Noncommutative geometry meets topological recursion

WWU Münster

The Harer–Zagier formula via intersection theory

joint work in progress with D. Lewański, P. Norbury

MPI Mathematics

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The Harer–Zagier formula

Consider the moduli space of curves

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \substack{C \text{ cmplx cmpct smooth curve} \\ \text{genus } g \text{ with } n \text{ marked pnts}} \right\} \Big/ \sim$$

which is a smooth complex orbifold of dimension $3g - 3 + n$.

Theorem (Harer–Zagier, '86)

The Euler characteristic of the moduli space of curves $\chi_{g,n} = \chi(\mathcal{M}_{g,n})$ is given by

$$\chi_{g,n} = (-1)^n (2g - 3 + n)! \frac{B_{2g}}{2g(2g-2)!}$$

New strategy: Gauss–Bonnet theorem

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Theorem (Generalised Gauss–Bonnet theorem)

Let $\overline{M}$ be a compact complex m-orbifold, $M \subset \overline{M}$ open such that $D = \overline{M} \setminus M$ is a "nice" divisor. Then

$$\chi(M) = \int_{\overline{M}} c_m\big(T_{\overline{M}}(-\log D)\big).$$

$T_{\overline{M}}(-\log D)$ is the log tangent bundle.

In our case,

$$\overline{M} = \overline{\mathcal{M}}_{g,n}, \qquad M = \mathcal{M}_{g,n}$$

and $T_{\overline{M}}(-\log D)$ is such that its fiber over $(C, p_1, \ldots, p_n)$ is

$$H^0\big(C, \omega_C^{\otimes 2}(p_1 + \cdots + p_n)\big)^\vee.$$

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Hodge integrals

The Chern classes of these types of bundles were computed by Chiodo (generalising Mumford's formula):

$$c\big(T_{\overline{M}}(-\log D)\big) = \Lambda^{\vee} \exp\Big(-\sum_{m\geqslant 1} \frac{1}{m}\kappa_m\Big).$$

Proposition

$$\chi_{g,n} = \sum_{\ell\geqslant 0} \frac{(-1)^{\ell}}{\ell!} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda^{\vee} \prod_{i=1}^{\ell} \sum_{\mu_i\geqslant 1} \psi_{n+i}^{\mu_i+1}$$

Upshot: we expressed $\chi_{g,n}$ in terms of Hodge integrals, which are well-studied integrals in algebraic geometry and mathematical physics.

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How to prove the HZ formula

Step 1. We proved that

$$\chi_{g,n+1} = -(2g-2+n)\,\chi_{g,n}$$

using string and dilaton equation.

Step 2. Prove the base cases:

$$g=0: \qquad \chi_{0,3} = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$$

$$g=1: \qquad \chi_{1,1} = -\int_{\overline{\mathcal{M}}_{1,1}} (\lambda_1 + \kappa_1) = -\frac{1}{12}$$

$$g \geqslant 2: \qquad \chi_{g,0} = \sum_{\ell \geqslant 0} \frac{(-1)^\ell}{\ell!} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda^\vee \prod_{i=1}^{\ell} \sum_{\mu_i \geqslant 1} \psi_i^{\mu_i+1} = \frac{B_{2g}}{2g(2g-2)}$$

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Topological recursion

Integrals of classes like the previous ones were linked to topological recursion by Lewański–Popolitov–Shadrin–Zvonkine.

Corollary

Consider the spectral curve

$$x(z) = \log(z) - z, \qquad y(z) = \frac{1}{z}, \qquad B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}.$$

Then

$$\chi_{g,n} = \left(\prod_{i=1}^{n} \operatorname*{Res}_{z_i=1} (1 - z_i) \right) \omega_{g,n}(z_1, \ldots, z_n).$$

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Thank you!

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