Fibered categories, II

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Plan of the talk

1. Recap: fibered categories
2. 2-Yoneda Lemma
3. Equivariant objects in fibered categories
4. Bibliography

Motivation: moduli space of elliptic curves

Definition

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Fact 1: Two elliptic curves are isomorphic if and only if they have the same *j*-invariant.

Fact 2: The map $j\colon \mathfrak{h} \to \mathbb{C}$ is a branched cover, with two branching points being $0, 1728 \in \mathbb{C}$.

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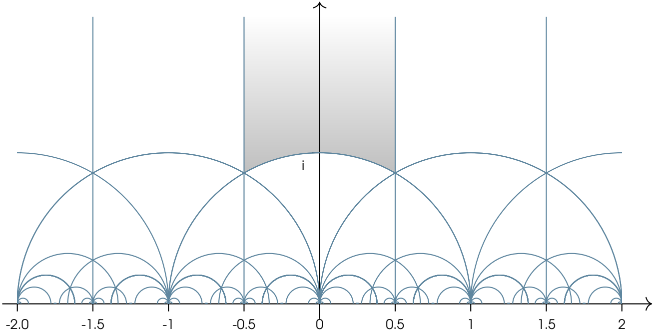

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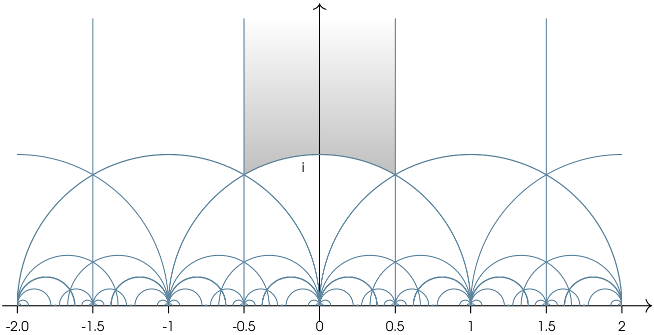

A natural conclusion is that $M_{\text{ell}} = \mathbb{C}$ is the moduli space of elliptic curves. However, $M_{\text{ell}} = \mathbb{C}$ lacks an important additional property and cannot be considered the *true* moduli space of elliptic curves.

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1. the map $j_X\colon B \to M_{\text{ell}}$ to be holomorphic,
2. there exists a universal family $p\colon \mathcal{E} \to M_{\text{ell}}$ such that every family $\pi\colon X \to B$ is isomorphic to the pullback of $\mathcal{E}$ via j_X:

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "\pi"'] & j_X^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ B \arrow[r, equal] & B \arrow[r, "j_X"] & M_{\text{ell}} \end{tikzcd}$$

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$M_{\text{ell}} = \mathbb{C}$ does not satisfy the second condition!

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There exists elliptic curves with non-trivial automorphism $\alpha\colon X_0 \to X_0$. This allows the construction of a non-trivial family $X \to B = S^1 \times S^1$ as follows: start with the trivial family $S^1 \times [0, 1] \times X_0$ over the cylinder and then glue the two ends by identifying the fiber using the automorphism α.

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The associated map j_X is constant, whith the value being $j(X_0)$, so the pullback is a trivial family $B \times X_0 \to B$. This contradicts the existence of a universal family, since X is locally trivial but globally non-trivial.

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We get a forgetful functor $\mathcal{M}_{\text{ell}} \to \underline{\text{Mfd}}_{\mathbb{C}}$, that gives to $\mathcal{M}_{\text{ell}}$ the structure of a category over $\underline{\text{Mfd}}_{\mathbb{C}}$ fibered in groupoids.

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We can also define another category $\mathcal{E}$, where the objects are families of elliptic curves $X \xrightarrow{\pi} B$ with a second section $O' \colon B \to X$, and morphism are like in $\mathcal{M}_{\mathrm{ell}}$. Forgetting O', we get a functor $\mathcal{E} \to \mathcal{M}_{\mathrm{ell}}$.

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Now a family of elliptic curves over a base space B is the same as functor $F\colon \underline{\mathrm{Mfd}}_{\mathbb{C}}/B \to \mathcal{M}_{\mathrm{ell}}$: by Yoneda, it is determined by

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Suggestion

Moduli problems are naturally formulated in terms of fibered categories.

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\zeta \arrow[drr, bend left, "\psi"] \arrow[dr, dashed, "\exists!\theta"'] \arrow[dd, mapsto] & & \\
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\begin{tikzcd}
\xi \arrow[r, dotted, "\exists\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
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A fibered category over $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f\colon U \to V$ in $\mathcal{C}$ and an object $\eta \in \mathcal{F}$ with $p_{\mathcal{F}}\eta = V$, there is a cartesian arrow $\phi\colon \xi \to \eta$ with $p_{\mathcal{F}}\phi = f$.

Examples of fibered categories

- A group G is a category with a single object, and elements of the group as arrows. For group homomorphism $p\colon G \to H$, every arrow in G is cartesian. Moreover, G is fibered over H if and only if p is surjective.

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- For a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we have the the slice category $\mathcal{C}/X$ whose objects are arrows $U \to X$ in $\mathcal{C}$ and morphisms are commutative diagrams
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- Let G be a topological group, and $\mathcal{B}G$ the category whose objects are principal G-bundles $P \to U$ and morphisms are diagrams
$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$
with ϕ equivariant. The forgetful functor $\mathcal{B}G \to \underline{\mathrm{Top}}$ projecting to the codomain is fibered.

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- Similarly, in the algebraic setting we have quasi-coherent sheaves $\underline{\text{QCoh}}/S \to \underline{\text{Sch}}/S$.

Fibers over objects

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This allows us to define a pseudo-functor

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where f^* is the choice of a pullback of f. It is not a functor, because in general $\mathrm{id}_U^* \neq \mathrm{id}_{\mathcal{F}(U)}$ and $(g \circ f)^* \neq f^* \circ g^*$, but we have natural transformations between them.

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Question

Which fibered categories over $\mathcal{C}$ corresponds to actual functors?

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Thus, the pseudo-functor $\mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ is actually a functor $\mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Set}}$.

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Question

Can we have a Yoneda's lemma involving fibered categories over $\mathcal{C}$?

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- 1-morphisms are *cartesian functors*: a functor $\Phi : \mathcal{F} \to \mathcal{G}$ such that $p_{\mathcal{G}} \circ \Phi = p_{\mathcal{F}}$ and sending cartesian arrows to cartesian arrows.

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We define the 2-Yoneda embedding as the functor

$$\mathcal{C} \xrightarrow{\mathbb{h}} \underline{\mathrm{Fib}}(\mathcal{C}) : \begin{cases} X & \mapsto \mathbb{h}_X := \mathcal{C}/X \\ f & \mapsto \mathbb{h}_f := f \circ - \end{cases}$$

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More precisely, $\mathbb{h}_X$ is the slice category $\mathcal{C}/X \to \mathcal{C}$, while for $X \xrightarrow{f} Y$, the cartesian functor $\mathbb{h}_f$ is defined on objects as

$$(U \xrightarrow{\phi} X) \longmapsto (U \xrightarrow{f\circ\phi} Y)$$

and on morphisms as

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "\phi"'] & & V \arrow[dl, "\psi"] \\ & X & \end{tikzcd} \longmapsto \begin{tikzcd} U \arrow[rr] \arrow[dr, "f\circ\phi"'] & & V \arrow[dl, "f\circ\psi"] \\ & Y & \end{tikzcd}$$

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Let $X \in \mathcal{C}$ and $\mathcal{F} \in \underline{\mathrm{Fib}}(\mathcal{C})$. There is an equivalence of categories $\underline{\mathrm{Fib}}(\mathcal{C})(\mathbb{h}_X, \mathcal{F}) \simeq \mathcal{F}(X)$, with the following being inverse equivalences.

$$\underline{\mathrm{Fib}}(\mathcal{C})(\mathbb{h}_X, \mathcal{F}) \to \mathcal{F}(X) : \begin{cases} F \mapsto F(\mathrm{id}_X) \\ \alpha \mapsto \alpha_{\mathrm{id}_X} \end{cases}$$

$$\mathcal{F}(X) \to \underline{\mathrm{Fib}}(\mathcal{C})(\mathbb{h}_X, \mathcal{F}) : \begin{cases} \xi \mapsto F_\xi : \begin{cases} U \xrightarrow{\phi} X & \mapsto \quad \phi^*\xi \\ \begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "\phi"'] & & V \arrow[dl, "\psi"] \\ & X & \end{tikzcd} & \mapsto \quad f^* : \phi^*\xi \to \psi^*\xi \end{cases} \\ \text{if you want, try to DIY} \end{cases}$$

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Here f^* is the unique arrow in $\mathcal{F}$ making the diagram on the right commutative.

$$\begin{tikzcd}
\phi^*\xi \arrow[dr, dashed, "\exists! f^*"] \arrow[drr, bend left] \arrow[dd] & & \\
& \psi^*\xi \arrow[r] \arrow[dd] & \xi \arrow[dd] \\
U \arrow[dr, "f"'] \arrow[drr, bend left, "\phi"] & & \\
& V \arrow[r, "\psi"'] & X
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Applying the cartesian functor F, we get a cartesian arrow between $F(\phi)$ and $F(\mathrm{id}_X) = \xi$. So there is a canonical isomorphism between $F(\phi)$ and $\phi^* \xi = F_\xi(\phi)$. Again, one can show that this defines an isomorphism of functors.

Representable fibered categories

Definition

A fibered category over $\mathcal{C}$ is representable if it is equivalent to a category of the form $\mathcal{C}/X$.

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Fact 3. A category $\mathcal{S}$ is equivalent to a set iff $\mathcal{S}$ is groupoid such that from a given object to another there is at most one arrow.

Splitting fibered categories

Definition

Consider a fibered category $\mathcal{F}$ over $\mathcal{C}$ with a class $\mathcal{K}$ of cartesian arrows such that for each arrow $f\colon U \to V$ in $\mathcal{C}$ and each object η in $\mathcal{F}(V)$, there exists a unique arrow in $\mathcal{K}$ with target η mapping to f in $\mathcal{C}$. Such class is called splitting if it contains all identities and is closed under composition.

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This defines a fibered category over $\mathcal{C}$, denoted $\mathcal{F}'$. By definition, it is splitting and it comes with a functor $\mathcal{F}' \to \mathcal{F}$ sending

$$h_X \xrightarrow{\Phi} \mathcal{F} \in \mathcal{F}'$$

to $\Phi(\mathrm{id}_X) \in \mathcal{F}(X)$. By the 2-Yoneda lemma, this defined an equivalence of categories on every fiber, and thus is an equivalence of fibered categories.

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Fix a category $\mathcal{C}$ with finite product and terminal object pt.

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Definition

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A group object structure on an object $G \in \mathcal{C}$ is equivalent to assigning three arrows

- the multiplication $m_G \colon G \times G \to G$
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$$\begin{tikzcd} \mathrm{pt} \times G \arrow[r, "e_G \times \mathrm{id}_G"] \arrow[dr, equal] & G \times G \arrow[d, "m_G"] \\ & G \end{tikzcd}$$

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Equivalently, f is G-equivariant if the diagram on the right commutes.

$$\begin{tikzcd} G \times X \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{id}_G \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\alpha_Y"'] & Y \end{tikzcd}$$

Equivariant objects in fibered categories

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A G-equivariant arrow of $\mathcal{F}(X)$ is an arrow $\phi \colon \rho \to \sigma$ of $\mathcal{F}(X)$ such that the induced function $h_\rho(\xi) \to h_\sigma(\xi)$ is $h_G(U)$-equivariant for all U and all $\xi \in \mathcal{F}(U)$.

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Definition

A G-equivariant object of $\mathcal{F}(X)$ is an object $\rho \in \mathcal{F}(X)$, together with a natural transformation $(h_G \circ p_{\mathcal{F}}) \times h_\rho \to h_\rho$ such that for any object $U \in \mathcal{C}$ and $\xi \in \mathcal{F}(U)$,

1. the induced function $h_G(U) \times h_\rho(\xi) \to h_\rho(\xi)$ is an action of the group $h_G(U)$ on the set $h_\rho(\xi)$,
2. the function $h_\rho(\xi) \to h_X(U)$ induced by $p_{\mathcal{F}}$ is $h_G(U)$-equivariant.

A G-equivariant arrow of $\mathcal{F}(X)$ is an arrow $\phi : \rho \to \sigma$ of $\mathcal{F}(X)$ such that the induced function $h_\rho(\xi) \to h_\sigma(\xi)$ is $h_G(U)$-equivariant for all U and all $\xi \in \mathcal{F}(U)$.

The G-equivariant objects over X are the objects of a category $\mathcal{F}^G(X)$, in which the arrows are the equivariant arrows in $\mathcal{F}(X)$.

Thank you!

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