MoSCATR VII

Multicurve count, Masur–Veech volumes and topological recursion

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1905.10352 [math.GT], 2010.11806 [math.DG]

MPI Mathematics

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June 3, 2021

Setup

- Σ a bordered surface (smooth connected oriented stable surface) of genus g with n labelled boundary components
- Teichmüller space

$$\mathcal{T}_\Sigma(\vec{L}) = \left\{ \begin{matrix} \text{hyperbolic metric } \sigma \text{ on } \Sigma \\ \text{s.t. } \ell_\sigma(\partial_i \Sigma) = L_i \end{matrix} \right\} \Big/ \text{isotopy}$$

  It has a natural symplectic form ω_{WP}
- A (primitive) multicurve $\gamma = (\gamma_1^{m_1}, \ldots, \gamma_N^{m_N})$ is collection of simple closed curves γ_i with multiplicity $m_i \in \mathbb{Z}_+$ s.t.
  - γ_i is not null-homotopic and not homotopic to a boundary component
  - γ_i is not homotopic to γ_j for $i \neq j$
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  Length of multicurves: $\ell_\sigma(\gamma) = \sum_{i=1}^{N} m_i \, \ell_\sigma(\gamma_i)$

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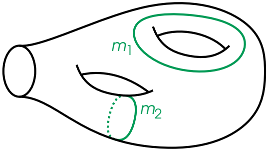

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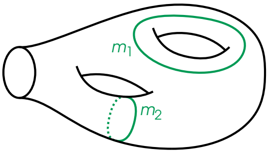

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Goal: count the number of multicurves

Consider the function $\mathcal{N}_\Sigma : \mathcal{T}_\Sigma(\vec{L}) \times \mathbb{R}_+ \to \mathbb{Z}_+$

$$\mathcal{N}_\Sigma(\sigma; t) = \#\{\ \gamma \text{ multicurve s.t. } \ell_\sigma(\gamma) \leqslant t\ \}$$

(Mirzakhani)

First goal

Compute the number of multicurves: $\mathcal{N}_\Sigma(\sigma; t)$

The function $\mathcal{N}_\Sigma$ is mapping class group invariant, so it descends to a function $\mathcal{N}_{g,n}$ on the moduli space $\mathcal{M}_{g,n}(\vec{L}) = \mathcal{T}_\Sigma(\vec{L}) / \operatorname{Mod}_\Sigma$ that we can integrate

$$\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t) = \int_{\mathcal{M}_{g,n}(\vec{L})} \mathcal{N}_{g,n}(X; t)\, d\mu_{\mathrm{WP}}(X), \qquad d\mu_{\mathrm{WP}} = \frac{\omega_{\mathrm{WP}}^{3g-3+n}}{(3g-3+n)!}$$

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Length statistics of multicurves

Multiplicative statistics of primitive multicurves $\Longleftarrow$ geometric/topological recursion (GR/TR)

(Andersen–Borot–Orantin)

Consider a function $f\colon \mathbb{R}_+ \to \mathbb{C}$ decaying fast enough at ∞. Define the functions

$$\Omega^f_\Sigma(\sigma) = \sum_{\gamma \text{ prmtv}} \prod_{i=1}^{k_\gamma} f(\ell_\sigma(\gamma_i)), \qquad \langle \Omega^f_{g,n} \rangle (\vec{L}) = \int_{\mathcal{M}_{g,n}(\vec{L})} \Omega^f_{g,n}\, d\mu_{\mathrm{WP}}(X)$$

Theorem (Andersen–Borot–Orantin)

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$$\Omega_{\Sigma}^{f}(\sigma) = \sum_{\gamma \text{ prmtv}} \prod_{i=1}^{k_\gamma} f(\ell_\sigma(\gamma_i))$$

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for explicit kernels B^f and C^f:

$$\begin{aligned} B^{f}(L, L', \ell) &= B_{MM}(L, L', \ell) + f(\ell) \\ C^{f}(L, \ell, \ell') &= C_{MM}(L, \ell, \ell') + f(\ell) f(\ell') \\ &\quad + B_{MM}(L, \ell, \ell') f(\ell) \\ &\quad + B_{MM}(L, \ell', \ell) f(\ell') \end{aligned}$$

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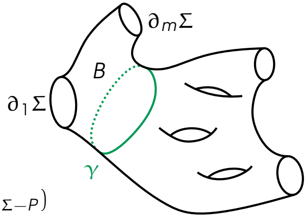

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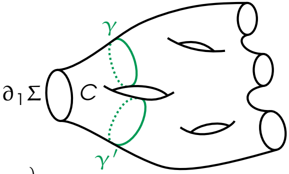

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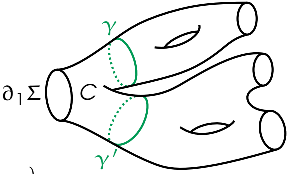

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$$\begin{cases} \frac{\partial}{\partial x} x\, B_{MM}(x, y, z) = \frac{H(z, x+y) + H(z, x-y)}{2} \\ \frac{\partial}{\partial x} x\, C_{MM}(x, y, z) = H(y+z, x) \\ H(x, y) = \left(1 + e^{\frac{x+y}{2}}\right)^{-1} + \left(1 + e^{\frac{x-y}{2}}\right)^{-1} \end{cases}$$

(Paul's talk)

TR for average of length statistics of multicurves

$$\Omega^f_\Sigma \text{ by GR} \qquad + \qquad \langle \Omega^f_{g,n} \rangle(\vec{L}) = \int_{\mathcal{M}_{g,n}(\vec{L})} \Omega^f_{g,n} \, d\mu_{\mathrm{WP}}(X)$$

Topological recursion: for $2g-2+n>1$

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and initial conditions $\langle \Omega^f_{0,3} \rangle(\vec{L}) = 1$, $\langle \Omega^f_{1,1} \rangle(L) = V^{\mathrm{WP}}_{1,1}(L) + \frac{1}{2}\int_0^{+\infty} f(\ell)\ell d\ell$.

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$\omega_{\mathrm{WP}} = \sum_i d\ell_i \wedge \tau_i$

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Stable graphs sums

Ω^f_Σ by definition $\quad + \quad \langle \Omega^f_{g,n} \rangle(\vec{L}) = \int_{\mathcal{M}_{g,n}(\vec{L})} \Omega^f_{g,n}\, d\mu_{\mathrm{WP}}(X)$

Sum over stable graphs: for $2g-2+n>0$

$$\langle \Omega^f_{g,n} \rangle(\vec{L}) = \sum_{\substack{\Gamma \text{ stble grph} \\ \text{type } (g,n)}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v \in V_\Gamma} V^{\mathrm{WP}}_{g(v),n(v)}\big((\ell_e)_{e \in E(v)}, (L_\lambda)_{\lambda \in \Lambda_v}\big) \prod_{e \in E_\Gamma} f(\ell_e)\ell_e d\ell_e$$

Stable graphs sums

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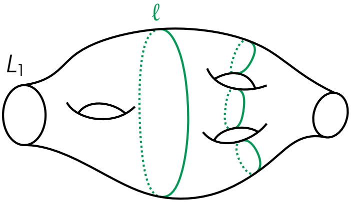

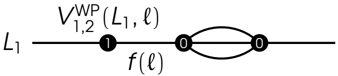

Mirzakhani's count vs. GR

$$\mathcal{N}_{\Sigma}(\sigma;t) = \#\left\{ \begin{smallmatrix} \gamma \text{ multicurve} \\ \text{s.t. } \ell_{\sigma}(\gamma) \leqslant t \end{smallmatrix} \right\} \qquad \text{vs.} \qquad \Omega^{f}_{\Sigma}(\sigma) = \sum_{\gamma \text{ prmtv}} \prod_{i=1}^{k_\gamma} f(\ell_\sigma(\gamma_i))$$

1. Additive ($\ell(\gamma) = \sum_i m_i \ell(\gamma_i)$) vs. multiplicative ($\prod_i f(\ell(\gamma_i))$)
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GR/TR and sum over stable graphs for counting multicurves

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(Mirzakhani)

TR for $\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t)$ can be expressed in the CEO form:

$$\begin{cases} \mathbb{P}^1 \\ x(z) = \frac{z^2}{2} \\ y(z) = -\frac{\sin(2\pi z)}{2\pi} \end{cases} \qquad B(z_1, z_2; s) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{\pi^2}{s^2 \sin^2 \frac{\pi(z_1 - z_2)}{s}} \right) \frac{dz_1 dz_2}{2}$$

$$\left(\prod_{i=1}^{n} \operatorname*{Res}_{\zeta_i = 0} \frac{1}{z_i - \zeta_i} \right) \omega_{g,n}(\vec{\zeta}; s) = s \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}_+} \langle \mathcal{N}_{g,n} \rangle (\vec{L}; t) e^{-st} dt \right) \prod_{i=1}^{n} e^{-z_i L_i} L_i dL_i.$$

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The average number of multicurves in low $2g-2+n$

(g,n)	$\langle \mathcal{N} \rangle_{g,n}(\vec{L};t)$
$(0,3)$	1
$(0,4)$	$\frac{1}{2}m_{(1)} + (2 + \frac{t^2}{4})\pi^2$
$(0,5)$	$\frac{1}{8}m_{(2)} + \frac{1}{2}m_{(1^2)} + (3 + \frac{t^2}{4})\pi^2 m_{(1)} + (10 + \frac{5t^2}{3} + \frac{t^4}{32})\pi^4$
$(1,1)$	$\frac{1}{48}m_{(1)} + (\frac{1}{12} + \frac{t^2}{24})\pi^2$
$(1,2)$	$\frac{1}{192}m_{(2)} + \frac{1}{96}m_{(1^2)} + (\frac{1}{12} + \frac{t^2}{48})\pi^2 m_{(1)} + (\frac{1}{4} + \frac{13t^2}{144} + \frac{t^4}{384})\pi^4$
$(1,3)$	$\frac{1}{1152}m_{(3)} + \frac{1}{192}m_{(2,1)} + \frac{1}{96}m_{(1^3)} + (\frac{1}{24} + \frac{13t^2}{2304})\pi^2 m_{(2)} + (\frac{1}{8} + \frac{t^2}{48})\pi^2 m_{(1^2)}$ $+ (\frac{13}{24} + \frac{13t^2}{96} + \frac{t^4}{384})\pi^4 m_{(1)} + (\frac{14}{9} + \frac{71t^2}{144} + \frac{61t^4}{3456} + \frac{11t^6}{69120})\pi^6$
$(2,1)$	$\frac{1}{442368}m_{(4)} + (\frac{29}{138240} + \frac{t^2}{27648})\pi^2 m_{(3)} + (\frac{139}{23040} + \frac{49t^2}{27648} + \frac{119t^4}{3317760})\pi^4 m_{(2)}$ $+ (\frac{169}{2880} + \frac{5t^2}{216} + \frac{119t^4}{138240} + \frac{t^6}{138240})\pi^6 m_{(1)} + (\frac{29}{192} + \frac{115t^2}{1728} + \frac{4199t^4}{1244160} + \frac{t^6}{18432} + \frac{29t^8}{103219200})\pi^8$

Here m_λ is the monomial symmetric polynomial associated to the partition λ, evaluated at $L_1^2, \ldots, L_n^2$.

Combinatorial models

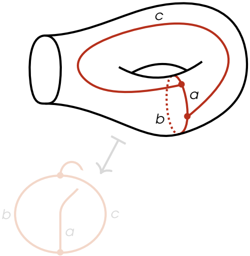
- Combinatorial Teichmüller space

$$\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L}) = \left\{ \begin{array}{c} \text{embedded metric ribbon graphs } \mathbb{G} \text{ in } \Sigma \\ \text{s.t. } \mathbb{G} \text{ is a retract of } \Sigma \text{ and } \ell_{\mathbb{G}}(\partial_i \Sigma) = L_i \end{array} \right\} \Big/ \text{isotopy}$$

It has a natural symplectic form ω_{K}. The quotient by the mapping class group $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) = \mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L}) / \mathrm{Mod}_{\Sigma}$ has finite volume

$$V_{g,n}^{\mathrm{K}}(\vec{L}) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)$$

(Mumford, Thurston, Harer, Penner, Kontsevich,...)



- We can measure length of simple closed curves (or multicurves) w.r.t $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$ by homotopying curves to the embedded metric ribbon graphs ("geodesic length")

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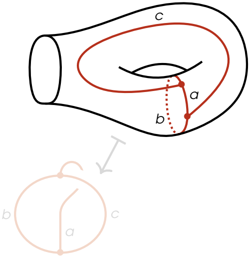

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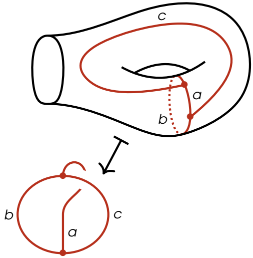

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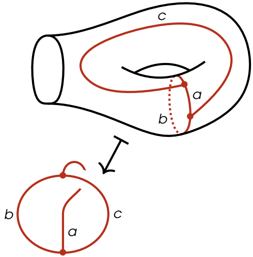

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Combinatorial Wolpert and Mirzakhani identity

Theorem

- For every choice of pants decomposition of Σ, there exist length-twist coord's that are Darboux (combinatorial Wolpert formula):

$$\omega_{\mathbb{K}} = \sum_{i=1}^{3g-3+n} d\ell_i \wedge d\tau_i$$

- There exists a combinatorial Mirzakhani–McShane identity

$$1 = \sum_{m=2}^{n} \sum_{P} B_{\mathbb{K}}(L_1, L_m, \ell_{\mathbb{G}}(\gamma)) + \frac{1}{2} \sum_{P} C_{\mathbb{K}}(L_1, \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma')) \qquad (*)$$

- Integrating $(*)$ against $d\mu_{\mathbb{K}}$, we get TR for $V^{\mathbb{K}}_{g,n}(\vec{L})$ (i.e. Witten conjecture/Kontsevich theorem)

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Counting combinatorial multicurves

Consider the function $\mathcal{N}_{\Sigma}^{\text{comb}} \colon \mathcal{T}_{\Sigma}^{\text{comb}}(\vec{L}) \times \mathbb{R}_+ \to \mathbb{Z}_+$

$$\mathcal{N}_{\Sigma}^{\text{comb}}(\mathbb{G}; t) = \#\{\ \gamma \text{ multicurve s.t. } \ell_{\mathbb{G}}(\gamma) \leqslant t\ \}$$

which descends to a function on the combinatorial moduli space

$$\langle \mathcal{N}_{g,n}^{\text{comb}} \rangle\, (\vec{L}; t) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})} \mathcal{N}_{g,n}^{\text{comb}}(G; t)\, d\mu_K(G)$$

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$$\langle \mathcal{N}_{g,n}^{\text{comb}} \rangle\,(\vec{L}; t) = \int_{\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})} \mathcal{N}_{g,n}^{\text{comb}}(G; t)\, d\mu_K(G)$$

Theorem

After Laplace transform $t \to s$:

- the number of multicurves $\mathcal{N}_{\Sigma}^{\text{comb}}(\mathbb{G}; t)$ is computed by GR,
- its average $\langle \mathcal{N}_{g,n}^{\text{comb}} \rangle(\vec{L}; t)$ is computed by TR and by a sum over stable graphs.

The CEO formulation

TR for $\langle \mathcal{N}^{\mathrm{comb}}_{g,n} \rangle (\vec{L}; t)$ can be computed from the spectral curve

$$\begin{cases} \mathbb{P}^1 \\ x(z) = \frac{z^2}{2} \\ y(z) = -z \end{cases} \qquad B(z_1, z_2; s) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{\pi^2}{s^2 \sin^2 \frac{\pi(z_1 - z_2)}{s}} \right) \frac{dz_1 \, dz_2}{2}$$

(g,n)	$\langle \mathcal{N}^{\mathrm{comb}}_{g,n} \rangle (\vec{L}; t)$
$(0,3)$	1
$(0,4)$	$\frac{1}{2} m_{(1)} + \frac{t^2 \pi^2}{4}$
$(0,5)$	$\frac{1}{8} m_{(2)} + \frac{1}{2} m_{(1^2)} + \frac{t^2 \pi^2}{4} m_{(1)} + \frac{t^4 \pi^4}{32}$
$(1,1)$	$\frac{1}{48} m_{(1)} + \frac{t^2 \pi^2}{24}$
$(1,2)$	$\frac{1}{192} m_{(2)} + \frac{1}{96} m_{(1^2)} + \frac{t^2 \pi^2}{48} m_{(1)} + \frac{t^4 \pi^4}{384}$
$(2,1)$	$\frac{1}{442368} m_{(4)} + \frac{t^2 \pi^2}{27648} m_{(3)} + \frac{119 t^4 \pi^4}{3317760} m_{(2)} + \frac{t^6 \pi^6}{138240} m_{(1)} + \frac{29 t^8 \pi^8}{103219200}$

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Masur–Veech volumes...

Consider the moduli space $\mathcal{Q}_{g,n}$ parametrising elements (C, q), where

- C is a Riemann surface of genus g,
- q is a meromorphic quadratic differential with n simple poles and $(4g - 4 + n)$ simple zeros.

Theorem (Masur, Veech)

There is a well-defined measure on $\mathcal{Q}_{g,n}$, and a notion volume (called Masur–Veech volume):

$$V_{g,n}^{\mathrm{MV}} < +\infty.$$

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...and asymptotic number of multicurves

Theorem (Mirzakhani)

The asymtotic number of multicurves coincides with the Masur–Veech volume:

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Lemma

The asymtotic number of hyperbolic and combinatorial multicurves coincides:

$$\lim_{t\to\infty} \frac{\langle \mathcal{N}_{g,n} \rangle (\vec{L};t)}{t^{6g-6+n}} = \lim_{t\to\infty} \frac{\langle \mathcal{N}_{g,n}^{\mathrm{comb}} \rangle (\vec{L};t)}{t^{6g-6+n}}$$

Corollary

Masur–Veech volumes are computed by (two different) TRs and by a sum over stable graphs.

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A different approach to MV: intersection theory

Compactify $\mathcal{Q}_{g,n}$ and express the MV measure as a top form.

Theorem (Chen–Möller–Sauvaget)

$$V^{\mathrm{MV}}_{g,n} = (-1)^{3g-3+n} \frac{2^{2g+1}\pi^{6g-6+2n}}{(6g-7+2n)!} \int_{\overline{\mathcal{M}}_{g,n}} s(\overline{\mathcal{Q}}_{g,n})$$

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Proposition (Lewański–Popolitov–Shadrin–Zvonkine, Borot–G–Lewański)

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computes the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} s(\overline{\mathcal{Q}}_{g,n}) \psi_1^{d_1} \cdots \psi_n^{d_n}$.

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Thank you!

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