# Stability conditions on quivers with potentials and triangulated surfaces

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May 01, 2020

# Goal

Following Bridgeland–Smith, the aim of the next talks is to

1) define a $CY_3$ $\Delta$-category $\mathcal{D}(S, M)$ associated to a marked bordered surface $(S, M)$,

2) characterise the associated space of stability conditions in terms of meromorphic quadratic differentials on the surface:

$$\mathsf{Stab}_{\triangle}(\mathcal{D}(S,M)) / \mathsf{Aut}_{\triangle}(\mathcal{D}(S,M)) \cong \mathsf{Quad}_{\heartsuit}(S,M),$$

3) give a precise link between the trajectory structure of flat surfaces and the theory of wall-crossing and DT invariants.

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# Plan of the talk

# Stability conditions

Let $\mathcal{D}$ be a $\Delta$-category s.t. $K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$ is free of finite rank.

**Definition**

A stability condition on $\mathcal{D}$ is a pair $(Z, \mathcal{P})$, where

- $Z\colon K_0(\mathcal{D}) \to \mathbb{C}$ is group homomorphism (*central charge*)
- $\forall \phi \in \mathbb{R}$, $\mathcal{P}(\phi) \subset \mathcal{D}$ is a full subcategory (of *semistable objects of phase* $\phi$)

satisfying the following axioms.

- For all $0 \neq E \in \mathcal{P}(\phi)$, $Z(E) \in \mathbb{R}_+ e^{i\pi\phi}$
- $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$
- For $\phi_1 > \phi_2$, then $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$
- For all $0 \neq E \in \mathcal{D}$, $\exists(!)$ a Harder–Narasimhan filatration

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# Space of stability conditions

Define the space of stability conditions on $\mathcal{D}$:

$$\mathsf{Stab}(\mathcal{D}) \coloneqq \left\{ (Z, \mathcal{P}) \;\middle|\; \begin{array}{c} \text{stability conditions on } \mathcal{D} \\ \text{satisfying the support property} \end{array} \right\}.$$

**Theorem (Bridgeland)**

The space $\mathsf{Stab}(\mathcal{D})$ has a natural Hausdorff topology, and the forgetful map

$$\mathsf{Stab}(\mathcal{D}) \longrightarrow \mathsf{Hom}(K_0(\mathcal{D}), \mathbb{C}) \cong \mathbb{C}^n, \qquad (Z, \mathcal{P}) \longmapsto Z$$

is a local homeomorphism. In particular, the space of stability conditions is an $n$-dimensional complex manifold.

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# Hearts of t-structures

**Definition**

A t-structure on $\mathcal{D}$ is a pair $(\mathcal{D}^{\geqslant 0}, \mathcal{D}^{\leqslant 0})$ of full subcategories, satisfying the following axioms

- $\mathcal{D}^{\geqslant 1} \coloneqq \mathcal{D}^{\geqslant 0}[-1] \subset \mathcal{D}^{\geqslant 0}$
- $\mathrm{Hom}(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1}) = 0$
- Any $0 \neq E \in \mathcal{D}$ fits in a DT $E^{\leqslant 0} \to E \to E^{\geqslant 1} \to E^{\leqslant 0}[1]$

A t-structure is bounded if $\mathcal{D} = \bigcup_n \mathcal{D}^{\geqslant -n} \cap \mathcal{D}^{\leqslant -n}$.

The heart of a t-structure is $\mathcal{D}^{\heartsuit} \coloneqq \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$.

**Facts:**

1) The heart of a t-structure is an abelian category
2) A bounded t-structure is determined by its heart
3) Every stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ has an associated heart

$$\mathcal{A} = \mathcal{P}((0, 1]).$$

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# Hearts of stability conditions as chambers

Let $\mathcal{A} \subset \mathcal{D}$ be a heart satisfying some finiteness assumptions:

- $\mathcal{A}$ is a finite-length heart, *i.e.* is artinian and noetherian as abelian category,
- $\mathcal{A}$ has $n$ simple objects $S_1, \dots, S_n$.

Denote by $\mathsf{Stab}(\mathcal{A}) \subset \mathsf{Stab}(\mathcal{D})$ the subset of consisting of those stability conditions whose heart is $\mathcal{A}$. The forgetful map is a bijection

$$\mathsf{Stab}(\mathcal{A}) \cong \left\{ Z \in \mathsf{Hom}(K_0(\mathcal{D}), \mathbb{C}) \;\middle|\; Z(S_i) \in \bar{\mathbb{H}} \right\} \cong \bar{\mathbb{H}}^n, \quad \mathbb{H} \cup \mathbb{R}_{<0}.$$

In other words every heart determines a chamber in $\mathsf{Stab}(\mathcal{D})$.

The way this cells are glued together is well-described by by means of tilts at simple objects.

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# Tilts from torsion pairs

**Definition**

Let $\mathcal{A}$ be an abelian category. A torsion pair for $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$,
- For all $E \in \mathcal{A}$, there exists a SES $0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

**Fact:** if $\mathcal{A} \subset \mathcal{D}$ is a heart and $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{A}$, then

$$\mathcal{A}^{\#} := \langle \mathcal{F}, \mathcal{T}[-1] \rangle \subset \mathcal{D}$$

is again a heart for $\mathcal{D}$, called the tilt of $\mathcal{A}$.

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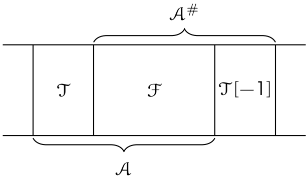

## Tilts at simple objects

Suppose that $\mathcal{A}$ is a finite-length heart and $S \in \mathcal{A}$ is a simple object. Let $\langle S \rangle \subset \mathcal{A}$ be the full subcategory consisting of objects whose simple factors are isomorphic to $S$. Define the full subcategories

$$S^{\perp} := \{ A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(S, A) = 0 \}, \qquad {}^{\perp}S := \{ A \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(A, S) = 0 \}.$$

**Lemma**

$(\langle S \rangle, S^{\perp})$ and $({}^{\perp}S, \langle S \rangle)$ are torsion pairs for $\mathcal{A}$. In other words,

$$\mu_S^{-}(\mathcal{A}) := \langle S[1], {}^{\perp}S \rangle, \qquad \mu_S^{+}(\mathcal{A}) := \langle S^{\perp}, S[-1] \rangle$$

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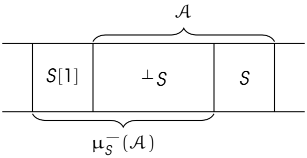


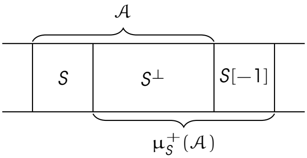

# Tilts and spaces of stability conditions

Tilts at simple objects controls how the chambers $\mathsf{Stab}(\mathcal{A})$ are glued together.

**Proposition (Bridgeland)**

Let $\mathcal{A} \subset \mathcal{D}$ be a finite-length heart with $n$ simple objects $S_1, \ldots, S_n$, and suppose that $(Z, \mathcal{P})$ is a stability condition lying in a codim-1 wall of the chamber $\mathsf{Stab}(\mathcal{A})$, *i.e.* $\operatorname{Im} Z(S_i) = 0$ for a unique simple object $S_i$. Assume that the tilts $\mu^{\pm}_{S_i}(\mathcal{A})$ are also of finite-length. Then there exists a neighbourhood $U \subset \mathsf{Stab}(\mathcal{D})$ of $(Z, \mathcal{P})$ such that

- $Z(S_i) \in \mathbb{R}_{<0}$ implies $U \subset \mathsf{Stab}(\mathcal{A}) \cup \mathsf{Stab}(\mu^{-}_{S_i}(\mathcal{A}))$,
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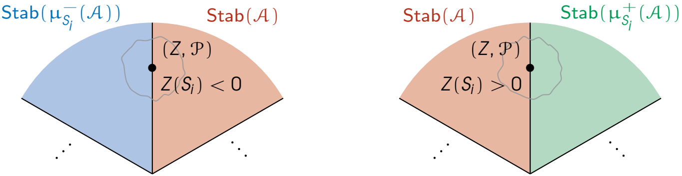

# Summary

**Distinguished components in $\mathrm{Stab}(\mathcal{D})$**

Let $\mathcal{D}$ be a $\Delta$-category equipped with a finite-length heart $\mathcal{A}$ with $n$ simple objects, defined up to tilts at simple objects.

Then $\mathrm{Stab}(\mathcal{D})$ is a complex manifold of dimension $n$, equipped with a distinguished connected component $\mathrm{Stab}_*(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$.

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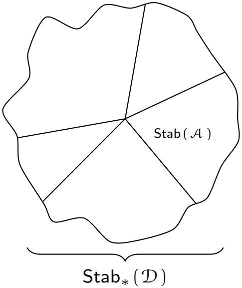


$\sqcup \cdots$

# Quivers

## Definition

A quiver $Q$ is a finite oriented graph. It is given by

- a finite set $Q_0$ (vertices),
- a finite set $Q_1$ (arrows),
- two maps $s\colon Q_1 \to Q_0$ (taking an arrow to its source) $t\colon Q_1 \to Q_0$ (taking an arrow to its target).

A simple example is the $\vec{A}_n$ quiver, which is an orientation of the $A_n$ Dynkin diagram:

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A natural operation on quivers is that of mutation at vertices, defined by Fomin–Zelevinsky. From now on, we assume that $Q$ has no loops or 2-cycles.

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Fix $i \in Q_0$. The mutation $\mu_i(Q)$ is the quiver obtained from $Q$ as follows.

- For each subquiver $j \xrightarrow{b} i \xrightarrow{a} k$, add a new arrow $j \xrightarrow{[ab]} k$.
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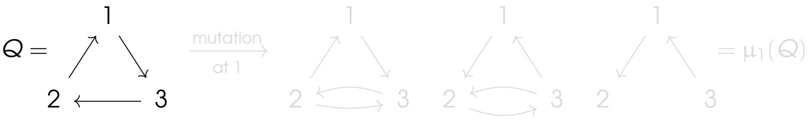

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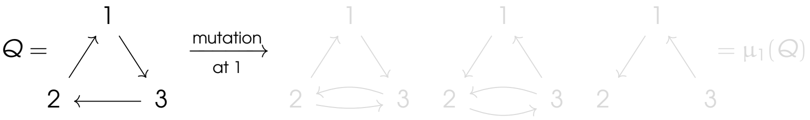

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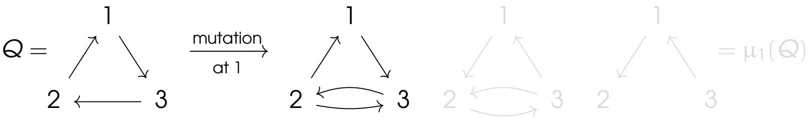

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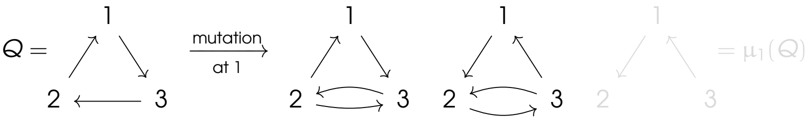

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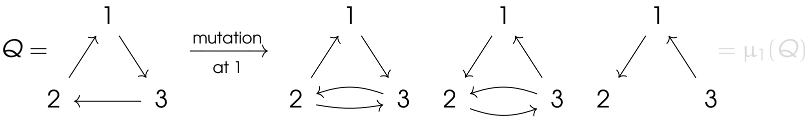

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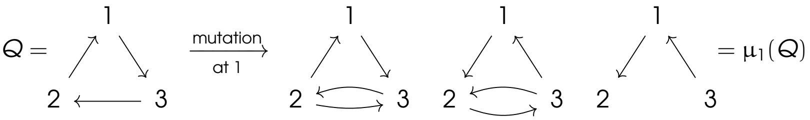

# The complete path algebra

A natural object one can attach to a quiver $Q$ is its complete path algebra: fix an algebraically closed field $k$, and set

$$\widehat{kQ} := \prod_{p \text{ path}} kp.$$

Consider its bounded derived category $\mathcal{D}^b(\mathrm{Mod}(\widehat{kQ}))$. One would like to obtain linear equivalences of this category under mutations, but it turns out that this is not the case.

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# Quivers with potential

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$$HH_0(Q) \coloneqq \frac{\widehat{kQ}}{[\widehat{kQ}, \widehat{kQ}]},$$

that is the set of infinite linear combination of cycles of $Q$. For each arrow $a \in Q_1$, we have the *cyclic derivative* $\partial_a \colon HH_0(Q) \to \widehat{kQ}$ such that any path $p$,

$$\partial_a p \coloneqq \sum_{p=uav} vu.$$

A potential on $Q$ is an element $W \in HH_0(Q)$ not involving cycles of length 0.

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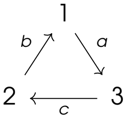


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Derksen–Weyman–Zelevinsky showed that mutations can be extended to quivers with potentials in a nice way.

**Theorem (Derksen–Weyman–Zelevinsky)**

The mutation operation $Q \mapsto \mu_i(Q)$ admits a good extension to quivers with potentials

$$(Q, W) \longmapsto \mu_i(Q, W) \coloneqq (Q', W'),$$

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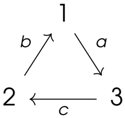
$Q =$ 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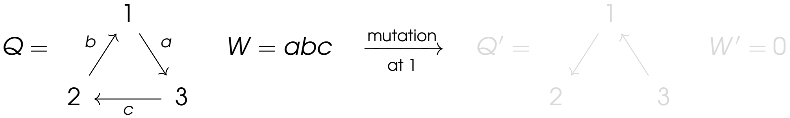

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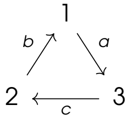
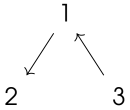
$Q =$ 

$W = abc \xrightarrow[\text{at } 1]{\text{mutation}}$ $Q' =$ 

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# The complete Ginzburg algebra

## Definition

Let $(Q, W)$ be a quiver with potential. Define a new quiver $\widetilde{Q}$, with $\widetilde{Q}_0 = Q_0$ and graded arrow as follows:

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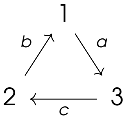


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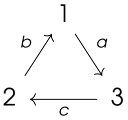


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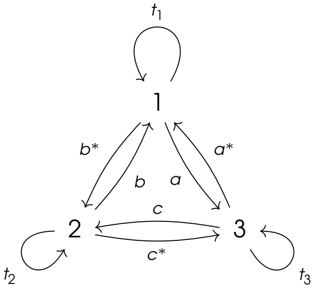


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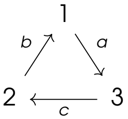


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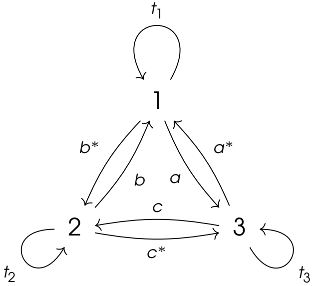


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The main result we need here is due to Keller and Yang.

**Theorem (Keller–Yang)**

Fix a quiver with potential $(Q, W)$, and consider the Δ-category $\mathcal{D}(Q, W) := \mathcal{D}^b(\text{dg-Mod}(\Gamma(Q, W)))$.

- $\mathcal{D}(Q, W)$ is of finite type over $k$ and $CY_3$.
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We can now see how mutations of quivers with potentials correspond to tilts in $\mathcal{D}(Q, W)$.

**Theorem (Keller–Yang)**

Let $(Q, W)$ be a quiver with potential and fix a vertex $i \in Q_0$. Let $(Q', W') = \mu_i(Q, W)$. There is a canonical pair of $k$-linear triangulated equivalences

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Moreover, if we denote by $S_i$ the simple object in $\mathcal{A}(Q, W)$ associated to the vertex $i$, we have

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# Ideal triangulations

A marked closed surface is a pair $(S, M)$ consisting of a compact, oriented close surface $S$ of genus $g$ and a finite non-empty set $M \subset S$ of marked points, also called punctures, of cardinality $\#M = m > 0$. For the purpose of the following discussion, we suppose that if $g = 0$, then $m \geqslant 5$.

An ideal triangulation $T$ of $(S, M)$ is a triangulation of $S$, whose vertex set is precisely $M$. Notice that ideal triangulations have always $6g - 6 + 3m$ edges. It is called non-degenerate if every vertex has valency $\geqslant 3$.

To a non-degenerate triangulation $T$, we associate a quiver $Q(T)$ with no loops and 2-cycles as follows.

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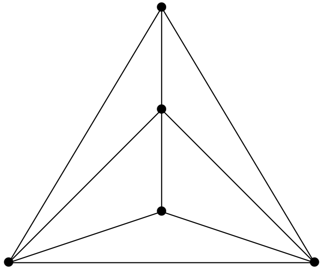

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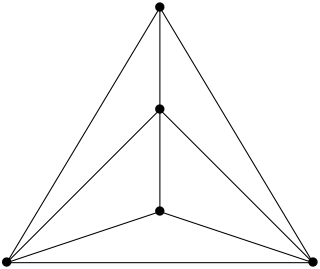

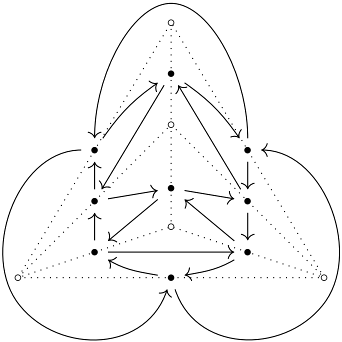

# The potential

Notice that there are two natural system of cycles in $Q(T)$.

- $\Delta$) Inside each triangle $\Delta$ of $T$, a clockwise 3-cycle $W_\Delta$.
- $p$) Around each puncture $p \in M$ of valency $d$, an anticlockwise $d$-cycle $W_p$.

We define the potential $W(T)$ on $Q(T)$ by taking the sum

$$W(T) := \sum_{\Delta \text{ triangle of } T} W_\Delta - \sum_{p \in M} W_p.$$

Thus, we obtain a quiver with potential $(Q(T), W(T))$ associated to a non-degenerate ideal triangulation of $(S, M)$, and consequently a $CY_3$ $\Delta$-category $\mathcal{D}(T)$ of finite type over $k$, equipped with a canonical t-structure whose canonical heart $\mathcal{A}(T)$ is of finite length and has $6g - 6 + 3m$ simple objects, in bijection with the vertices of $Q(T)$ and the edges of $T$.

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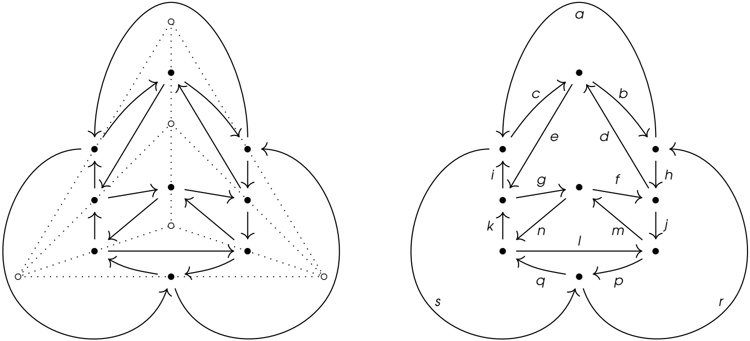


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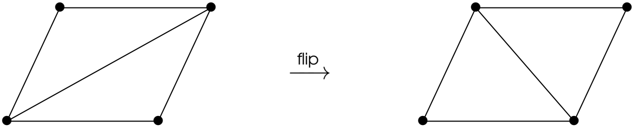
# Flips and mutations

We say that two non-degenerate ideal triangulations $T_1$ and $T_2$ are related by a flip, if they differ locally in a quadrilateral by a flip of the diagonal.




**Lemma**

If two non-degenerate ideal triangulations $T_1$ and $T_2$ are related by a flip, then the corresponding quivers with potential $(Q(T_1), W(T_1))$ and $(Q(T_2), W(T_2))$ are related by a mutation at the vertex corresponding to the flipped edge.

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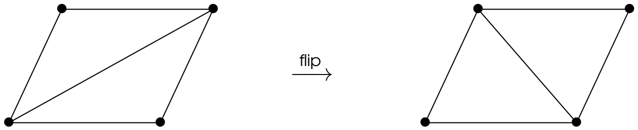


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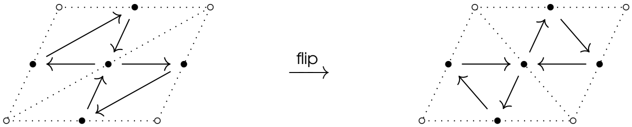


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# Problem with non-degenerate triangulations

It is not true that all non-degenerate triangulations are related by flips through non-degenerate triangulations.

Labardini-Fragoso extended the correspondence between ideal triangulations and quivers with potential to the larger class of triangulations containing vertices of valency $\leqslant 2$, proving the much more difficult result that mutations flips induce mutations in this general context. His result applies to marked surface with boundary as well.

Since every ideal triangulation is connected by a finite chain of flips, we find

**Theorem (Labardini-Fragoso)**

Let $(S, M)$ be a marked bordered surface.

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## Distinguished component in $\mathsf{Stab}(\mathcal{D}(S, M))$

Let $(S, M)$ be a marked bordered surface.

- There is an associated $CY_3$ $\Delta$-category $\mathcal{D}(S, M)$ of finite type
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(We have to exclude from the statement some degenerate topologies)

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# Thank you!


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