series is absolutely convergent for the supremum norm over any compact subset of $\mathcal{T}_\Sigma$;
- Ω_Σ is $\mathrm{Mod}_\Sigma^\partial$-invariant, and descends to a function $\Omega_{g,n}$ on $\mathcal{M}_{g,n}$;
- if the initial data are continuous, Ω_Σ is also continuous.

From now on, we will only consider continuous initial data.

From GR to TR

Define (whenever the integral makes sense)

$$V\Omega_{g,n}(L_1,\ldots,L_n) \coloneqq \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \Omega_{g,n}\, d\mu_{\mathrm{WP}}.$$

Theorem (Andersen, Borot, Orantin '17)

Let (A, B, C, D) be strongly admissible. Then $V\Omega_{g,n}$ is well-defined and satisfy TR: for any $2g-2+n \geqslant 2$

$$\begin{aligned} V\Omega_{g,n}(L_1,L_2,\ldots,L_n) &= \sum_{m=2}^{n}\int_{\mathbb{R}_+} d\ell\,\ell\,B(L_1,L_m,\ell)\,V\Omega_{g,n-1}(\ell,L_2,\ldots,\widehat{L_m},\ldots,L_n) \\ &+ \frac{1}{2}\int_{\mathbb{R}_+^2} d\ell\,d\ell'\,\ell\ell'\,C(L_1,\ell,\ell')\Big(V\Omega_{g-1,n+1}(\ell,\ell',L_2,\ldots,L_n) \\ &+ \sum_{\substack{h+h'=g \\ J\sqcup J'=\{L_2,\ldots,L_n\}}} V\Omega_{h,1+|J|}(\ell,J)\,V\Omega_{h',1+|J'|}(\ell',J')\Big) \end{aligned}$$

with initial conditions $V\Omega_{0,3} = A$ and $V\Omega_{1,1}(L_1) = VD(L_1)$.

Summary

Input:

$$A, B, C \in \mathcal{C}^0(\mathbb{R}^3_+), \qquad D_T \in \mathcal{C}^0(\mathcal{T}_T)^{\mathrm{Mod}^\partial_T}$$

GR output:

$$\Omega_\Sigma \in \mathcal{C}^0(\mathcal{T}_\Sigma)^{\mathrm{Mod}^\partial_\Sigma}$$

TR output:

$$V\Omega_{g,n}(L) \coloneqq \int_{\mathcal{M}_{g,n}(L)} \Omega_{g,n} \, d\mu_{\mathrm{WP}} \;\in\; \mathcal{C}^0(\mathbb{R}^n_+)$$

Example: Mirzakhani–McShane identity

The following initial data are admissible

$$A^{\mathrm{M}}(L_1, L_2, L_3) = 1,$$

$$B^{\mathrm{M}}(L_1, L_2, \ell) = 1 - \frac{1}{L_1} \ln\left(\frac{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1+\ell}{2}\right)}{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1-\ell}{2}\right)}\right),$$

$$C^{\mathrm{M}}(L_1, \ell, \ell') = \frac{2}{L_1} \ln\left(\frac{e^{\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}\right),$$

$$D^{\mathrm{M}}_T(\sigma) = \sum_{\gamma \text{ simple closed curve}} C^{\mathrm{M}}\big(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)\big),$$

and $\Omega^{\mathrm{M}}_\Sigma \equiv 1$ is the constant function 1 on $\mathcal{T}_\Sigma$. Thus, $V\Omega^{\mathrm{M}}_{g,n}(L) = V^{\mathrm{M}}_{g,n}(L)$ are the Weil–Petersson volumes.

Kontsevich amplitudes

Set $[x]_+ = \max(x, 0)$. The following initial data are admissible

$$A^{\mathrm{K}}(L_1, L_2, L_3) = 1,$$
$$B^{\mathrm{K}}(L_1, L_2, \ell) = \frac{1}{2L_1}\big([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+\big),$$
$$C^{\mathrm{K}}(L_1, \ell, \ell') = \frac{1}{L_1}\,[L_1 - \ell - \ell']_+,$$
$$D^{\mathrm{K}}_T(\sigma) = \sum_{\gamma \text{ simple closed curve}} C^{\mathrm{K}}\big(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)\big).$$

and $V\Omega^{\mathrm{K}}_{g,n}(L)$ are the Kontsevich volumes

$$V\Omega^{\mathrm{K}}_{g,n}(L) = \int_{\overline{\mathfrak{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right) =: V^{\mathrm{K}}_{g,n}(L).$$

Masur–Veech polynomials: definition

For $2g-2+n>0$ define $V^{\mathrm{MV}}_{g,n}(L_1,\ldots,L_n)$ as

$$\sum_{\Gamma\in\mathcal{G}_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|}\int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v\in V_\Gamma} V^{\mathrm{K}}_{g(v),n(v)}\big((\ell_e)_{e\in E(v)},(L_\lambda)_{\lambda\in\Lambda(v)}\big)\prod_{e\in E_\Gamma}\frac{\ell_e d\ell_e}{e^{\ell_e}-1},$$

which is a polynomial in $L_1^2,\ldots,L_n^2$ of total degree $3g-3+n$.

Theorem (Delecroix, Goujard, Zograf, Zorich ’19 & Andersen, Borot, Charbonnier, Delecroix, G., Lewański, Wheeler ’19)

$$MV_{g,n}=V^{\mathrm{MV}}_{g,n}(0,\ldots,0).$$

The proof of DGZZ is based on combinatorial methods, while our proof is based on geometric recursion.

Theorem (TR for MV volumes. ABCDGLW'19)

The spectral curve on $\mathcal{C}^{\mathrm{MV}} = \mathbb{P}^1$

$$x^{\mathrm{MV}}(z) = \frac{z^2}{2}, \qquad y^{\mathrm{MV}}(z) = -z,$$

$$\omega_{0,2}^{\mathrm{MV}}(z_1, z_2) = \frac{\zeta(2; z_1 - z_2) + \zeta(2; -z_1 + z_2)}{2} \, dz_1 \, dz_2$$

produces TR output

$$\omega_{g,n}^{\mathrm{MV}}(z_1, \ldots, z_n) = \sum_{d_1 + \cdots + d_n \leqslant 3g-3+n} F_{g,n}[d_1, \ldots, d_n] \prod_{i=1}^{n} \zeta(2d_i + 2; z_i) \, dz_i$$

where $F_{g,n}$ are the coefficients of $V_{g,n}^{\mathrm{MV}}$ in the expansion

$$V_{g,n}^{\mathrm{MV}}(L_1, \ldots, L_n) = \sum_{d_1 + \cdots + d_n \leqslant 3g-3+n} F_{g,n}[d_1, \ldots, d_n] \prod_{i=1}^{n} \frac{L_i^{2d_i}}{(2d_i + 1)!}.$$

Theorem (GR for MV volumes. ABCDGLW'19)

The GR initial data

$$A^{\mathrm{MV}}(L_1, L_2, L_3) = 1$$

$$B^{\mathrm{MV}}(L_1, L_2, \ell) = \frac{1}{e^{\ell} - 1} + B^{\mathrm{K}}(L_1, L_2, \ell),$$

$$C^{\mathrm{MV}}(L_1, \ell, \ell') = \frac{1}{(e^{\ell} - 1)(e^{\ell'} - 1)} + C^{\mathrm{K}}(L_1, \ell, \ell') + \frac{1}{e^{\ell} - 1} B^{\mathrm{K}}(L_1, \ell, \ell') + \frac{1}{e^{\ell'} - 1} B^{\mathrm{K}}(L_1, \ell', \ell)$$

$$D_T^{\mathrm{MV}}(\sigma) = D_T^{\mathrm{K}}(\sigma) + \sum_{\gamma \text{ multicurve}} e^{-\ell_{\sigma}(\gamma)}$$

are admissible, and the associated TR amplitudes equals the Masur–Veech polynomials:

$$V\Omega_{g,n}^{\mathrm{MV}}(L) = V_{g,n}^{\mathrm{MV}}(L).$$

Idea of the proof: curve counting

There are two "curve counting" functions defined on the Teichmüller space of Σ that play an important role.

1)
$$B_\Sigma(\sigma) = \lim_{\beta\to\infty} \frac{\#\{\gamma \text{ multicurve} \mid \ell_\sigma(\gamma) \leqslant \beta\}}{\beta^{6g-6+2n}}.$$
Due to Mirzakhani, $VB_{g,n}(0,\ldots,0)$ is proportional to $MV_{g,n}$.

2) For a test function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ and $\beta > 0$, consider
$$N_\Sigma^{\phi,\beta}(\sigma) = \sum_{\gamma \text{ multicurve}} \phi\left(\frac{\ell_\sigma(\gamma)}{\beta}\right).$$
For good enough test function, $N_\Sigma^{\phi,\beta}$ is computed by GR.

Idea of the proof: curve counting

1) $VB_{g,n}(0,\ldots,0) \simeq MV_{g,n}$
2) $N_{\Sigma}^{\phi,\beta}$ is computed by GR

Theorem (ABCDGLW'19)

For an admissible test function ϕ, the limit

$$\lim_{\beta\to\infty} \frac{VN_{g,n}^{\phi,\beta}(L)}{\beta^{6g-6+2n}}$$

is proportional to $VB_{g,n}(L)$.

Considering $\phi(\ell) = e^{-\ell}$, we obtain the TR for Masur–Veech polynomials. Kontsevich initial data are appearing in the limit of Mirzakhani's ones.

The computational power of TR allowed us to make some conjectures about the behaviour of $MV_{g,n}$.

Conjecture (MV volumes for fixed genus and asymptotics)

There exist polynomials a_g and b_g with rational coefficients of degrees

$$\deg(a_g) = \lfloor (g-1)/2 \rfloor \quad \text{and} \quad \deg(b_g) = \lfloor g/2 \rfloor$$

such that, for $2g-2+n>0$,

$$\frac{MV_{g,n}}{(2\pi^2)^{3g-3+n}} = \Big((2g-3+n)!\, a_g(n) + (4g-5+2n)!!\, b_g(n) \Big).$$

Only the genus zero case is proved (with $a_0 = 0$ and $b_0 = 1$), while the higher genera case has been proved by by Chen, Möller and Sauvaget (article in preparation).

Other results: connections with area Siegel–Veech constants and conjectures for behaviour in fixed genus and asymptotics.

Open question

Are the MV polynomials connected to some integrable hierarchy problem?

Thank you!

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2. J. E. Andersen, G. Borot, S. Charbonnier, A. Giacchetto, D. Lewański, and C. Wheeler. *On the Kontsevich geometry of the combinatorial Teichmüller space.* In preparation.
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4. V. Delecroix, E. Goujard, P. Zograf, and A. Zorich *Masur–Veech volumes, frequencies of simple closed geodesics and intersection numbers of moduli spaces of curves* (2019). math.GT/1908.068611