The Kontsevich geometry of the combinatorial Teichmüller spaces

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MAX-PLANCK-GESELLSCHAFT

IMPRS Moduli Spaces

Moduli space of curves

For $g \geqslant 0$, $n \geqslant 0$, and $2g - 2 + n > 0$, consider the moduli space of curves

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} C \text{ is a complex} \\ \text{compact curve of genus } g \\ \text{with } n \text{ distinct marked points} \end{array} \right\} \Big/ \sim .$$

It parametrises complex curves of a fixed genus g, with n marked points, up to isomorphism.

It is a smooth complex orbifold of dimension $3g - 3 + n$. It admits a compactification $\overline{\mathcal{M}}_{g,n}$.

Fundamental problem

Understand $H^\bullet(\mathcal{M}_{g,n})$ and $H^\bullet(\overline{\mathcal{M}}_{g,n})$ in terms of generators and relations.

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Moduli space of curves

For example, one can define some natural cohomology classes

$$\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}), \ i = 1, \dots, n$$

$$\kappa_m \in H^{2m}(\overline{\mathcal{M}}_{g,n}), \ m = 1, \dots, 3g-3+n$$

$$\lambda_j \in H^{2j}(\overline{\mathcal{M}}_{g,n}), \ j = 1, \dots, g$$

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Which are the relations between such classes in the cohomology ring $H^\bullet(\overline{\mathcal{M}}_{g,n})$? For instance,

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Intersection theory

Since $\overline{\mathcal{M}}_{g,n}$ is a complex compact orbifold, we can talk about intersection theory on it: for every $\alpha \in H^{6g-6+2n}(\overline{\mathcal{M}}_{g,n})$, it makes sense to consider

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q}.$$

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Understand the intersection theory of $\overline{\mathcal{M}}_{g,n}$.

For instance, how to compute the following number?

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \qquad d_1 + \cdots + d_n = 3g - 3 + n.$$

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Theorem (Witten conjecture, Kontsevich theorem '92)

We have a recursion on $2g - 2 + n$:

$$\begin{aligned} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = & \sum_{i=2}^{n} \frac{(2d_1 + 2d_i - 1)!!}{(2d_1 + 1)!!\,(2d_i - 1)!!} \langle \tau_{d_1+d_i-1} \tau_{d_2} \cdots \widehat{\tau_{d_i}} \cdots \tau_{d_n} \rangle_g \\ & + \frac{1}{2} \sum_{a+b=d_1-2} \frac{(2a+1)!!\,(2b+1)!!}{(2d_1+1)!!} \Big(\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\ & + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \{\tau_{d_2}, \ldots, \tau_{d_n}\}}} \langle \tau_a J_1 \rangle_{g_1} \langle \tau_b J_2 \rangle_{g_2} \Big) \end{aligned}$$

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Ribbon graphs

Definition

A ribbon graph is a graph G with a cyclic order of the edges at each vertex.

From the geometric realisation $|G|$, we have a well-defined genus $g \geqslant 0$ and number of boundary components $n \geqslant 1$ of G. We call (g, n) the type of G.

We assume ribbon graphs to be connected, with vertices of valency $\geqslant 3$, and labeled boundaries.

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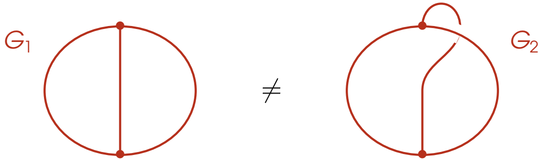

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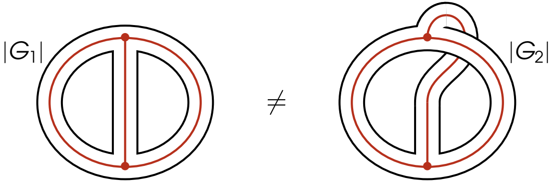

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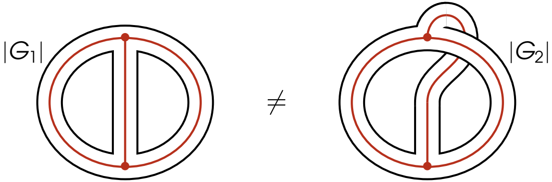

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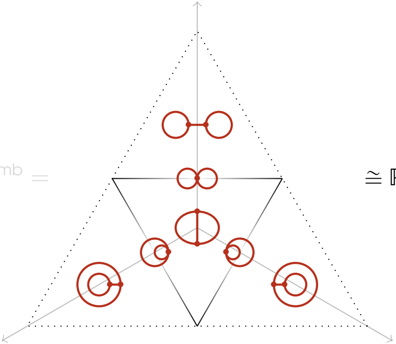
$\mathcal{M}_{0,3}^{\mathrm{comb}} =$  $\cong \mathbb{R}_+^3$

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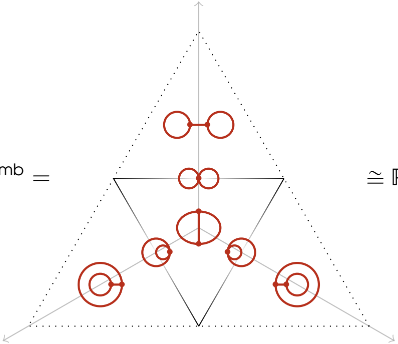

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The combinatorial moduli space

Define the combinatorial moduli space

$$\mathcal{M}_{g,n}^{\mathrm{comb}} = \bigcup_{\substack{G \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{E_G}}{\mathrm{Aut}(G)},$$

where we glue orbicells through degeneration of edges.

We have a map $p\colon \mathcal{M}_{g,n}^{\mathrm{comb}} \to \mathbb{R}_+^n$, assigning to each metric ribbon graph the length of the labeled faces. We set $\mathcal{M}_{g,n}^{\mathrm{comb}}(L) = p^{-1}(L)$.

Proposition (Jenkins '57, Strebel '67, Harer '86)

$\mathcal{M}_{g,n}^{\mathrm{comb}}(L)$ is a real orbicell complex of dimension $6g-6+2n$, and there exists a homeomorphism of topological orbifolds

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where $(f, G) \sim (f', G')$ iff there exists a MRG isomorphism $\phi \colon G \to G'$ such that $|\phi| \circ f$ is homotopic to f'. We will denote $[f, G]$ by $\mathbb{G}$ and call it a embedded MRG. We have a map $\pi \colon \mathcal{T}_\Sigma^{\mathrm{comb}} \to \mathcal{M}_{g,n}^{\mathrm{comb}}$, $\pi(\mathbb{G}) = G$.

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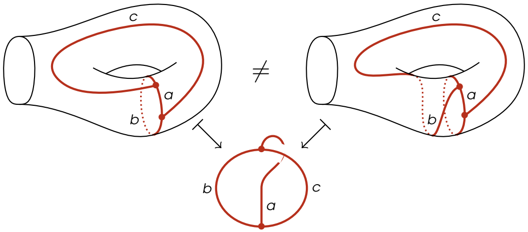

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Proposition

$\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$ is a real cell complex of dimension $6g-6+2n$. The pure mapping class group $\mathrm{Mod}_{\Sigma} = \mathrm{Homeo}^{+}(\Sigma, \partial\Sigma)/\,\mathrm{Homeo}_{0}(\Sigma)$ is acting on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(L)$, and

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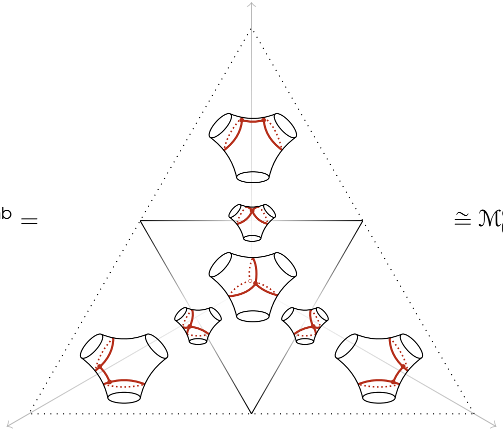
Example 1: a pair of pants

$\mathcal{T}_P^{\mathrm{comb}} =$  $\cong \mathcal{M}_{0,3}^{\mathrm{comb}} \cong \mathbb{R}_+^3$

Example 2: a one-holed torus

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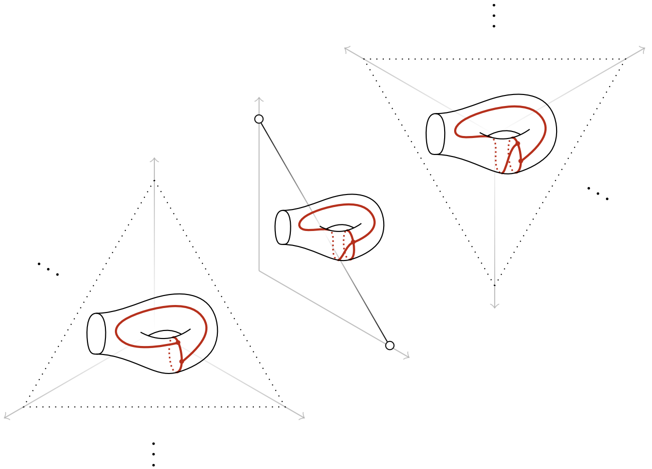

Length of simple closed curve

Consider a (homotopy class of a) simple closed curve γ in Σ, and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$. By homotoping the curve to the embedded graph and measuring summing up the length of the edges it travels through, we obtain the length of γ with respect to $\mathbb{G}$: $\ell_{\mathbb{G}}(\gamma) \in \mathbb{R}_+$.

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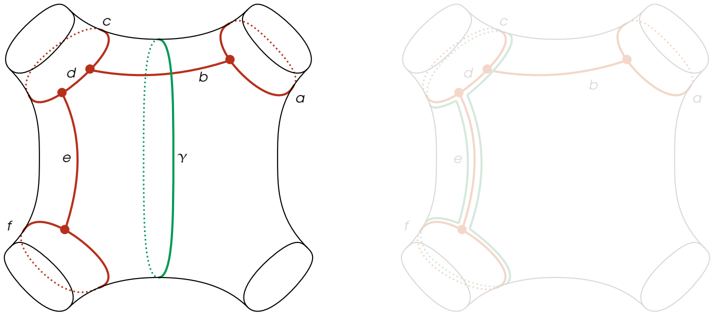

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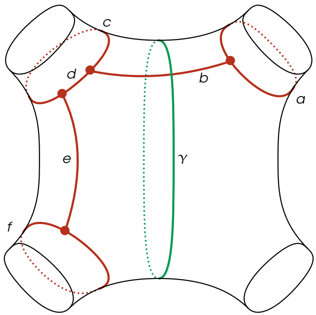

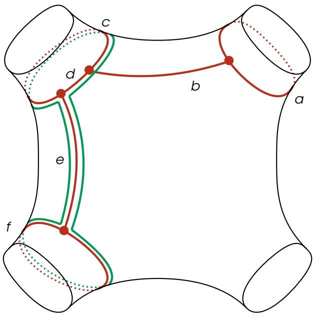

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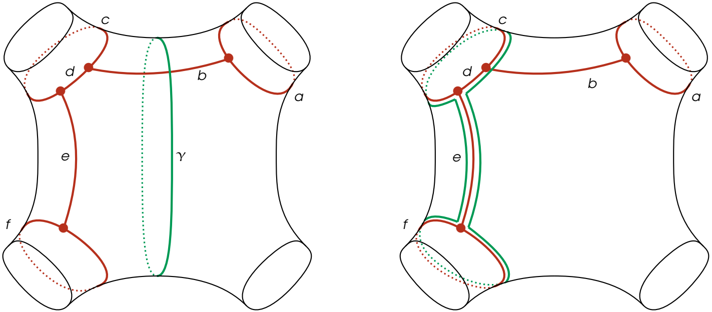

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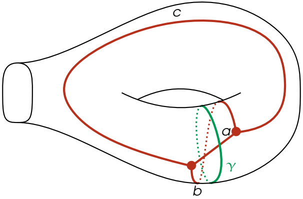

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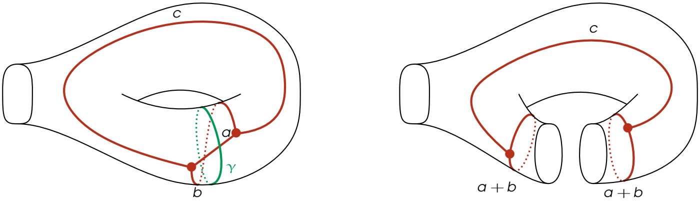

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Then for a.e. $\tau \in \mathbb{R}$, it is possible to glue $\mathbb{G}_1$ and $\mathbb{G}_2$ along $\partial_i\Sigma \sim \partial_j\Sigma'$ with twist τ, and obtain an embedded MRG on the glued surface.

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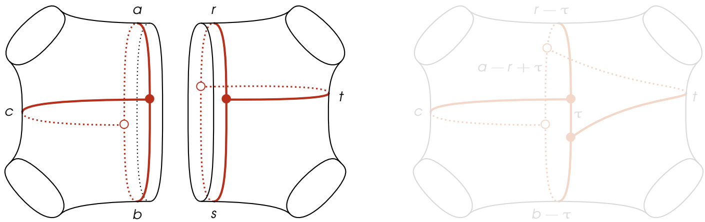

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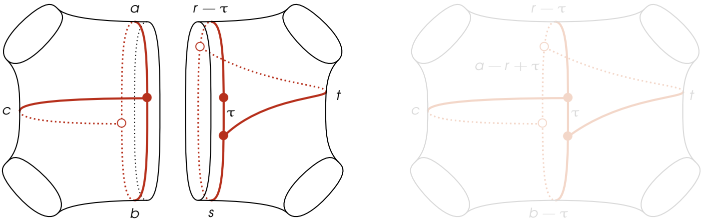

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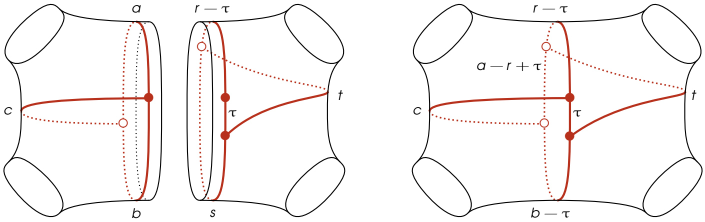

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Combinatorial Fenchel–Nielsen coordinates

Fix a pants decomposition $\mathscr{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$ of Σ, together with a set coordinate curves $\mathcal{S} = (\beta_j)_{j \in J}$. We have a map

$$\begin{aligned} \mathrm{FN}\colon \mathcal{T}^{\mathrm{comb}}_{\Sigma}(L) &\longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \\ \mathbb{G} &\longmapsto \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_{i=1}^{3g-3+n} \end{aligned}$$

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$$\begin{aligned} \mathrm{FN}(\mathbb{G}) &= \big(\ell_{\mathbb{G}}(\gamma), \tau_{\mathbb{G}}(\gamma)\big) \\ &= (a+b, -a) \end{aligned}$$

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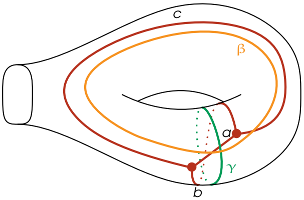
$\mathbb{G} =$ 

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Theorem (Andersen, Borot, Charbonnier, AG, Lewański, Wheeler)

For every choice $(\mathscr{P}, \mathcal{S})$, the map

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is a homeomorphism onto its image, with an open dense image.

Upshot. To talk about length, cutting and gluing, we have to consider markings of MRGs (they do not make sense at the level of the combinatorial moduli space). Thanks to this, we found global coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ on $\mathcal{T}^{\mathrm{comb}}_{\Sigma}(L)$.

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The Kontsevich form

There exists natural 2-form ω_K on $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$, called the Kontsevich form, defined on each cell by

$$\omega_K = \frac{1}{2} \sum_{i=1}^{n} \sum_{e_a^{[i]} \prec e_b^{[i]}} d\ell_{e_a^{[i]}} \wedge d\ell_{e_b^{[i]}},$$

where $e_1^{[i]}, e_2^{[i]}, \ldots$ are the edges around the ith face of the ribbon graph underlying the cell, and $\prec$ is the order on the edges induced by the orientation of the surface.

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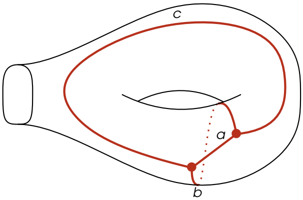

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Theorem (Kontsevich, '92)

The form ω_K on $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$ is symplectic on $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$, is mapping class group invariant, and the symplectic volume of $\mathcal{M}_{g,n}^{\mathrm{comb}}(L)$, denoted $V_{g,n}(L)$, is finite and given by

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(L)} \frac{\omega_K^{\wedge(3g-3+n)}}{(3g-3+n)!} = \sum_{d_1+\cdots+d_n=3g-3+n} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g \prod_{i=1}^{n} \frac{L_i^{2d_i}}{2^{d_i} d_i!}.$$

Upshot: the computation of all $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g$ is equivalent to the computation of the symplectic volume $V_{g,n}(L)$.

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For every choice of pants decomposition and coordinate curves on Σ, we have a global coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ on $\mathcal{T}_\Sigma^{\mathrm{comb}}(L)$. Then

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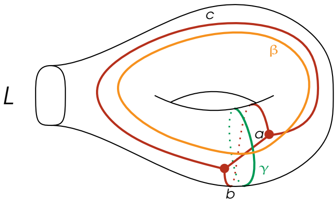

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Consider the following auxiliary functions $\mathscr{D}, \mathscr{R} \colon \mathbb{R}_+^3 \to \mathbb{R}_+$:

$$\mathscr{D}(L, \ell_1, \ell_2) = [L - \ell_1 - \ell_2]_+$$
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where $[x]_+ = \max(x, 0)$.

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For any $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}(L)$, we have

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The Kontsevich volumes are computed recursively by

$$V_{g,n}(L_1,\ldots,L_n) = \sum_{i=2}^{n} \int_{\mathbb{R}_+} d\ell\, \ell\, \frac{\mathscr{R}(L_1,L_i,\ell)}{L_1}\, V_{g,n-1}(\ell,L_2,\ldots,\widehat{L_i},\ldots,L_n)$$

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with initial conditions $V_{0,3}(L_1,L_2,L_3)=1$ and $V_{1,1}(L)=\frac{L^2}{48}$.

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integral structure $\mathcal{M}_{g,n}^{\mathrm{comb},\mathbb{Z}}$

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recursion for $N_{g,n}(L) = \sum_{G \in \mathcal{M}_{g,n}^{\mathrm{comb},\mathbb{Z}}(L)} \frac{1}{\mathrm{Aut}(G)}$

The number of integral MRGs are computed recursively by

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More from these combinatorial spaces?

Define $\mathcal{N}_\Sigma : \mathcal{T}_\Sigma^{\mathrm{comb}} \times \mathbb{R} \to \mathbb{N}$ the counting function,

$$\mathcal{N}_\Sigma(\mathbb{G}, t) = \#\{\gamma \mid \text{multicurve in } \Sigma \text{ with } \ell_{\mathbb{G}}(\gamma) \leqslant t\}.$$

Theorem (ABCGLW)

The Laplace transform of $\mathcal{N}_\Sigma$ in the cut-off variable t, namely

$$\int_{\mathbb{R}_+} \mathcal{N}_\Sigma(\mathbb{G}, t)\, e^{-ts}\, dt,$$

is computed recursively by a Mirzakhani-type recursion (geometric recursion). Moreover

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Thank you!

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