Noncommutative geometry meets topological recursion WWU Münster

The Harer–Zagier formula via intersection theory

joint work in progress with D. Lewański, P. Norbury



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The Harer–Zagier formula

Consider the moduli space of curves

$$\mathcal{M}_{g,n} = \left\{ \left(C, p_1, \dots, p_n \right) \middle| \begin{array}{c} C \text{ cmplx cmpct smooth curve} \\ \text{genus } g \text{ with } n \text{ marked pnts} \end{array} \right\} \middle/ \sim$$

which is a smooth complex orbifold of dimension 3g - 3 + n.

Theorem (Harer–Zagier, '86)

The Euler characteristic of the moduli space of curves $\chi_{g,n} = \chi(\mathcal{M}_{g,n})$ is given by

$$\chi_{g,n} = (-1)^n (2g - 3 + n)! \frac{B_{2g}}{2g(2g - 2)!}$$

New strategy: Gauss-Bonnet theorem

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Let \overline{M} be a compact complex *m*-orbifold, $M \subset \overline{M}$ open such that $D = \overline{M} \setminus M$ is a "nice" divisor. Then

$$\chi(M) = \int_{\overline{M}} C_m \big(T_{\overline{M}}(-\log D) \big).$$

 $T_{\overline{M}}(-\log D)$ is the log tangent bundle.

In our case,

$$\overline{M} = \overline{\mathcal{M}}_{g,n}, \qquad M = \mathcal{M}_{g,n}$$

and $T_{\overline{M}}(-\log D)$ is such that its fiber over (C, p_1, \dots, p_n) is

 $H^0(C, \omega_C^{\otimes 2}(p_1 + \cdots + p_n))^{\vee}.$

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Hodge integrals

The Chern classes of these types of bundles were computed by Chiodo (generalising Mumford's formula):

$$C(T_{\overline{M}}(-\log D)) = \Lambda^{\vee} \exp\left(-\sum_{m \ge 1} \frac{1}{m} \kappa_m\right).$$

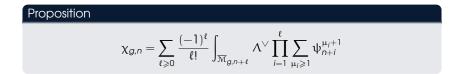
Proposition $\chi_{g,n} = \sum_{\ell \ge 0} \frac{(-1)^{\ell}}{\ell!} \int_{\overline{\mathcal{M}}_{g,n+\ell}} \Lambda^{\vee} \prod_{i=1}^{\ell} \sum_{\mu_i \ge 1} \psi_{n+i}^{\mu_i+1}$

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How to prove the HZ formula

Step 1. We proved that

$$\chi_{g,n+1} = -(2g-2+n)\chi_{g,n}$$

using string and dilaton equation.

Step 2. Prove the base cases:

$$g = 0: \qquad \chi_{0,3} = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$$

$$g = 1: \qquad \chi_{1,1} = -\int_{\overline{\mathcal{M}}_{1,1}} (\lambda_1 + \kappa_1) = -\frac{1}{12}$$

$$g \ge 2: \qquad \chi_{g,0} = \sum_{\ell \ge 0} \frac{(-1)^{\ell}}{\ell!} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda^{\vee} \prod_{i=1}^{\ell} \sum_{\mu_i \ge 1} \psi_i^{\mu_i + 1} = \frac{B_{2g}}{2g(2g - 2)}$$

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Integrals of classes like the previous ones were linked to topological recursion by Lewański–Popolitov–Shadrin–Zvonkine.

Corollary

Consider the spectral curve

$$x(z) = \log(z) - z,$$
 $y(z) = \frac{1}{z},$ $B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$

Then

$$\chi_{g,n} = \left(\prod_{i=1}^n \operatorname{Res}_{z_i=1} (1-z_i)\right) \omega_{g,n}(z_1,\ldots,z_n).$$

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Thank you!

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