Integrable systems and Hurwitz theory

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Plan of the talk

Natural definition (?)

Imagine your Geometry teacher introducing Grassmannians as follows.

Definition

The Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ is <u>defined</u> as the set of points

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^5$$

satisfying the relation

$$x_0 x_1 - x_2 x_3 + x_4 x_5 = 0.$$

One can give a similar definition of KP tau functions.

Definition

A tau function of the KP hierarchy is a (formal) function

$$\tau \in \mathbb{C}[t_1, t_2, \ldots]$$

satisfying an infinite system of PDEs: for $u = 2\partial_{t_1}^2 \log(\tau)$,

$$3\partial_{t_2}^2 u + \partial_{t_1}(-4\partial_{t_3} u + 6u\partial_{t_1} u + \partial_{t_1}^3 u) = 0$$

$$\ldots$$

Sato Grassmannian

Denote $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. Consider the vector spaces over $\mathbb{C}$

$$V = \operatorname{span}\{\, e_s \mid s \in \mathbb{Z}' \,\} \qquad V_+ = \operatorname{span}\{\, e_s \mid s \in \mathbb{Z}'_{>0} \,\}$$

and define the (big cell of the) Sato Grassmannian

$$\mathrm{Gr}^0(V) = \{\, W \subseteq V \mid \pi_W \colon W \to V_+ \text{ is an iso} \,\}.$$

Define the Fock space as the space of semi-infinite wedges on V that stabilises on the right:

$$\mathcal{F} = \operatorname{span}\left\{ e_{s_1} \wedge e_{s_2} \wedge \cdots \;\middle|\; \substack{\exists c \in \mathbb{Z} \text{ s.t. for } k \gg 0 \\ s_k + \frac{1}{2} - k = c} \right\} \Big/ \sim .$$

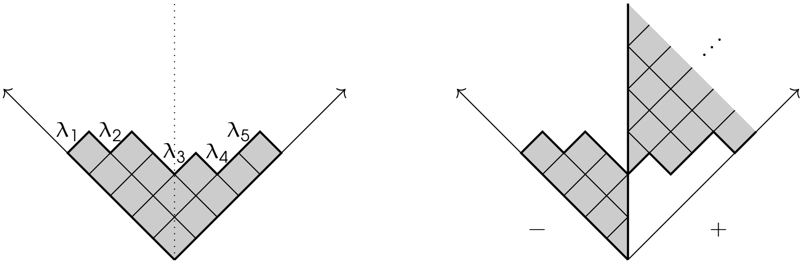
The value c is called the charge. Denote by $\mathcal{F}^0$ the charge-zero subspace. It contains the vacuum vector

$$|0\rangle = e_{1/2} \wedge e_{3/2} \wedge e_{5/2} \wedge \cdots .$$

Semi-infinite wedge and partitions

More generally, for any partition λ, we have and element of $\mathcal{F}^0$:

$$|\lambda\rangle = e_{1/2-\lambda_1} \wedge e_{3/2-\lambda_2} \wedge e_{5/2-\lambda_3} \wedge \cdots$$



$$|(5\,4\,2\,1\,1)\rangle = e_{-9/2} \wedge e_{-5/2} \wedge e_{1/2} \wedge e_{5/2} \wedge e_{7/2} \wedge e_{11/2} \wedge \cdots$$

Plücker relations

Definition

Define the Plücker embedding $\mathrm{Gr}^0(V) \to \mathbb{P}\mathcal{F}^0$

$$W \longmapsto \pi_W^{-1}(e_{1/2}) \wedge \pi_W^{-1}(e_{3/2}) \wedge \cdots$$

Define the operators

$$\psi_s = e_s \wedge, \qquad \psi_s^\dagger = \iota_{e^*_{-s}}$$

called the creation (for $s < 0$) and annihilation ($s > 0$) operators.

Theorem

An element $|\omega\rangle \in \mathbb{P}\mathcal{F}^0$ represents a point of $\mathrm{Gr}^0(V)$ iff it satisfies the fermionic Plücker relations

$$\sum_{s \in \mathbb{Z}'} \psi_s |\omega\rangle \otimes \psi_s^\dagger |\omega\rangle = 0.$$

Boson-fermion correspondence

Define the operators

$$H_n = \sum_{s \in \mathbb{Z}'} :\psi_{-s}\psi^{\dagger}_{s+n}: \qquad n \in \mathbb{Z}.$$

The boson-fermion correspondence is the isomorphism $T\colon \mathcal{F}^0 \to \mathbb{C}[t_1, t_2, \dots]$ defined as

$$T(|\omega\rangle) = \left\langle 0 \middle| e^{H(\boldsymbol{t})} \middle| \omega \right\rangle, \qquad H(\boldsymbol{t}) = \sum_{n>0} t_n H_n.$$

Example

Since $H_n |0\rangle = 0$ for $n > 0$, we have $e^{H(\boldsymbol{t})} |0\rangle = |0\rangle$. Thus,

$$T(|0\rangle) = \langle 0|0\rangle = 1.$$

Another example is $|\omega\rangle = (e_{1/2} + e_{-5/2}) \wedge e_{3/2} \wedge e_{5/2} \wedge \cdots$. The associated function is

$$T(|\omega\rangle) = \left\langle 0 \middle| \left(1 + \frac{t_1^3}{6} H_1^3 + \frac{t_1 t_2}{2} (H_1 H_2 + H_2 H_1) + t_3 H_3\right) \middle| \omega \right\rangle = 1 + \frac{t_1^3}{6} + \frac{t_1 t_2}{2} + t_3.$$

Plücker relations in the bosonic formalism

Theorem

An element $\tau \in \mathbb{C}[t_1, t_2, \dots]$ represents a point of $\mathrm{Gr}^0(V)$ iff it satisfies the bosonic Plücker relations

$$\oint \exp\left(2\sum_{k>0} z^k u_k\right) \exp\left(-2\sum_{k>0} \frac{z^{-k}}{k}\partial_{u_k}\right) \tau(\mathbf{t}+\mathbf{u})\tau(\mathbf{t}-\mathbf{u})dz = 0,$$

order by order in Taylor expansions in u.

Every coefficient of monomials in the u_k's is then a PDE satisfied by $\tau(\mathbf{t})$: the KP hierarchy.

Example

The element $|\omega\rangle = (e_{1/2} + e_{-5/2}) \wedge e_{3/2} \wedge e_{5/2} \wedge \cdots$ belongs to $\mathrm{Gr}^0(V)$. Thus, the associated function

$$\tau(\mathbf{t}) = 1 + \frac{t_1^3}{6} + \frac{t_1 t_2}{2} + t_3$$

satisfies the KP hierarchy.

$\widehat{\mathrm{GL}}(\infty)$-action

Define the Lie algebra

$$\widehat{\mathfrak{gl}}(\infty) = \left\{ C + \sum_{r,s \in \mathbb{Z}'} X_{rs} :\psi_{-r}\psi_s^\dagger: \;\middle|\; C \in \mathbb{C},\ X_{rs} = 0 \text{ for } |r-s| \gg 0 \right\}.$$

The associated Lie group $\widehat{\mathrm{GL}}(\infty) = \left\{ e^{g_1} \cdots e^{g_k} \;\middle|\; g_i \in \widehat{\mathfrak{gl}}(\infty) \right\}$ acts transitively on $\mathrm{Gr}^0(V)$. Thus,

$$\mathrm{Gr}^0(V) = \left\{ e^{g_1} \cdots e^{g_k} |0\rangle \;\middle|\; g_i \in \widehat{\mathfrak{gl}}(\infty) \right\}.$$

Corollary

An element $\tau \in \mathbb{C}[t_1, t_2, \ldots]$ is a KP tau function iff it can be expressed as

$$\tau(\mathbf{t}) = \left\langle 0 \middle| e^{H(\mathbf{t})} e^{g_1} \cdots e^{g_k} \middle| 0 \right\rangle$$

for some $g_i \in \widehat{\mathfrak{gl}}(\infty)$.

The main example

Natural element of $\widehat{\mathfrak{gl}}(\infty)$ are given by the diagonal elements

$$\mathcal{F}_m = \sum_{s \in \mathbb{Z}'} s^m :\psi_{-s}\psi_s^\dagger: .$$

We can then construct the (1-parameter families of) tau functions

$$\tau_m(\beta; \mathbf{t}) = \left\langle 0 \middle| e^{H(\mathbf{t})} e^{\beta \frac{\mathcal{F}_m}{m}} e^{H_{-1}} \middle| 0 \right\rangle ,$$

which has deep connections with the representation theory of the symmetric group and (spoiler alert) Hurwitz theory.

Lemma

The following relations hold:

$$(H_{-1})^d |0\rangle = \sum_{\lambda \vdash d} \dim(\lambda) |\lambda\rangle \qquad \mathcal{F}_m |\lambda\rangle = p_m(\lambda) |\lambda\rangle \qquad H_{\mu_1} \cdots H_{\mu_n} |\lambda\rangle = \chi_\lambda(\mu) |0\rangle ,$$

where $\chi_\lambda(\mu)$ are the irreducible characters of the symmetric group, and

$$p_m(\lambda) = \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \tfrac{1}{2} \right)^m - \left(-i + \tfrac{1}{2} \right)^m \right] .$$

The main example

Proposition

The tau function τ_m can be expressed as

$$\tau_m(\beta;\mathbf{t}) = \sum_{n,b\geqslant 0} \frac{1}{n!} \sum_{\mu_1,\dots,\mu_n=0}^{\infty} a_m(b;\mu)\frac{\beta^b}{b!}\prod_{i=1}^{n} t_{\mu_i},$$

where

$$a_m(b;\mu) = \sum_{\lambda\vdash|\mu|} \chi_\lambda(\mu)\left(\frac{p_m(\lambda)}{m}\right)^b \frac{\dim(\lambda)}{|\mu|!}.$$

Proof. From the lemma,

$$e^{H_{-1}}|0\rangle = \sum_{d\geqslant 0}\sum_{\lambda\vdash d}\frac{\dim(\lambda)}{d!}|\lambda\rangle.$$

Applying $e^{\beta\frac{\mathcal{F}_m}{m}}$, we get

$$e^{\beta\frac{\mathcal{F}_m}{m}}e^{H_{-1}}|0\rangle = \sum_{b\geqslant 0}\left(\sum_{d\geqslant 0}\sum_{\lambda\vdash d}\left(\frac{p_m(\lambda)}{m}\right)^b\frac{\dim(\lambda)}{d!}|\lambda\rangle\right)\frac{\beta^b}{b!}.$$

Applying $e^{H(\mathbf{t})}$, we get the thesis.

Hurwitz covers

Hurwitz theory is the theory of computing the number of ramified coverings of the Riemann sphere with specified ramifications.

Definition

Fix $d \geqslant 0$ and $\mu^1, \ldots, \mu^k \vdash d$. A Hurwitz cover of type $(\mu^1, \ldots, \mu^k)$ is a d-fold covering map $f \colon C \to \mathbb{P}^1$, where

- C is a connected, compact Riemann surface,
- f has k branch points $x_1, \ldots, x_k$ of ramification profile $\mu^1, \ldots, \mu^k$.

For a given Hurwitz cover $f \colon C \to \mathbb{P}^1$, the genus of C is determined by the Riemann–Hurwitz formula:

$$2 - 2g = 2d - \sum_{i=1}^{k} (d - \ell(\mu^i)).$$

Hurwitz numbers

Definition

Fix $d \geqslant 0$ and $\mu^1, \dots, \mu^k \vdash d$. Define the Hurwitz numbers

$$H(\mu^1, \dots, \mu^k) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|},$$

where the sum runs over all isomorphism classes of Hurwitz covers $f\colon C \to \mathbb{P}^1$ of type $(\mu^1, \dots, \mu^k)$. Denote by $H^\bullet(\mu^1, \dots, \mu^k)$ the same count, but allowing the covering surface to be disconnected.

For instance, one has

$$H((2),(2)) = \tfrac{1}{2},$$

corresponding to the cover $\mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^2$ which has an automorphism group of order 2.

Monodromy representation

Consider a degree d Hurwitz cover $f\colon C \to \mathbb{P}^1$ of type $(\mu^1, \dots, \mu^k)$ ramified over $B = \{x_1, \dots, x_k\} \subset \mathbb{P}^1$. Fix a point $p \notin B$ and label its preimages by $p_1, \dots, p_d$. We define the monodromy representation:

$$\rho\colon \pi_1(\mathbb{P}^1 \setminus B, p) \longrightarrow S_d = S_{\{p_1,\dots,p_d\}}$$
$$\gamma \longmapsto \sigma_\gamma = \left[p_m \mapsto \widetilde{\gamma}_m(1)\right],$$

where $\widetilde{\gamma}_m$ is the unique lift of γ starting at p_m.

Notice that

- a different choice of labeling corresponds to composing ρ with an inner automorphism of S_d,
- if γ^i is a loop winding once around x_i, the cycle type of σ_{γ^i} is μ^i,
- if $\sigma_{\gamma^1} \cdots \sigma_{\gamma^k} = \mathrm{id}$,
- C is connected iff $\langle \sigma_{\gamma^1} \cdots \sigma_{\gamma^k} \rangle$ acts transitively.

Viceversa, a monodromy representation contains enough information to reconstruct the Hurwitz cover (up to automorphism).

Hurwitz numbers for group theorists

Proposition

Fix $\mu^1, \dots, \mu^k \vdash d$. The corresponding disconnected Hurwitz numbers are given by the following permutation count in S_d

$$H^\bullet(\mu^1, \dots, \mu^k) = \frac{1}{d!} \left| \left\{ (\sigma_1, \dots, \sigma_k) \;\middle|\; \begin{smallmatrix} \sigma_1, \dots, \sigma_k \in S_d \\ \sigma_i \text{ has cycle type } \mu^i \\ \sigma_1 \cdots \sigma_k = \mathrm{id} \end{smallmatrix} \right\} \right|.$$

The connected count can be obtained by imposing the transitivity condition.

We can recast the computation of Hurwitz numbers as a multiplication problem in the symmetric group algebra.

Hurwitz numbers for representation theorists

Consider the symmetric group algebra $\mathbb{C}S_d$. For a partition $\mu \vdash d$, define the elements

$$C_\mu = \sum_{\substack{\sigma \in S_d \\ \sigma \text{ of cycle type } \mu}} \sigma.$$

Hurwitz numbers are the coefficient of the identity in the appropriate product of elements of the symmetric group algebra.

Corollary

Fix $\mu^1, \ldots, \mu^k \vdash d$. The corresponding disconnected Hurwitz numbers are given by the following multiplication count in $\mathbb{C}S_d$:

$$H^\bullet(\mu^1, \ldots, \mu^k) = \frac{1}{d!} [\mathrm{id}] C_{\mu^1} \cdots C_{\mu^k}.$$

$H^\bullet\big((3),(3)\big)$

For a group theorist,

$$H^\bullet\big((3),(3)\big) = \frac{1}{3!}\left|\left\{ \begin{array}{c} \big((1\,2\,3),(1\,3\,2)\big) \\ \big((1\,3\,2),(1\,2\,3)\big) \end{array} \right\}\right| = \frac{1}{3}.$$

For a representation theorist, in $\mathbb{C}S_3$ we have

$$C_{(3)} = (1\,2\,3) + (1\,3\,2),$$

so that

$$C_{(3)} \cdot C_{(3)} = 2\,\mathrm{id} + (1\,2\,3) + (1\,3\,2).$$

Thus, we find

$$H^\bullet\big((3),(3)\big) = \frac{1}{3!}[\mathrm{id}]\,C_{(3)} \cdot C_{(3)} = \frac{1}{3}.$$

Centre of the symmetric group algebra

The elements C_μ for $\mu \vdash d$ are central elements in $\mathbb{C}S_d$. More precisely, they form a basis:

$$\mathcal{Z}(\mathbb{C}S_d) = \bigoplus_{\mu \vdash d} \mathbb{C}.C_\mu$$

Theorem (Maschke)

$\mathcal{Z}(\mathbb{C}S_d)$ is a semisimple algebra: there exists a basis e_λ such that

$$e_\lambda \cdot e_{\lambda'} = \delta_{\lambda,\lambda'} e_\lambda.$$

Moreover, the change of bases essentially given by the character table:

$$e_\lambda = \frac{\dim(\lambda)}{d!} \sum_{\mu \vdash d} \chi_\lambda(\mu) C_\mu \qquad C_\mu = |C_\mu| \sum_{\lambda \vdash d} \frac{\chi_\lambda(\mu)}{\dim(\lambda)} e_\lambda.$$

Burnside character formula

We have

$$e_\lambda = \frac{\dim(\lambda)}{d!}\sum_{\mu\vdash d}\chi_\lambda(\mu)C_\mu \qquad C_\mu = |C_\mu|\sum_{\lambda\vdash d}\frac{\chi_\lambda(\mu)}{\dim(\lambda)}e_\lambda.$$

Thus, we find

$$C_{\mu^1}\cdots C_{\mu^k} = \sum_{\lambda\vdash d}\left(\prod_{i=1}^{k}|C_{\mu^i}|\frac{\chi_\lambda(\mu^i)}{\dim(\lambda)}\right)e_\lambda.$$

On the other hand, $e_\lambda = \frac{\dim(\lambda)}{d!}\chi_\lambda((1^d))\,\mathrm{id} + \cdots = \frac{\dim(\lambda)^2}{d!}\,\mathrm{id} + \cdots$. As a consequence, we obtain the following result.

Theorem (Burnside character formula)

Fix $\mu^1,\dots,\mu^k \vdash d$. The corresponding Hurwitz numbers are given by

$$H^\bullet(\mu^1,\dots,\mu^k) = \sum_{\lambda\vdash d}\left(\frac{\dim(\lambda)}{d!}\right)^2\left(\prod_{i=1}^{k}|C_{\mu^i}|\frac{\chi_\lambda(\mu^i)}{\dim(\lambda)}\right).$$

Simple Hurwitz numbers

Define the simple Hurwitz numbers as Hurwitz numbers with a single ramification point of arbitrary ramification profile, and simple ramification otherwise:

$$h_{g,\mu} = |\mathrm{Aut}(\mu)|\, H(\mu, \underbrace{(2,1^{d-1}), \cdots, (2,1^{d-1})}_{b \text{ times}}) \qquad b = 2g-2+\ell(\mu)+d.$$

The Burnside character formula specialises to

$$h^{\bullet}_{g,\mu} = \frac{1}{\prod_{i\geqslant 1}\mu_i} \sum_{\lambda\vdash d} \chi_\lambda(\mu) \left(|C_{(2,1^{d-1})}| \frac{\chi_\lambda((2,1^{d-1}))}{\dim(\lambda)} \right)^b \frac{\dim(\lambda)}{d!}.$$

Simple Hurwitz numbers and KP

Lemma

The following identity holds:

$$|C_{(2,1^{d-1})}|\frac{\chi_\lambda((2,1^{d-1}))}{\dim(\lambda)} = \frac{1}{2}\sum_{i=1}^{\infty}\left[\left(\lambda_i - i + \tfrac{1}{2}\right)^2 - \left(-i+\tfrac{1}{2}\right)^2\right] = \frac{p_2(\lambda)}{2}.$$

As a consequence, we find the following formula for disconnected simple Hurwitz numbers:

$$h^\bullet_{g,\mu} = \frac{1}{\prod_{i\geqslant 1}\mu_i}\sum_{\lambda\vdash d}\chi_\lambda(\mu)\left(\frac{p_2(\lambda)}{2}\right)^b\frac{\dim(\lambda)}{d!}.$$

Theorem (Okounkov)

The generating series of disconnected simple Hurwitz numbers is a tau function of the KP hierarchy:

$$\sum_{g,n\geqslant 0}\frac{1}{n!}\sum_{\mu_1,\dots,\mu_n=0}^{\infty} h^\bullet_{g,\mu}\frac{\beta^b}{b!}\prod_{i=1}^{n}\mu_i t_{\mu_i} = \left\langle 0\middle| e^{H(\mathbf{t})}e^{\beta\frac{\mathcal{F}_2}{2}}e^{H_{-1}}\middle|0\right\rangle.$$

Generalisations

- Burnside's character formula can be generalised to counting of covers with a base of genus γ.

$$H^{\bullet}_{\gamma}(\mu^1, \dots, \mu^k) = \sum_{\lambda \vdash d} \left(\frac{\dim(\lambda)}{d!} \right)^{2-2\gamma} \left(\prod_{i=1}^{k} |C_{\mu^i}| \frac{\chi_\lambda(\mu^i)}{\dim(\lambda)} \right).$$

- Through semi-infinite wedge, one can consider tau functions of the 2D Toda hierarchy, which depend on two infinite sets of times $\mathbf{t} = (t_1, t_2 \dots)$ and $\mathbf{t}' = (t'_1, t'_2 \dots)$. Okounkov proved that the generating series of disconnected double Hurwitz numbers

$$h_{g,\mu,\nu} = |\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)| H(\mu, \nu, \underbrace{(2, 1^{d-1}), \cdots, (2, 1^{d-1})}_{b \text{ times}})$$

is a tau function of the 2D Toda hierarchy.

Generalisations

- One can give a Hurwitz number interpretation to the tau function
$$\left\langle 0 \middle| e^{H(\mathbf{t})} e^{\beta \frac{\mathcal{F}_r}{r}} e^{H_{-1}} \middle| 0 \right\rangle$$
in terms of Hurwitz number with *r*-completed cycles (Okounkov–Pandharipande).

- One can change H_{-1} to $\frac{H_{-q}}{q}$ for any $q \geqslant 1$. The corresponding tau function can be expressed in terms of *q*-orbifold Hurwitz number with *r*-completed cycles.

- One can impose more conditions on the count (monotonicity and strict monotonicity conditions). The associated partition functions are still tau functions of the KP or $2D$ Toda hierarchies (Harnad–Orlov).

- One can construct a Fock space from uncharged fermion. Through boson-fermion correspondence, one can obtain tau functions of the BKP hierarchy. They are connected to spin Hurwitz numbers.

More properties & applications

- Hurwitz numbers satisfy many more interesting properties
  - evolution equation (cut-and-join equation),
  - topological recursion and connection with matrix models,
  - expression in terms of intersection theory on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ (ELSV formula).
- The ELSV formula, together with the KP result, can be used to prove Witten's conjecture: the partition function

$$Z(\mathbf{t}) = \sum_{2g-2+n>0} \frac{1}{n!} \sum_{\mu_1,\dots,\mu_n=0}^{\infty} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \prod_{i=1}^{n} t_{\mu_i}$$

  is a tau function of the KdV hierarchy (Kazarian–Lando).
- The ELSV formula, together with the KP result, can be used to prove other results concerning other invariants: Gromov–Witten of $\mathbb{P}^1$, Masur–Veech volumes, the Euler characteristic of $\mathcal{M}_{g,n}$, ...

Thank you!

1. G. Borot. "An introductory walk in the integrable woods". Mini-course, Leibniz Universität, Hannover (2015).
2. T. Miwa, M. Jimbo, E. Date. *Solitons: Differential equations, symmetries and infinite dimensional algebras.* Vol. 135. Cambridge Tracts in Mathematics. Cambridge University Press, 2000.
3. R. Cavalieri, E. Miles. *Riemann surfaces and algebraic curves. A first course in Hurwitz theory.* Vol. 87. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2016.
4. A. Okounkov. "Toda Equations for Hurwitz Numbers". Math. Res. Lett. 7 (2000), pp. 447-453.
5. A. Okounkov, R. Pandharipande, "Gromov–Witten theory, Hurwitz theory, and completed cycles". Ann. of Math. 163.2 (2006), pp. 517–560.