Euler classes & negative powers of the canonical class

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Overview

§ 1. Ω - classes

§ 2. Euler characteristic of $\mathcal{M}_{g,n}$

§ 3. Θ - class

§ 1. Ω-classes

r,k integers, $r \geq 1$

$$\overline{\mathcal{M}}^{r,k}_{g;a} = \text{proper moduli stack of r-th roots of } \omega_{\log}^{\otimes k}\left(-\textstyle\sum_{i=1}^{n} a_i p_i\right)$$

$a \in \mathbb{Z}^n$ s.t. $\sum_i a_i = k(2g-2+n) \pmod r$

Universal curve	$\mathcal{C} \xrightarrow{\pi} \overline{\mathcal{M}}^{r,k}_{g;a}$
— r-th root	$\mathcal{L} \to \mathcal{C}$
Forgetful map	$\overline{\mathcal{M}}^{r,k}_{g;a} \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n}$

$$\mathcal{L} \to \mathcal{C} \to \overline{\mathcal{M}}^{r,k}_{g;a} \to \mathcal{M}_{g,n}$$

Thm (Chiodo '08)

$$\mathrm{ch}_m(R^\bullet \pi_* \mathcal{L}) = \frac{B_{m+1}(\frac{k}{r})}{(m+1)!}\kappa_m - \sum_{i=1}^{n} \frac{B_{m+1}(\frac{a_i}{r})}{(m+1)!}\psi_i^m + \frac{r}{2}\sum_{a=0}^{r-1}\frac{B_{m+1}(\frac{a}{r})}{(m+1)!} j_{a*}\frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}$$

$$\Rightarrow \quad \Omega^{r,k}_{g,n}(a_1,\dots,a_n) = \epsilon_* c(-R^\bullet \pi_* \mathcal{L})$$
$$= \epsilon_* \exp\Big(\sum_{m \geq 1} (-1)^m (m-1)!\, \mathrm{ch}_m(R^\bullet \pi_* \mathcal{L})\Big) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

is a CohFT, has an expression in terms of stable graphs, ...

Applications.

	r	k	deg	Enumerative geom.
Hodge class →	1	1		simple Hurwitz numbers (ELSV formula) } GW of $\mathbb{P}^1$
	pq	$q \geq 1$		q-orbifold Hurwitz number w/ (p+1)-completed cycles } GW of $\mathbb{P}^1$
	r	1	top	Witten r-spin class

"r=0"	0	g	double ramif. cycle,
	1	2 top	Masur-Veech volumes (quadratic diff's)
	⋮		

§2. Euler characteristic of $\mathcal{M}_{g,n}$ $(r=1, k=-1)$

Thm (Harer-Zagier '86)

$$\chi_{g,n} = \begin{cases} (-1)^{n-3}(n-3)! & g=0,\ n\geq 3 \\ (-1)^n \dfrac{(n-1)!}{12} & g=1,\ n\geq 1 \\ (-1)^n (2g-3+n)!\ \dfrac{B_{2g}}{2g(2g-2)!} & g>1,\ n\geq 0 \end{cases}$$

New strategy: Gauss-Bonnet formula.

Prop (Costantini-Möller-Zachhuber). $\overline{M}$ smooth cmct m-dim orbifold, $D \subset \overline{M}$ a normal crossing divisor, $M = \overline{M} \setminus D$.

$$\chi(M) = \int_{\overline{M}} e\left(T_{\overline{M}}(\log D)\right)$$

log tangent bundle

$D = \bigcup_{i=1}^{d} D_i$, $U \subseteq \overline{M}$ neigh. of a pnt where $D_1, \dots, D_k$ meet $\perp$; choose local coord's s.t. $D_j \overset{loc}{=} \{x_j = 0\}$

$$\Rightarrow \Omega^1_{\overline{M}}(\log D)(U) = \left\langle \frac{dx_1}{x_1}, \dots, \frac{dx_k}{x_k}, dx_{k+1}, \dots, dx_m \right\rangle$$

as an $\mathcal{O}_{\overline{M}}(U)$-module

Our case: $\overline{M} = \overline{\mathcal{M}}_{g,n}$, $D = \partial\overline{\mathcal{M}}_{g,n}$ $\Rightarrow$ $M = \mathcal{M}_{g,n}$

$$\rightsquigarrow \quad T_{\overline{\mathcal{M}}_{g,n}}(\log \partial\overline{\mathcal{M}}_{g,n})\Big|_{(C,p_1,\dots,p_n)} = H^0\left(C, \omega_C^{\otimes 2}\left(\textstyle\sum_i p_i\right)\right)^*$$

Take the Ω-class construction for $r=1$, $k=-1$, $a=0^n$:

$$\Omega_{g,n}^{r=1,k=-1}(0^n) = c(\underbrace{-R^\bullet\pi_*\mathcal{L}}) = c(T_{\overline{\mathcal{M}}_{g,n}}(\log \partial\overline{\mathcal{M}}_{g,n}))\,.$$

$\omega_{\log}^{\otimes -1}(-\Sigma_i p_i)$

fiber over $(C,p_1,\dots,p_n)$: $\quad H^1(C,\omega_{\log}^{\otimes -1}) \; - \; \underbrace{H^0(C,\omega_{\log}^{\otimes -1})}_{=0 \text{ for deg reasons}}$

$\cong$ Serre

$H^0(C,\omega^{\otimes 2}(\Sigma_i p_i))^*$

RR: $\chi = h^1 - h^0$
$\quad = +(2g-2+n)+g-1$
$\quad = 3g-3+n$

Chiodo's formula

$$\Omega_{g,n}^{r=1,k=-1}(0^n) \overset{\text{Chiodo's formula}}{=} \exp\left[\sum_{m\geq 1}(-1)^m\left(\frac{B_{m+1}(-1)}{m(m+1)}\kappa_m - \sum_{i=1}^n \frac{B_{m+1}(0)}{m(m+1)}\psi_i^m + \frac{1}{2}\frac{B_{m+1}(0)}{m(m+1)} j_*\frac{(\psi')^m-(-\psi'')^m}{\psi'+\psi''}\right)\right]$$

k/r $\qquad a_i/r$

$$= \Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$$

$B_{m+1}(0) = B_{m+1}$
$B_{m+1}(-1) = B_{m+1} - (-1)^m(m+1)$
$+$ Mumford's form.

Proposition. $\displaystyle \chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$

$\log(1-x) = -\sum_{m\geq 1}\frac{x^m}{m}$

$\underbrace{= \cancel{1} - \psi_{n+1}}$ $\quad 0$

Corollary. The HZ formula holds true.

Proof: $\displaystyle \chi_{g,n+1} = \int_{\overline{\mathcal{M}}_{g,n+1}} p^*\left(\Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)\right)\exp\left(-\sum_{m\geq 1}\frac{\psi_{n+1}^m}{m}\right)$

$\Lambda(-1) = p^*\Lambda(-1)$
$\kappa_m = p^*\kappa_m + \psi_{n+1}^m$

$$= \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)\underbrace{p_*(-\psi_{n+1})}_{=-(2g-2+n)}$$

$$= -(2g-2+n)\chi_{g,n}$$

$\chi_{0,3} = 1 \quad \Rightarrow \quad \boxed{\chi_{0,3} = (-1)^{n+3}(n-3)!}$

$$\chi_{1,1} = \int_{\overline{\mathcal{M}}_{1,1}} (-\lambda_1 - \kappa_1) = -\frac{1}{24} - \frac{1}{24} = -\frac{1}{12} \quad \Rightarrow \quad \boxed{\chi_{1,1} = (-1)^n \frac{(n-1)!}{12}}$$

$$\chi_{g,0} = \sum_{\ell \geq 1} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda(-1) \prod_{i=1}^{\ell} \psi_i^{\mu_i+1} = \frac{B_{2g}}{2g(2g-2)}$$

Dubrovin-Yang-Zagier '17
using ELSV + Toda eqn
for Hurwitz numbers

$$\Downarrow$$

$$\boxed{\chi_{g,n} = (-1)^n (2g-3+n)! \, \frac{B_{2g}}{2g(2g-2)!}}$$

§3. Θ-class $(r=2, k=-1)$

$$\Theta_{g,n} = (-1)^n 2^{g-1} \left[\Omega_{g,n}^{r=2,k=-1}(a = 1^n) \right]^{top} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$$

$$-R^\bullet \pi_* \mathcal{L} \big|_{(C,p_1,\dots,p_n,L)} = \underbrace{H^1(C,L)} - \cancel{H^0(\ )} \qquad 0 \text{ for deg reasons}$$

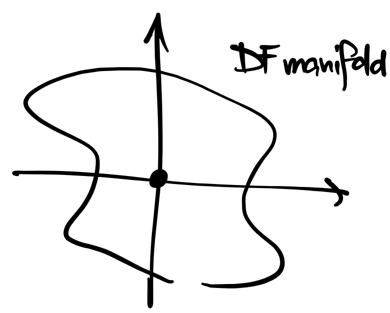
$$L^{\otimes 2} \cong \omega_{\log}^{\otimes -1}\left(-\textstyle\sum_i p_i\right)$$

$$\mathrm{rk} = -\frac{-(2g-2+n)-n}{2} + g - 1 = g-1+n+g-1 = 2g-2+n$$

Prop (Norbury) $(\Theta_{g,n})_{2g-2+n>0}$ is a CohFT (w/o flat unit) satisfying

$$\psi_{n+1} \cdot p^* \Theta_{g,n} = \Theta_{g,n+1}$$

Conjecture. $Z^{\Theta}(\underline{t},\hbar) = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2}}{n!} \sum_{k_1,\dots,k_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{k_i} t_{k_i} \right)$ is a KdV τ-fctn.



Rmrk. Θ is not semisimple! Worst: the whole DF mfld is non-semisimple.

$\rightsquigarrow$ Deform the DF structure to a semisimple one, then apply Teleman

Defn/Thm.

$p_m: \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$

$$\Theta^{\epsilon}_{g,n} = (-1)^n 2^{g-1} \sum_{m\geq 0} \frac{(-\epsilon)^m}{m!} p_{m,*}\left[\Omega^{r=2,k=-1}_{g,n+m}(1^n, 0^m)\right]^{top}$$

Then

1) $\Theta^{\epsilon}_{g,n} = \Theta_{g,n} + \epsilon \cdot H^{4g-6+4n}(\overline{\mathcal{M}}_{g,n})$

$1 \bullet_{\epsilon} 1 = -\epsilon^2 1$

$E = \frac{1}{2}(\partial_t - t\partial_t)$

conf. dim $= 3$

2) If $\epsilon \neq 0$, $\Theta^{\epsilon}_{g,n}$ is a semi-simple, homogeneous CohFT

3) $\Theta^{\epsilon}_{g,n} = (-1)^n \epsilon^{4g-4+2n} \exp\left(\sum_{m\geq 1} (-1)^m \epsilon^{2m} s_m \kappa_m\right)$

where $\exp\left(-\sum_{m\geq 1} s_m u^m\right) = \sum_{k\geq 0} (-1)^k (2k+1)!!\, u^m$. ← From Teleman's rec. thm

$T(u) = u(1 - R^{-1}(u)v(u))$, $\quad v' + \frac{\mu + \delta/2\,\mathrm{Id}}{u} v = -\frac{\phi}{u^2}(v-1)$

Corollary (Kazarian-Norbury conj)

Conj. Holds in $A^{\bullet}(\overline{\mathcal{M}}_{g,n})$

i) $\left[\exp\left(\sum_{m\geq 1} s_m \kappa_m\right)\right]^d = 0$ in $H^{2d}(\overline{\mathcal{M}}_{g,n})$ $\forall d > 2g-2+n$

ii) $\Theta_{g,n} = \left[\exp\left(\sum_{m\geq 1} s_m \kappa_m\right)\right]^{2g-2+n}$

Corollary (Norbury's conj). Z^{Θ} is a KdV tau-fnct.

1) $Z^{\Theta^{\epsilon}}$ is computed by topological recursion

2) TR is equivalent to certain Virasoro cnstrnts: $L^{\epsilon}_k Z^{\Theta^{\epsilon}} = 0$ $\forall k \geq 0$

3) Taking $\epsilon \to 0$, the Virasoro cnstrnts L^0_k have a unique solution, Brézin-Gross-Witten function, which is a KdV τ-fnctn.