A gentle introduction to moduli spaces

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MAX-PLANCK-GESELLSCHAFT

IMPRS Moduli Spaces

Classifying objects

Imagine you want to classify your wardrobe, sorting your clothes by type: you will fill a drawer with socks, one with your underwear, another one with trousers, etc.

$$\begin{array}{c} \text{set of objects} \\ + \\ \text{equivalence relation} \end{array} \implies \text{equivalence classes}$$

Another example: classifying finite groups, up to isomorphism.

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Moduli spaces

When do we talk about moduli spaces?

We talk about moduli space when the set of equivalence classes has a "natural" geometric structure.

Example (Moduli of sphere)

A sphere is a set

$$\left\{ (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i=0}^{d} (x_i - p_i)^2 = R^2 \right\} \subset \mathbb{R}^{d+1},$$

for some $d \in \mathbb{N}_+$, $p = (p_0, \ldots, p_d) \in \mathbb{R}^{d+1}$ and $R \in \mathbb{R}_+$.

Then the moduli of spheres is

$$\left\{\text{spheres}\right\} \big/ \text{isometry} = \mathbb{N}_+ \times \mathbb{R}_+,$$

where $\mathbb{N}_+$ accounts for the dimension and $\mathbb{R}_+$ accounts for the radius.

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Moduli spaces

Example (Projective spaces)

Consider the set of lines through the origin in $\mathbb{C}^{d+1}$: the complex projective space.

$$\mathbb{C}P^d = \frac{\mathbb{C}^{d+1} \setminus \{0\}}{\sim}$$

where $v \sim \lambda v$ for some $\lambda \neq 0$. It is a complex, compact manifold of complex dimension d.

In many cases, moduli spaces are endowed with more geometric structure than just the topology. For example, $\mathbb{C}P^d$ has a natural symplectic form, called the Fubini–Study form ω_{FS}, and

$$\mathrm{Vol}(\mathbb{C}P^d) = \int_{\mathbb{C}P^d} \frac{\omega_{FS}^d}{d!} = \frac{\pi^d}{d!}.$$

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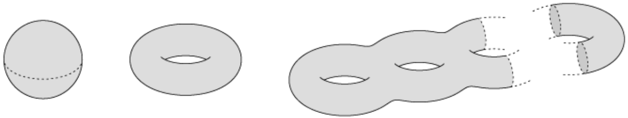
Riemann surfaces

Consider connected, compact, complex curves (also called Riemann surfaces) with n marked, labeled, pairwise distinct points.

Every compact complex curve has an underlying structure of a 2-dimensional oriented smooth compact surface, that is uniquely characterized by its genus g.



We will call (g, n) the type of the Riemann surface.

Goal

Classify Riemann surfaces of fixed type, up to isomorphisms that respect the marked points.

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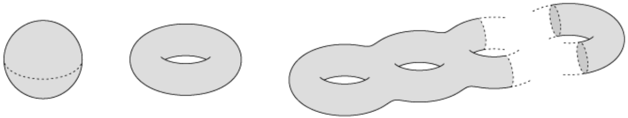

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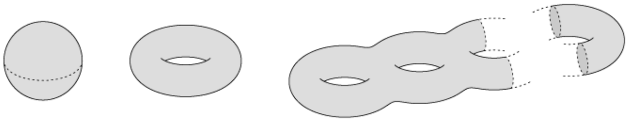

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Classify Riemann surfaces of fixed type, up to isomorphisms that respect the marked points.

A first example

$(g,n)=(0,0)$. The sphere possesses a unique structure of Riemann surface up to isomorphism: that of a complex projective line $\mathbb{C}P^1$. The automorphism group of $\mathbb{C}P^1$ is $\mathrm{PSL}(2,\mathbb{C})$, acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

$(g,n)=(0,3)$. Consider now $\mathbb{C}P^1$ with three marked points, p_1, p_2, p_3. There exists an automorphism of the sphere, such that the marked points are mapped to $0, 1, \infty$. In other words, there is only one Riemann surface of type $(0,3)$ up to isomorphism.

$(g,n)=(0,4)$. Consider now $\mathbb{C}P^1$ with four marked points, p_1, p_2, p_3, p_4. There exists an automorphism of the sphere, such that the marked points are mapped to $0, 1, \infty, \lambda$ for some $\lambda \in \mathbb{C}P^1 \setminus \{0, 1, \infty\}$. In other words, the set of Riemann surface of type $(0,4)$ up to isomorphism is

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The moduli space of Riemann surfaces

For arbitrary (g, n), one can define the moduli space of Riemann surfaces

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; C \text{ has genus } g \right\} \Big/ \text{iso}.$$

We already saw that

$$\mathcal{M}_{0,3} = \{*\}, \qquad \mathcal{M}_{0,4} = \mathbb{C}P^1 \setminus \{0, 1, \infty\}.$$

In general, we have the following natural geometric structure on $\mathcal{M}_{g,n}$.

Theorem

If $2g - 2 + n > 0$, then $\mathcal{M}_{g,n}$ is a smooth complex orbifold of complex dimension $3g - 3 + n$.

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Afternoon's talk

After an intersection-theoretic motivation (and the presentation of Witten's conjecture/Kontsevich theorem), I will introduce a family of combinatorial models of the moduli space of Riemann surfaces:

$$\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) \cong \mathcal{M}_{g,n}, \qquad \vec{L} \in \mathbb{R}_+^n.$$

All the element in the family are homeomorphic

$$S^1(R) \cong \mathbb{S}^1,$$

but have different volumes

$$\mathrm{Vol}(S^1(R)) = 2\pi R.$$

Every space $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ is endowed with a symplectic form ω_{K}, called the Kontsevich form, and I will present a method to compute the symplectic volume

$$V_{g,n}(\vec{L}) = \int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \frac{\omega_{\mathrm{K}}^{3g-3+n}}{(3g-3+n)!}$$

by induction on $2g-2+n$, based on a Mirzakhani-type identity.

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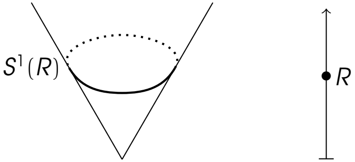

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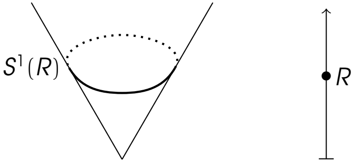

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