Cones, Segre classes, and deformation to the normal cone

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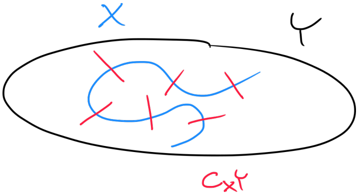
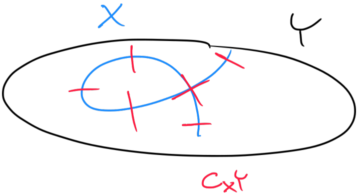
Plan of the talk

Motivation: normal bundle for singular subvarieties

Smooth case:



Singular case:



NORMAL BUNDLE $\rightsquigarrow$ NORMAL CONE

In other words, in the singular setting cones are the right substitute of vector bundles.

Cones: definition

All schemes are of finite type over a field k.

Definition

Let $S^\bullet = S^0 \oplus S^1 \oplus \cdots$ be a sheaf of graded $\mathcal{O}_X$-algebras on a scheme X, such that

- the canonical map $\mathcal{O}_X \to S^0$ is an iso,
- $S^\bullet$ is locally generated as an $\mathcal{O}_X$-algebra by S^1.

Define the cone

$$C := \underline{\mathrm{Spec}}(S^\bullet), \qquad \pi\colon C \to X,$$

and the projective cone

$$\mathbb{P}(C) := \underline{\mathrm{Proj}}(S^\bullet), \qquad p\colon \mathbb{P}(C) \to X.$$

It comes with a canonical line bundle $\mathcal{O}(1)$. The morphism p is proper.

Vector bundles are cones

Lemma

Consider a rank r vector bundle $E \to X$. Denote by $\mathcal{E}$ the sheaf of sections. Then

$$E = \underline{\mathrm{Spec}}(\mathrm{Sym}^\bullet \mathcal{E}^\vee).$$

In other words, the total space E is the cone associated to the sheaf $\mathrm{Sym}^\bullet \mathcal{E}^\vee$ of graded $\mathcal{O}_X$-algebras.

Idea of the proof. Consider an affine subset $U = \mathrm{Spec}(A) \subset X$ such that $\mathcal{E}^\vee(U) = (\mathcal{O}_X(U)^{\oplus r})^\vee = \mathrm{Hom}(A^{\oplus r}, A)$. Then

$$\mathrm{Sym}^\bullet \mathcal{E}^\vee(U) = A[x_1, \ldots, x_r] \quad \Longrightarrow \quad \mathrm{Spec}\big(\mathrm{Sym}^\bullet \mathcal{E}^\vee(U)\big) = \mathbb{A}^r \times U.$$

Normal cones

Let $X \hookrightarrow Y$ be a closed imbedding (*i.e.* is isomorphic to a closed subscheme) and $\mathcal{I}$ the ideal sheaf of X in Y.

Definition

The normal cone of X in Y is the cone defined by the graded sheaf of $\mathcal{O}_X$-algebra $\bigoplus_{n\geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}$:

$$C_X Y := \underline{\mathrm{Spec}}\Big(\bigoplus\nolimits_{n\geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}\Big), \qquad \pi\colon C_X Y \to X.$$

If $X \hookrightarrow Y$ is regular, then $\mathcal{I}/\mathcal{I}^2$ is locally free and the natural map $\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2) \to \bigoplus_{n\geqslant 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. In particular,

$$C_X Y = \underline{\mathrm{Spec}}\big(\mathrm{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2)\big)$$

is the normal bundle, also denote by $N_X Y$.

Blow-ups

Let $X \hookrightarrow Y$ be a closed imbedding and $\mathcal{I}$ the ideal sheaf of X in Y.

Definition

The blow-up of Y along X is the projective cone defined by the graded sheaf of $\mathcal{O}_Y$-algebra $\bigoplus_{n\geqslant 0} \mathcal{I}^n$:

$$Bl_X Y := \underline{\mathbf{Proj}}\Big(\bigoplus_{n\geqslant 0} \mathcal{I}^n\Big), \qquad p\colon Bl_X Y \to Y.$$

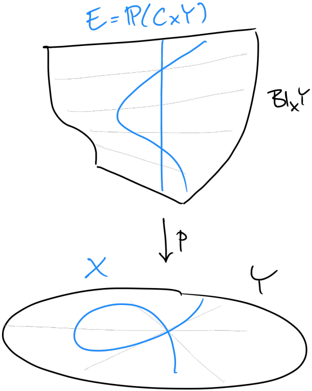
The canonical line bundle $\mathcal{O}(1)$ on $Bl_X Y$ is the ideal sheaf of $E := p^{-1}(X)$, which is therefore a Cartier divisor of $Bl_X Y$ called the exceptional divisor.

By construction, the exceptional divisor is the projective cone of $(\bigoplus_n \mathcal{I}^n) \otimes_{\mathcal{O}_Y} \mathcal{O}_X = \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$. Thus,

$$E = \mathbb{P}(C_X Y).$$

A schematic picture of a blow-up

$$\begin{tikzcd} E = \mathbb{P}(C_X Y) \arrow[r, hook] \arrow[d] & Bl_X Y \arrow[d, "p"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$



Proposition

If X is nowhere dense in Y, then $p \colon Bl_X Y \to Y$ is birational. More precisely, p is an iso between $Bl_X Y - E$ and $Y - X$.

Applications of blow-ups

Slogan

Blow-up is the "most economic way" to turn a subscheme into a Cartier divisor.

Blow-ups are fundamental constructions in algebraic geometry, with essential applications in:

- *birational geometry*: every birational morphism between projective varieties is a blow-up,
- Hironaka's *resolution of singularities*: resolutions are constructed by repeated blow-ups,
- *stability conditions and geometric PDEs*.

Segre class for vector bundles

Recall from Campbell's talk. Consider $p\colon \mathbb{P}(E) \to X$ the projective bundle associated to a rank $(e+1)$ vector bundle E. Its Segre classes is the operation

$$s_i(E)\cap : A_k X \to A_{k-i}X,$$

defined by

$$s_i(E)\cap\alpha := p_*(c_1(\mathcal{O}(1))^{e+i}\cap p^*\alpha).$$

Question

Can we generalise it to cones?

Problem

For a general cone C, $p\colon \mathbb{P}(C) \to X$ is not necessarily flat, so we cannot bull-back classes to $\mathbb{P}(C)$. However, there is a canonical class on $\mathbb{P}(C)$, that is the fundamental class.

Naïve definition:

$$s(C)\text{``}=\text{''}p_*\left(\sum_{i\geqslant 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C)]\right) \in A_\bullet X.$$

Segre class for cones

Let $C = \underline{\mathrm{Spec}}(S^\bullet)$ be a cone over X, and define its projective completion to be

$$\mathbb{P}(C \oplus \mathbf{1}) := \underline{\mathrm{Proj}}(S^\bullet[z]), \qquad S^k[z] := S^k \oplus S^{k-1}z \oplus \cdots \oplus S^0 z^k,$$

together with the proper morphism $q\colon \mathbb{P}(C \oplus \mathbf{1}) \to X$. Note that $\mathbb{P}(C \oplus \mathbf{1}) = C \cup \mathbb{P}(C)$, with $\mathbb{P}(C)$ called the hyperplane at infinity.

Definition

Define the Segre class $s(C) \in A_\bullet X$ of a cone C as

$$s(C) := q_*\left(\sum\nolimits_{i \geqslant 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})]\right).$$

Why adding the trivial factor $\mathbf{1}$ is the right thing to do?

If $S^\bullet = \mathcal{O}_X$ is trivial, then $\mathbb{P}(C)$ is empty while $\mathbb{P}(C \oplus \mathbf{1})$ is not; we get different answers. With definition, we get $s(C \oplus \mathbf{1}) = s(C)$.

Properties of the Segre class

Proposition

1. If $E \to X$ is a vector bundle,
$$s(E) = c(E)^{-1} \cap [X].$$
2. Let $C_1, \dots, C_r$ be the irreducible components of C, m_i the geometric multiplicity of C_i in C. Then
$$s(C) = \sum_{i=1}^{r} m_i \, s(C_i).$$

In other words, the definition agree with the previous one on vector bundles, and respects irreducible components of cones.

Properties of the Segre class

Recall

VBs: $$c(E)^{-1} \cap [X] = p_* \left(\sum_{i \geqslant 0} c_1(\mathcal{O}(1))^i \cap p^*[X] \right)$$

Cones: $$s(C) = q_* \left(\sum_{i \geqslant 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})] \right)$$

Sketch of the proof.

1. Notice that $[P(E \oplus \mathbf{1})] = q^*[X]$, so that

$$s(E) = c(E \oplus \mathbf{1})^{-1} \cap [X].$$

   By Whitney formula, $c(E \oplus \mathbf{1}) = c(E)$.

2. It can be shown that each C_i (which is a cone) is an open dense in $\mathbb{P}(C_i \oplus \mathbf{1})$. Thus,

$$[\mathbb{P}(C \oplus \mathbf{1})] = \sum_{i=1}^{r} m_i \, [\mathbb{P}(C_i \oplus \mathbf{1})]$$

   and the assertion follows.

Segre class of subschemes

Consider $X \hookrightarrow Y$ a closed immersion, $C_X Y \to X$ its normal cone:

$$C_X Y = \underline{\mathrm{Spec}}\left(\bigoplus_{n \geqslant 0} \mathcal{I}^n / \mathcal{I}^{n+1}\right).$$

Define the Segre class of X in Y as

$$s(X, Y) := s(C_X Y) \in A_\bullet X.$$

We have that for X regularly imbedded, $s(X, Y) = c(N_X Y)^{-1} \cap [X]$.

Properties of the Segre class of subschemes

Theorem

Let $f\colon Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' := f^{-1}(X)$ the inverse image scheme, $g\colon X' \to X$ the induced morphism.

1. If f is proper, Y irreducible, and f maps each irreducible component of Y' onto Y, then
$$g_* s(X', Y') = \deg(Y'/Y)\, s(X, Y).$$
2. If g is flat,
$$g^* s(X, Y) = s(X', Y').$$

Corollary

When f is birational, *i.e.* $\deg(Y'/Y) = 1$, we obtain birational invariance of Segre classes.

Moreover, if both X and X' are regular, we obtain birational invariance of Chern classes of normal bundles.

Proof of (1)

Assume f proper, Y and Y' irreducible ($s(-)$ and $\deg(-)$ behaves well with respect to irreducible components). Set $d := \deg(Y'/Y)$.

$$
\begin{tikzcd}
& E' := \mathbb{P}(C_{X'}Y' \oplus \mathbf{1}) \arrow[rr, hook] \arrow[dl, "q'"'] \arrow[dd, "G"'] & & M' := Bl_{X' \times 0}(Y' \times \mathbb{A}^1) \arrow[dl] \arrow[dd, "F"] \\
X' \arrow[rr, hook, crossing over] \arrow[dd, "g"'] & & Y' & \\
& E := \mathbb{P}(C_X Y \oplus \mathbf{1}) \arrow[rr, hook] \arrow[dl, "q"'] & & M := Bl_{X \times 0}(Y \times \mathbb{A}^1) \arrow[dl] \\
X \arrow[rr, hook] & & Y \arrow[from=uu, crossing over, "f"] &
\end{tikzcd}
$$

We have $G^* \mathcal{O}_E(1) = \mathcal{O}_{E'}(1)$ and

$$
f_*[Y'] = d\,[Y] \quad \overset{(\dagger)}{\Longrightarrow} \quad F_*[M'] = d\,[M] \quad \overset{(\ddagger)}{\Longrightarrow} \quad G_*[E'] = d\,[E]
$$

($\dagger$) LHS is computed on open dense, so $\times \mathbb{A}^1$ and blow-up does not change it,

($\ddagger$) proper push-forward commutes with intersecting with a Cartier divisor.

Proof of (1)

If f is proper,

$$\begin{array}{ccc} E' = \mathbb{P}(C_{X'}Y' \oplus \mathbf{1}) & \xrightarrow{q'} & X' \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ E = \mathbb{P}(C_X Y \oplus \mathbf{1}) & \xrightarrow[q]{} & X \end{array} \quad \rightsquigarrow \quad \left.\begin{array}{l} G^* \mathcal{O}_E(1) = \mathcal{O}_{E'}(1) \\ G_*[E'] = d\,[E] \end{array}\right\} \quad (\sharp)$$

Thus, we find

$$\begin{aligned} g_* s(X', Y') &= g_* q'_* \Big(\sum\nolimits_i c_1(\mathcal{O}_{E'}(1))^i \cap [E'] \Big) && \text{defn of } s(X', Y') \\ &= q_* G_* \Big(\sum\nolimits_i c_1(\mathcal{O}_{E'}(1))^i \cap [E'] \Big) && g_* q'_* = q_* G_* \\ &= q_* \Big(\sum\nolimits_i c_1(\mathcal{O}_E(1))^i \cap d\,[E] \Big) && \text{proj formula} + (\sharp) \\ &= d\, s(X, Y) && \text{defn of } s(X, Y). \end{aligned}$$

Proof of (2)

If g is flat,

$$\begin{tikzcd} E' = \mathbb{P}(C_{X'}Y' \oplus \mathbf{1}) \arrow[r, "q'"] \arrow[d, "G"'] & X' \arrow[d, "g"] \\ E = \mathbb{P}(C_X Y \oplus \mathbf{1}) \arrow[r, "q"'] & X \end{tikzcd} \quad \rightsquigarrow \quad \left.\begin{array}{l} G^*\mathcal{O}_E(1) = \mathcal{O}_{E'}(1) \\ G^*[E] = [E'] \end{array}\right\} \quad (\flat)$$

Thus, we find

$$\begin{aligned} g^* s(X,Y) &= g^* q_* \Big(\sum\nolimits_i c_1(\mathcal{O}_E(1))^i \cap [E]\Big) && \text{defn of } s(X,Y) \\ &= q'_* G^* \Big(\sum\nolimits_i c_1(\mathcal{O}_E(1))^i \cap [E]\Big) && g^* q_* = q'_* G^* \\ &= q'_* \Big(\sum\nolimits_i c_1(\mathcal{O}_{E'}(1))^i \cap [E']\Big) && \text{flat-bullback formula} + (\flat) \\ &= s(X', Y') && \text{defn of } s(X', Y'). \end{aligned}$$

Applications

- Let X be a scheme which can be imbedded as a closed subscheme of a non-singular variety Y. Then the class

  $$c(X) := c(T_Y|_X) \cap s(X, Y) \in A_\bullet X$$

  does not depend on the choice of imbedding. It is called the canonical class of X.

- For an irreducible subvariety X of a variety Y, the coefficient $e_X Y$ of $[X]$ in the class $s(X, Y)$ is called the multiplicity of Y along X.

- Geometry of linear systems: if a subscheme is the base locus of a linear system, its Segre class is related to important invariants of the system.

See Fulton for further readings.

Deformation of closed subschemes

Let $X \hookrightarrow Y$ be a closed subscheme, with normal cone $C := C_X Y$.

Theorem

There exists a deformation scheme $M = M_X Y$ together with a closed imbedding $X \times \mathbb{P}^1 \hookrightarrow M$ and a morphism $\rho\colon M \to \mathbb{P}^1$ s.t. the diagram commutes and

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "\mathrm{pr}"'] & & M \arrow[dl, "\rho"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

- ρ is flat,
- over $t \in \mathbb{P}^1 - \{\infty\}$, the fiber is $M_t := \rho^{-1}(t) = Y$ and the imbedding is the given one $X \hookrightarrow Y$,
- over $t = \infty$, we have $M_\infty := \rho^{-1}(\infty)$ is the sum of two effective Cartier divisors in M:
$$M_\infty = \mathbb{P}(C \oplus \mathbf{1}) + Bl_X Y$$
and the imbedding $X = X \times \{\infty\} \hookrightarrow M_\infty$ is the imbedding of X in C as the zero section, followed by the imbedding of C in $\mathbb{P}(C \oplus \mathbf{1})$,
- the two components of M_∞ intersect at $\mathbb{P}(C)$.

Deformation of closed subschemes

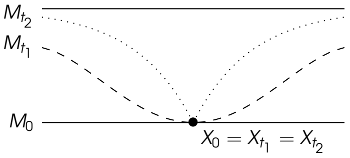
$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "\mathrm{pr}"'] & & M \arrow[dl, "\rho"] \\ & \mathbb{P}^1 & \end{tikzcd} \quad \rightsquigarrow \quad \begin{cases} X_t & := \mathrm{pr}^{-1}(t) = X \times \{t\} \\ M_t & := \rho^{-1}(t) \end{cases}$$

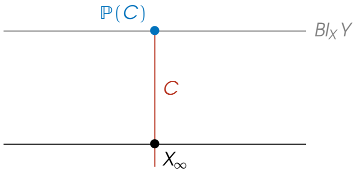
At $t \in \mathbb{P}^1 - \{\infty\}$:

$$[X_t \hookrightarrow M_t] = [X \hookrightarrow Y]$$

At $t = \infty$:

$$[X_\infty \hookrightarrow M_\infty] = [X \hookrightarrow \mathbb{P}(C \oplus \mathbf{1}) + Bl_X Y]$$





The construction

$$\boxed{M := Bl_{X_\infty}(Y \times \mathbb{P}^1)}$$

- *Flat morphism* ρ*:* it is the composition of flat morphisms

$$M = Bl_{X_\infty}(Y \times \mathbb{P}^1) \xrightarrow{p} Y \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}} \mathbb{P}^1.$$

- *Closed imbedding* $X \times \mathbb{P}^1 \hookrightarrow M$*.* From the sequence of closed imbeddings

$$X = X_\infty \hookrightarrow X \times \mathbb{P}^1 \hookrightarrow Y \times \mathbb{P}^1,$$

  it follows that $Bl_{X_\infty}(X \times \mathbb{P}^1) \hookrightarrow Bl_{X_\infty}(Y \times \mathbb{P}^1)$. Since $X_\infty \hookrightarrow X \times \mathbb{P}^1$ is a Cartier divisor, $Bl_{X_\infty}(X \times \mathbb{P}^1)$ is identified with $X \times \mathbb{P}^1$. Thus,

$$X \times \mathbb{P}^1 \hookrightarrow Bl_{X_\infty}(Y \times \mathbb{P}^1) = M$$

- Since $M \xrightarrow{p} Y \times \mathbb{P}^1$ is an iso away from $Y \times \{\infty\}$, we find that for $t \in \mathbb{P}^1 - \{\infty\}$

$$[X_t \hookrightarrow M_t] = [X \hookrightarrow M].$$

- For $t = \infty$, see Fulton.

Specialisation to the normal cone

Let $X \hookrightarrow Y$ be a closed imbedding, with normal cone $C := C_X Y$.

Definition

Define the homomorphisms $\sigma : Z_k Y \to Z_k C$ by

$$\sigma[V] := [C_{X \cap V} V]$$

and extended linearly.

Proposition

If $\alpha \sim 0$, then $\sigma(\alpha) \sim 0$. Hence, we define the specialisation homomorphism

$$\sigma : A_k Y \to A_k C.$$

Specialisation to the normal cone

Proposition

If $\alpha \sim 0$, then $\sigma(\alpha) \sim 0$.

Sketch of the proof. Let $M^\circ = M_X^\circ Y$ be the complement of $Bl_X Y$ inside $M_X Y = Bl_{X\times\{\infty\}} Y \times \mathbb{P}^1$.

$$
\begin{aligned}
i &\colon C \hookrightarrow M^\circ \\
j &\colon Y \times \mathbb{A}^1 \hookrightarrow M \\
Y \times \mathbb{A}^1 &= M^\circ - C
\end{aligned}
\quad \Longrightarrow \quad
\begin{tikzcd}
A_{k+1}C \arrow[r, "i_*"] & A_{k+1}M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"'] & A_{k+1}Y \times \mathbb{A}^1 \arrow[r] & 0 \\
& A_k C & A_k Y \arrow[u, "\mathrm{pr}^*"'] \arrow[l, dashed, "\tilde{\sigma}"] &
\end{tikzcd}
$$

- exact sequence from (see Reinier's talk)
- i^* is the Gysin map for divisors, and $i^* i_*(\alpha) = c_1(\mathcal{O}(C)) \cap \alpha$ (see Nitin's talk)
- *Fact:* $c_1(\mathcal{O}(C))$ is trivial, so $i^* i_* = 0$

Thus, we have a well-defined map $\tilde{\sigma}\colon A_k Y \to A_k C$. We have to show that it takes $[V]$ to $[C_{X\cap V} V]$.

Specialisation to the normal cone

$$
\begin{tikzcd}
A_{k+1}C \arrow[r, "i_*"] & A_{k+1}M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"'] & A_{k+1}Y \times \mathbb{A}^1 \arrow[r] & 0 \\
& A_k C & A_k Y \arrow[l, dashed, "\tilde{\sigma}"] \arrow[u, "\mathrm{pr}^*"'] &
\end{tikzcd}
$$

By definition, $\mathrm{pr}^*[V] = [\mathrm{pr}^{-1}(V)] = [V \times \mathbb{A}^1]$.

Fact 1: $M^\circ_{X \cap V} V$ is a closed subvariety of $M^\circ = M^\circ_X Y$ which restricts to $V \times \mathbb{A}^1$.

Thus, we have

$$\tilde{\sigma}[V] = i^*[M^\circ_{X \cap V} V].$$

Fact 2: The Cartier divisor $C = M^\circ \cap M_\infty$ intersects $M^\circ_{X \cap V} V$ in $C_{X \cap V} V$.

Thus by definition of i^*,

$$i^*[M^\circ_{X \cap V} V] = [C_{X \cap V} V].$$

Application: Gysin homomorphism

Let $i \colon X \hookrightarrow Y$ be a regular closed imbedding of codimension d, with normal bundle $N = N_X Y$.

Definition

Define the Gysin homomorphism

$$i^* : A_k Y \to A_{k-d} X$$

by composing the specialisation homomorphism $\sigma \colon A_k Y \to A_k N$ with the Gysin homomorphism for bundles $s_N^* \colon A_k N \to A_{k-d} X$ (see Campbell's talk).

Next time: intersection product!

Thank you!

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