Seminar “Moduli spaces of complex curves” @ WWU

The combinatorial geometry of the moduli space of curves

j/w J.E. Andersen, G. Borot, S. Charbonnier, D. Lewański, C. Wheeler

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Moduli space of curves

For $g, n \geqslant 0$ such that $2g - 2 + n > 0$, consider the moduli space of curves

$$\mathcal{M}_{g,n} \coloneqq \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} C \text{ cmplx cmpct curve} \\ \text{genus } g \text{ with } n \text{ marked pnts} \end{array} \right\} \Big/ \sim$$

which is a smooth complex orbifold of dimension $3g - 3 + n$. It admits a compactification $\overline{\mathcal{M}}_{g,n}$.

Fundamental problem

Understand $H^\bullet(\mathcal{M}_{g,n})$, $H^\bullet(\overline{\mathcal{M}}_{g,n})$ and its intersection theory:

- generators and relations,
- differential forms representing cohomology classes,
- efficient computation of intersection numbers,
- enumerative-geometric interactions (*e.g.* ELSV)
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Different geometric models

There exists various modular interpretation of $\mathcal{M}_{g,n}$. Alternative modular definitions lead to different geometric structures.

- Moduli space $\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})$ of metric ribbon graphs equipped with the Kontsevich symplectic form ω_{K}.
- Moduli space $\mathcal{M}_{g,n}^{\text{hyp}}(\vec{L})$ of hyperbolic surfaces equipped with the Weil–Petersson symplectic form ω_{WP}.
- Moduli space $\mathcal{M}_{g,n}^{\text{flat}}(\vec{\alpha})$ of flat surfaces equipped with the Veech volume form.
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$$\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) := \left\{ G \;\middle|\; \begin{array}{c} G \text{ metric ribbon graph} \\ \text{genus } g \text{ with } n \text{ bndrs} \\ \text{of length } \vec{L} \end{array} \right\} \Big/ \text{isometry}$$

has a natural symplectic form ω_K.

Theorem (Jenkins–Strebel '60s, Kontsevich '92, Zvonkine '02)

- For every $\vec{L} \in \mathbb{R}_+^n$, there is an orbifold isomorphism $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) \cong \mathcal{M}_{g,n}$.
- The symplectic volumes are finite and given by

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \exp(\omega_K) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2} \psi_i \right).$$

- The symplectic volumes are computed recursively on $2g-2+n$ (Witten's conjecture).

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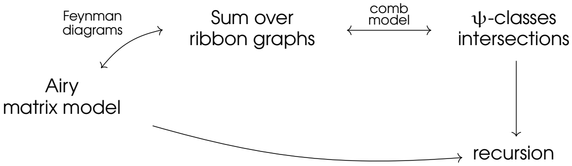
Proves comparison

Kontsevich's proof of the recursion is based on matrix model techniques.



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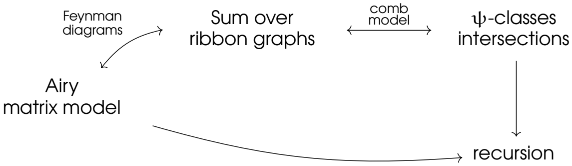

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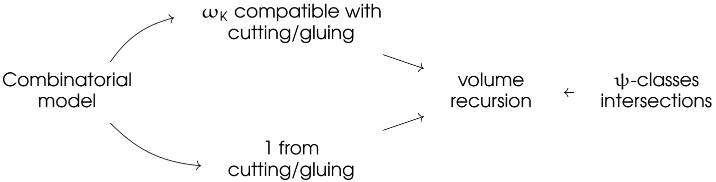

Ribbon graphs

Definition

A ribbon graph is a graph G with a cyclic order of the edges at each vertex.

We have well-defined

- *genus* $g \geqslant 0$,
- number of *boundary components* $n \geqslant 1$.

We call (g, n) the type of the ribbon graph. Boundaries are assumed to be labeled.

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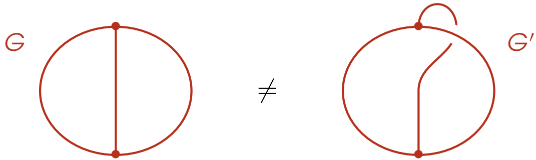

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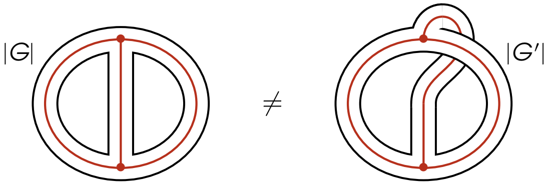

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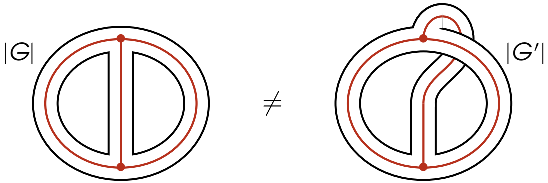

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Metric ribbon graphs

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A metric ribbon graph is a ribbon graph G with an assignment $\ell \colon E_G \to \mathbb{R}_+$. The space of such metrics is $\mathbb{R}_+^{E_G}$.

$\ell(\partial_1 G) = 57 + \pi$

$\ell(\partial_2 G) = \pi + \sqrt{2}$

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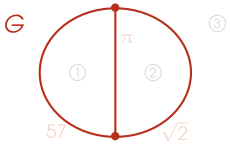

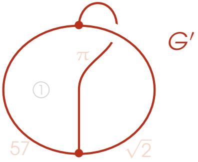

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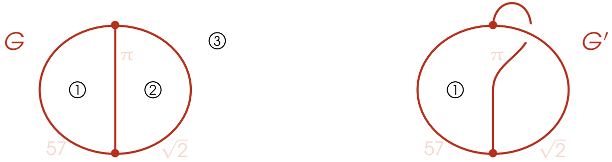

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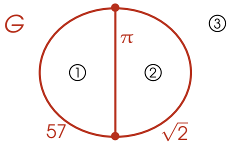

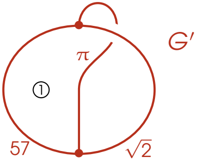

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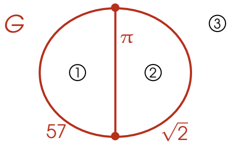

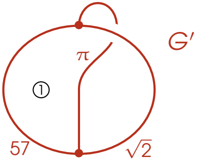

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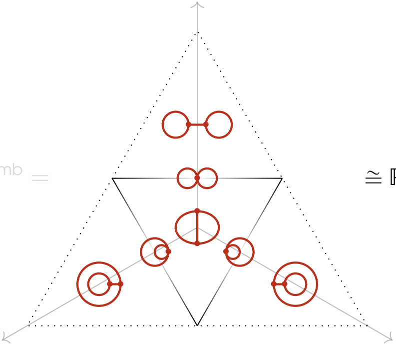

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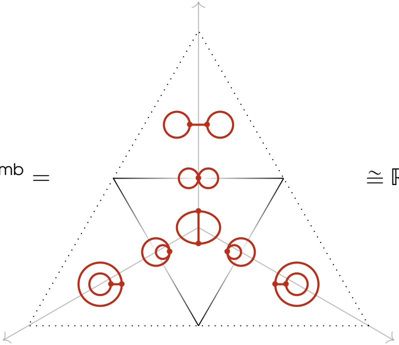

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The combinatorial moduli space

Define the combinatorial moduli space

$$\mathcal{M}_{g,n}^{\mathrm{comb}} := \bigcup_{\substack{G \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{E_G}}{\mathrm{Aut}(G)},$$

where we glue orbicells through degeneration of edges.

We have a map $p \colon \mathcal{M}_{g,n}^{\mathrm{comb}} \to \mathbb{R}_+^n$, assigning to each metric ribbon graph the length of the labeled faces. We set $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) := p^{-1}(\vec{L})$.

Proposition (Jenkins '57, Strebel '67, Zvonkine '02)

$\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ is a real topological orbifold of dimension $6g - 6 + 2n$, and there exists a homeomorphism of topological orbifolds

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The combinatorial Teichmüller space

Consider a topological compact surface Σ of genus $g \geqslant 0$, with $n \geqslant 1$ labeled boundaries $\partial_1\Sigma, \ldots, \partial_n\Sigma$.

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$$\mathcal{T}_\Sigma^{\mathrm{comb}} := \left\{ G \hookrightarrow \Sigma \;\middle|\; \begin{array}{l} G \text{ is a MRG embedded into } \Sigma \\ \text{s.t. } G \text{ is a deformation retract of } \Sigma \end{array} \right\} \Big/ \sim$$

where two embedded MRGs are identified iff

- they are isometric as MRGs,
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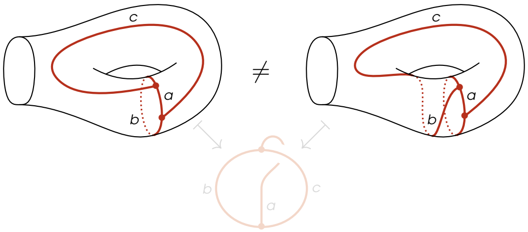

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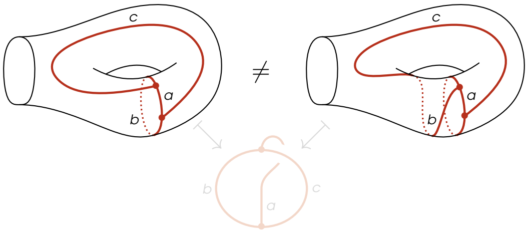

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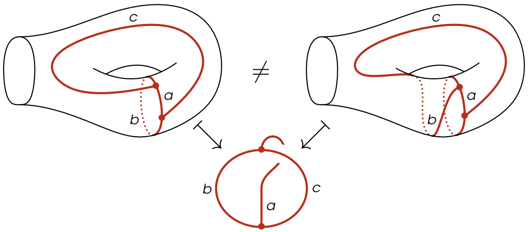

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Proposition

- $\mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$ is a real topological manifold of dimension $6g-6+2n$.
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Length of simple closed curve

Fix a simple closed curve γ in Σ, and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$. Define the length of γ with respect to $\mathbb{G}$:

- homotope γ to the embedded graph,
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$$\ell_{\mathbb{G}}(\gamma) = c + d + 2e + f.$$

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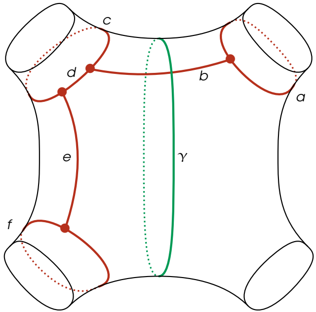

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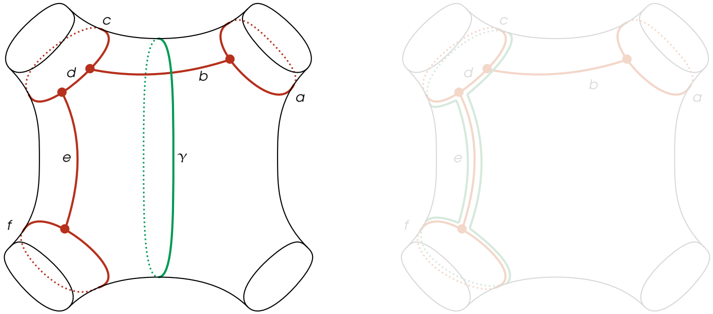

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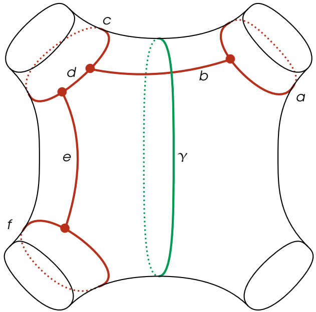

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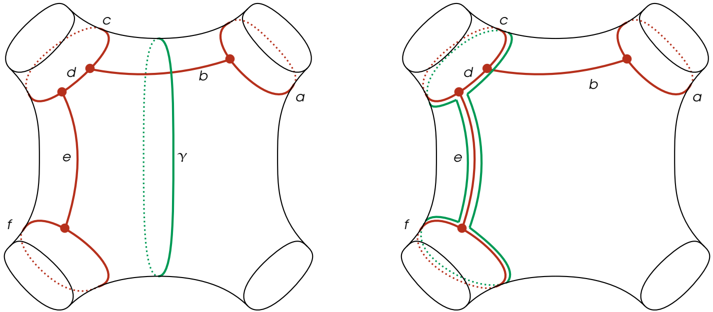

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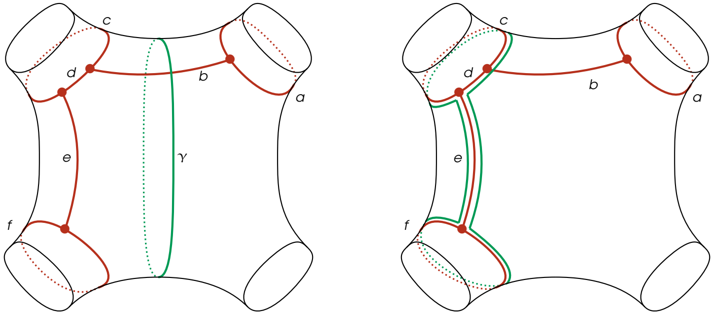

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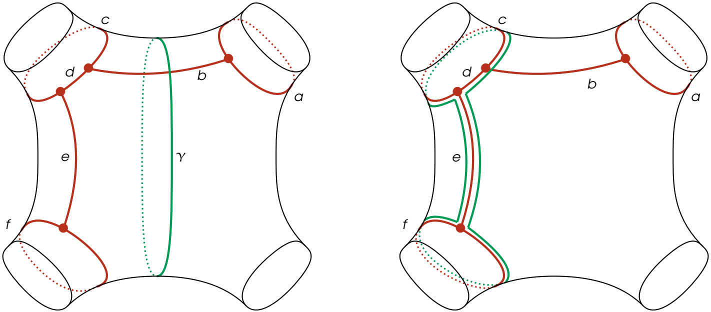

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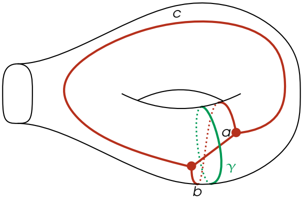

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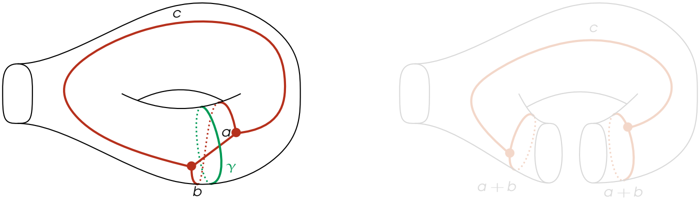

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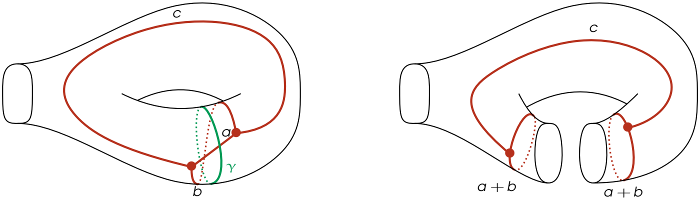

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For a.e. $\tau \in \mathbb{R}$, it is possible to glue $\mathbb{G}$ and $\mathbb{G}'$ along $\partial_i\Sigma \sim \partial_j\Sigma'$ with twist τ, and obtain an embedded MRG on the glued surface.

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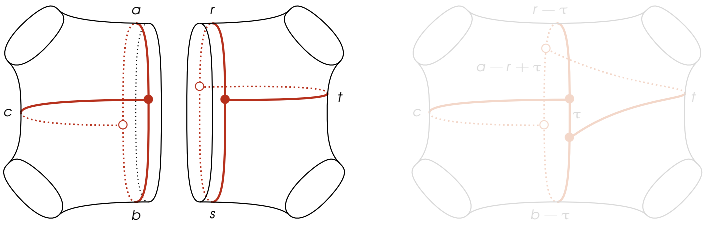

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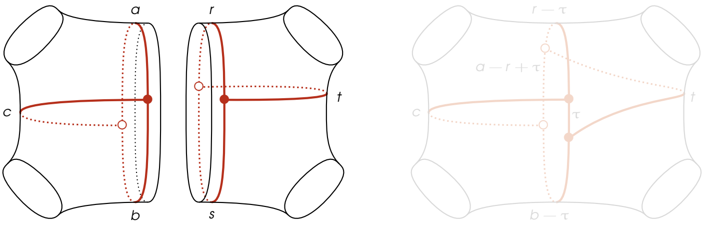

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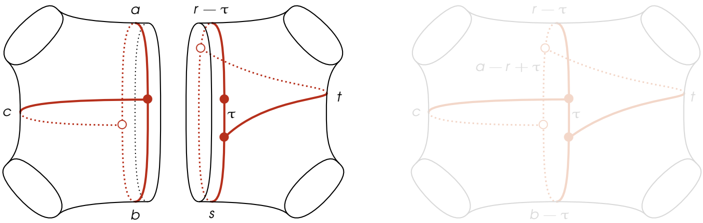

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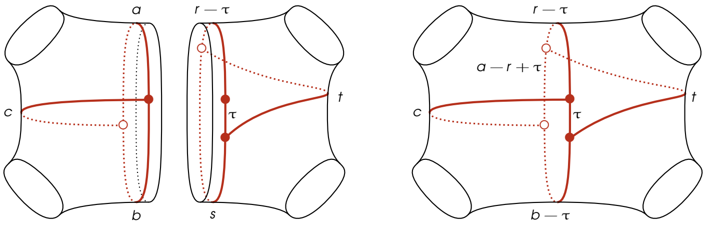

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Combinatorial Fenchel–Nielsen coordinates

Fix a pants decomposition $\mathscr{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$ of Σ. We have a map

$$\mathrm{FN}\colon \mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L}) \longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$$
$$\mathbb{G} \longmapsto \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_{i=1}^{3g-3+n}$$

called the combinatorial Fenchel–Nielsen coordinates.

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Question

Does $(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i))$ determine $\mathbb{G}$?

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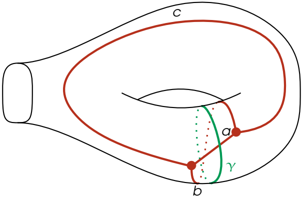
$\mathbb{G} =$ 

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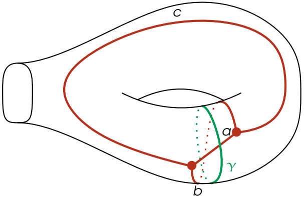
$\mathbb{G} =$ 

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Combinatorial Fenchel–Nielsen coordinates

Theorem (Andersen, Borot, Charbonnier, AG, Lewański, Wheeler)

For every choice of $\mathscr{P}$, the map

$$\mathrm{FN}\colon \mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L}) \longrightarrow (\mathbb{R}_{+} \times \mathbb{R})^{3g-3+n}$$

is a homeomorphism onto its image, with an open dense image.

The Kontsevich form

Define the Kontsevich 2-form ω_K on each cell of $\mathcal{T}^{\mathrm{comb}}_{\Sigma}(\vec{L})$ by

$$\omega_K := \sum_{i=1}^{n} \frac{L_i^2}{2}\Psi_i, \qquad \Psi_i := \sum_{e_i^{[a]} \prec e_i^{[b]}} \frac{d\ell_{e_i^{[a]}}}{L_i} \wedge \frac{d\ell_{e_i^{[b]}}}{L_i},$$

where $e_i^{[1]}, e_i^{[2]}, \ldots$ are the edges around the ith face of the ribbon graph underlying the cell, and $\prec$ is the order on the edges induced by the orientation of the surface.

$e^{[1]} = a,\ e^{[2]} = b,\ e^{[3]} = c$
$e^{[4]} = a,\ e^{[5]} = b,\ e^{[6]} = c$

$$\Psi_1 = \frac{2}{L^2}(da \wedge db + da \wedge dc + db \wedge dc)$$

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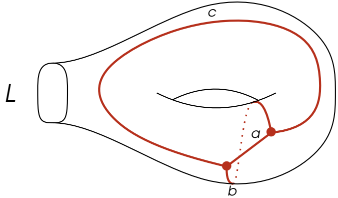

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Theorem (Kontsevich '92, Zvonkine '02)

- The form ω_K on $\mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$ is symplectic and MCG invariant
- The symplectic volume $V_{g,n}(\vec{L})$ of $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ is finite and given by

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \exp(\omega_K) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\Big(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\Big).$$

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For every choice of pants decomposition on Σ, we have a global coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$. Then

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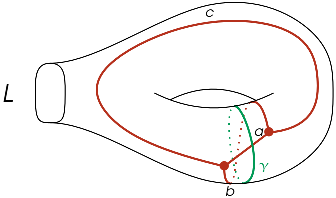

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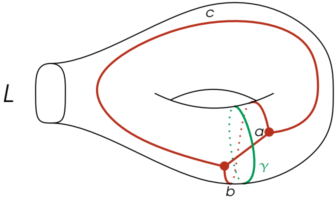

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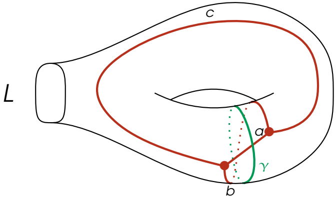

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Let T be a torus with one boundary component.

Theorem (ABCGLW '20)

For any $\mathbb{G} \in \mathcal{T}_T^{\mathrm{comb}}(L)$, we have

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Here $[x]_+ := \max(x, 0)$.

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A combinatorial McShane identity

Let T be a torus with one boundary component.

Theorem (ABCGLW '20)

For any $\mathbb{G} \in \mathcal{T}_T^{\mathrm{comb}}(L)$, we have

$$1 = \sum_{\substack{\gamma \\ \text{simple closed curve}}} \frac{[L - 2\ell_{\mathbb{G}}(\gamma)]_+}{L}.$$

Here $[x]_+ := \max(x, 0)$.

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Here, the first sum is over simple closed curves γ bounding a pair of pants with $\partial_1\Sigma$ and $\partial_i\Sigma$, and the second sum is over all pairs of simple closed curves γ, γ' bounding a pair of pants with $\partial_1\Sigma$.

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Witten–Kontsevich recursion

combinatorial Wolpert formula

combinatorial Mirzakhani identity

recursion for $V_{g,n}(\vec{L})$

The Kontsevich volumes are computed recursively by

$$
\begin{aligned}
V_{g,n}(L_1,\dots,L_n) &= \sum_{i=2}^{n} \int_{\mathbb{R}_+} d\ell\, \ell\, \frac{\mathcal{R}(L_1,L_i,\ell)}{L_1}\, V_{g,n-1}(\ell, L_2,\dots,\widehat{L_i},\dots,L_n) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} d\ell d\ell'\, \ell\ell'\, \frac{\mathcal{D}(L_1,\ell,\ell')}{L_1} \Big(V_{g-1,n+1}(\ell,\ell',L_2,\dots,L_n) \\
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with initial conditions $V_{0,3}(L_1,L_2,L_3)=1$ and $V_{1,1}(L)=\frac{L^2}{48}$.

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Integral structure

Definition

A metric ribbon graph G is called integral if the length of every edge is a positive integer.

$$\mathbb{Z}\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) := \left\{ \begin{array}{c} \text{integral MRGs} \\ \text{type } (g,n) \text{ and boundary } \vec{L} \end{array} \right\} \subset \mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}).$$

We can count integral points as

$$N_{g,n}(\vec{L}) := \sum_{G \in \mathbb{Z}\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \frac{1}{\mathrm{Aut}(G)}.$$

Idea. $N_{g,n}(\vec{L})$ is the volume of the combinatorial moduli space w.r.t the "counting measure", that is Dirac deltas at the integral points.

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Norbury recursion from Mirzakhani

integral structure $\mathbb{Z}\mathcal{M}_{g,n}^{\text{comb}}$

combinatorial Mirzakhani identity

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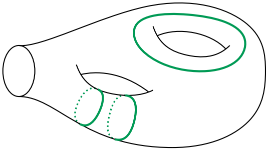
Multicurve count

Define $\mathcal{N}_\Sigma : \mathcal{T}^{\text{comb}}_\Sigma \times \mathbb{R}_+ \to \mathbb{N}$ the counting function,

$$\mathcal{N}_\Sigma(\mathbb{G}; t) \coloneqq \# \left\{ \gamma \;\middle|\; \begin{array}{c}\text{multicurve in } \Sigma \\ \text{with } \ell_{\mathbb{G}}(\gamma) \leqslant t\end{array} \right\}.$$



Theorem (ABCGLW '20)

- The counting function $\mathcal{N}_\Sigma(\mathbb{G}; t)$ is computed by a Mirzakhani-type recursion (geometric recursion).
- It is MCG invariant, and its mean value

$$\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t) \coloneqq \int_{\mathcal{M}^{\text{comb}}_{g,n}(\vec{L})} \mathcal{N}_{g,n}(G; t) \, \frac{\omega_K^{3g-3+n}}{(3g-3+n)!}$$

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- Taking the asymptotic as $t \to \infty$, we get the Masur–Veech volumes of the moduli space of quadratic differentials.

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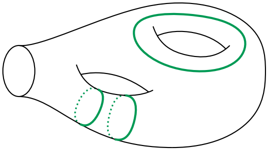

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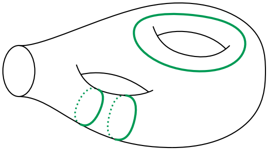

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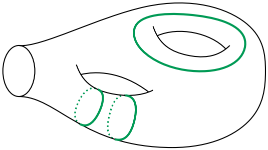

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To conclude we obtained:

- global length/twist coord's on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$
- a combinatorial Wolpert formula for ω_K
- a Mirzakhani identity, from which we gave a geometric proof of:
  - Witten–Kontsevich recursion for symplectic volumes/ψ-intersections
  - Norbury's recursion for lattice pnts
- a recursion for the multicurve counting and Masur–Veech volumes

* a PL manifold structure on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$

* a flow $\sigma^t : \mathcal{T}_{\Sigma}^{\mathrm{hyp}}(\vec{L}) \to \mathcal{T}_{\Sigma}^{\mathrm{hyp}}(\vec{L})$ that limits to $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$

Thank you!

1. J.E. Andersen, G. Borot, S. Charbonnier, A. Giacchetto, D. Lewański, C. Wheeler. "On the Kontsevich geometry of the combinatorial Teichmüller space". (2020) arXiv:2010.11806 [math.DG].

2. J.E. Andersen, G. Borot, N. Orantin. "Geometric recursion". (2017) arXiv:1711.04729 [math.GT].

3. M. Kontsevich "Intersection theory on the moduli space of curves and the matrix Airy function". *Commun. Math. Phys.* 147 (1992).

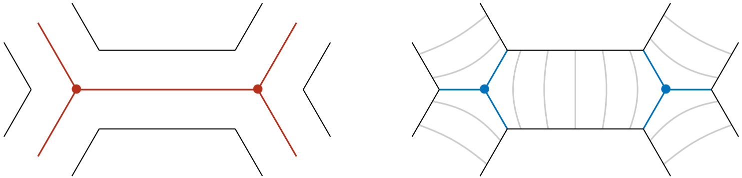
4. M. Mirzakhani "Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces". *Invent. Math.* 167.1 (2007).

5. P. Norbury "Counting lattice points in the moduli space of curves". *Math. Res. Lett.* 17 (2010).

6. E. Witten "Two-dimensional gravity and intersection theory on moduli space". *Surv. Diff. Geom.* 1.1 (1990).

Embedded MRGs and measured foliations

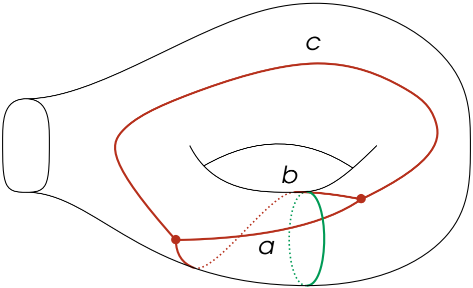
Every embedded MRG $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ defines an (isotopy class of) measured foliations on Σ. Locally:



Measured foliations dual to embedded MRGs

- are always transverse to $\partial\Sigma$,
- do not contain saddle connections (i.e. singular leaves connecting singular points).

Example of cutting/gluing



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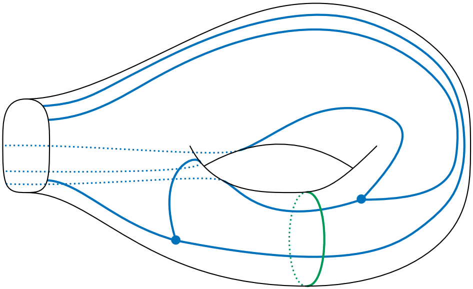

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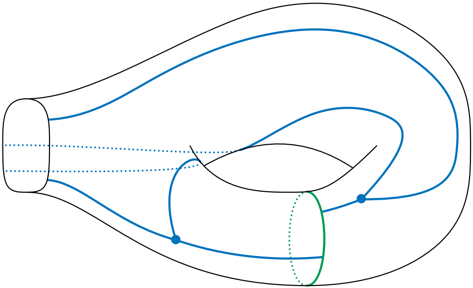

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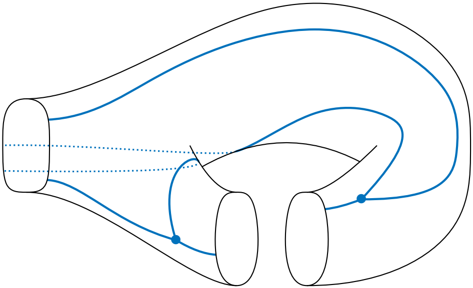

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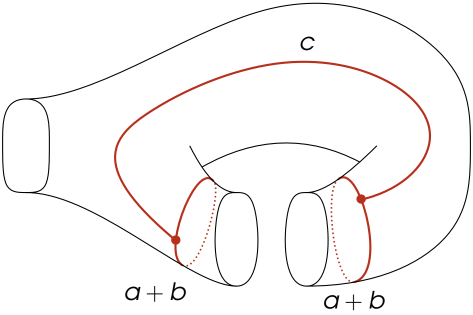
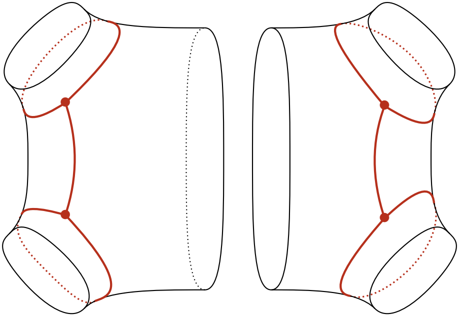


Non-admissible gluing



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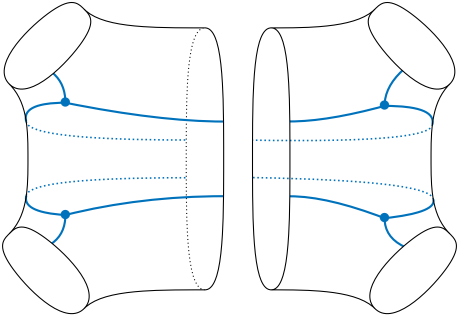

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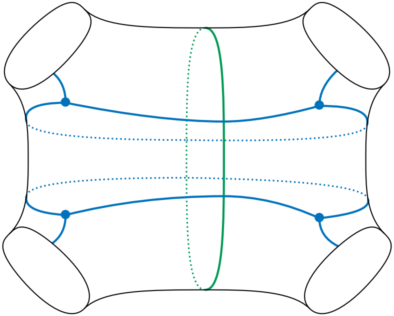

Geometric kernels

Lemma

For a fixed pair of pants P, identify $\mathbb{R}_+^3 \cong \mathcal{T}_P^{\mathrm{comb}}$.

- The function
$$\mathcal{D}(L, \ell, \ell') := [L - \ell - \ell']_+$$
associates to a point $(L, \ell, \ell') \in \mathcal{T}_P^{\mathrm{comb}}$ the fraction of $\partial_1 P$ that is not common with $\partial_2 P \cup \partial_3 P$ (once retracted to the graph).
- The function
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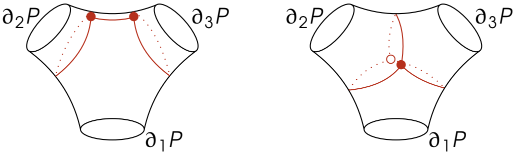

Spectral curves

- Symplectic volumes $V_{g,n}(\vec{L})$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = \frac{z^2}{2} \qquad y(z) = z \qquad B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- Lattice point count $N_{g,n}(\vec{L})$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = z + \frac{1}{z} \qquad y(z) = z \qquad B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- Average number of multicurves $\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t)$ of length $\leqslant t$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = \frac{z^2}{2} \qquad y(z) = z$$

$$B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{(s\pi)^2}{\sin^2\big(s\pi(z_1 - z_2)\big)} \right) \frac{dz_1\, dz_2}{2}$$