Mini-rencontre in Bordeaux

Geometry of combinatorial moduli spaces

j/w J.E. Andersen, G. Borot, S. Charbonnier, D. Lewański, C. Wheeler

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April 11th, 2022

Moduli space of curves

For $g, n \geqslant 0$ such that $2g - 2 + n > 0$, consider the moduli space of curves

$$\mathcal{M}_{g,n} := \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} C \text{ cmplx cmpct curve} \\ \text{genus } g \text{ with } n \text{ marked pnts} \end{array} \right\} \Big/ \sim$$

which is a smooth complex orbifold of dimension $3g - 3 + n$. It admits a compactification $\overline{\mathcal{M}}_{g,n}$.

Fundamental problem

Understand $H^\bullet(\mathcal{M}_{g,n})$, $H^\bullet(\overline{\mathcal{M}}_{g,n})$ and its intersection theory:

- generators and relations,
- differential forms representing cohomology classes,
- efficient computation of intersection numbers,
- enumerative-geometric interactions

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Different geometric models

There exists various modular interpretation of $\mathcal{M}_{g,n}$. Alternative modular definitions lead to different geometric structures.

- Moduli space $\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})$ of metric ribbon graphs equipped with the Kontsevich symplectic form ω_{K}.
- Moduli space $\mathcal{M}_{g,n}^{\text{hyp}}(\vec{L})$ of hyperbolic surfaces equipped with the Weil–Petersson symplectic form ω_{WP}.
- Moduli space $\mathcal{M}_{g,n}^{\text{flat}}(\vec{\alpha})$ of flat surfaces equipped with the Veech volume form.
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has a natural symplectic form ω_K.

Theorem (Jenkins–Strebel '60s, Kontsevich '92, Zvonkine '02)

- For every $\vec{L} \in \mathbb{R}_+^n$, there is an orbifold isomorphism $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) \cong \mathcal{M}_{g,n}$.
- The symplectic volumes are finite and given by

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \exp(\omega_K) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\right).$$

- The symplectic volumes are computed recursively on $2g-2+n$ (Witten's conjecture).

Today's talk: new proof of Witten's conjecture, based on the *geometry of* $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ and parallel to Mirzakhani's proof in the hyperbolic setting

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Ribbon graphs

Definition

A ribbon graph is a graph G with a cyclic order of the edges at each vertex.

We have well-defined

- *genus* $g \geqslant 0$,
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We call (g, n) the type of the ribbon graph. Boundaries are assumed to be labeled.

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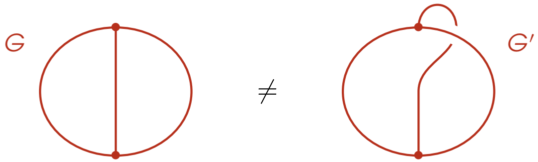

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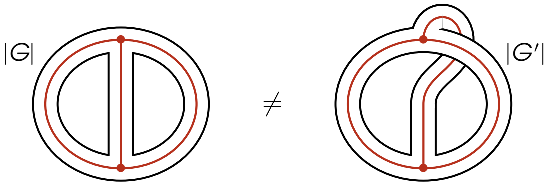

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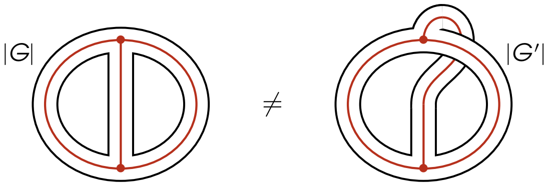

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Metric ribbon graphs

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A metric ribbon graph is a ribbon graph G with an assignment $\ell\colon E_G \to \mathbb{R}_+$. The space of such metrics is $\mathbb{R}_+^{E_G}$.

$\ell(\partial_1 G) = 57 + \pi$

$\ell(\partial_2 G) = \pi + \sqrt{2}$ $\qquad$ $\ell(\partial_1 G') = 2(57 + \pi + \sqrt{2})$

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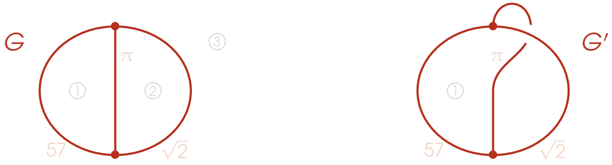

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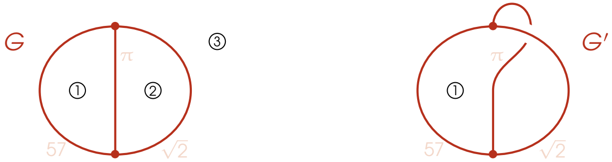

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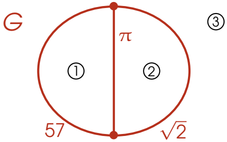

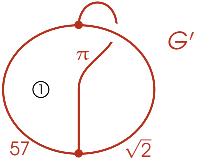

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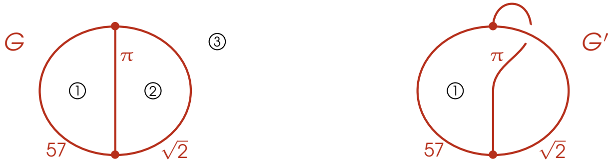

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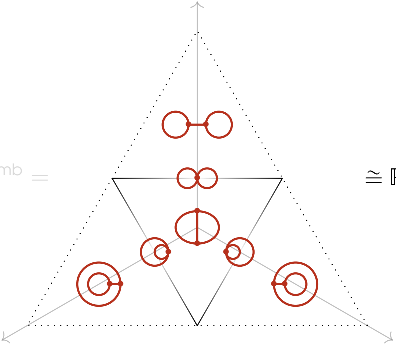

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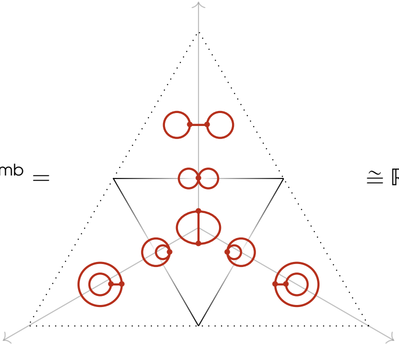

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$$\mathcal{M}_{g,n}^{\mathrm{comb}} := \bigcup_{\substack{G \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{E_G}}{\mathrm{Aut}(G)},$$

where we glue orbicells through degeneration of edges.

We have a map $p \colon \mathcal{M}_{g,n}^{\mathrm{comb}} \to \mathbb{R}_+^n$, assigning to each metric ribbon graph the length of the labeled faces. We set $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L}) := p^{-1}(\vec{L})$.

Proposition (Jenkins '57, Strebel '67, Zvonkine '02)

$\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ is a real topological orbifold of dimension $6g-6+2n$, and there exists a homeomorphism of topological orbifolds

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Consider a topological, compact, oriented surface Σ of genus $g \geqslant 0$, with $n \geqslant 1$ labeled boundaries $\partial_1\Sigma, \ldots, \partial_n\Sigma$.

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$$\mathcal{T}_\Sigma^{\text{comb}} := \left\{ G \hookrightarrow \Sigma \;\middle|\; \text{s.t.}\ \begin{array}{l} G \text{ is a MRG embedded into } \Sigma \\ G \text{ is a deformation retract of } \Sigma \end{array} \right\} \Big/ \sim$$

where two embedded MRGs are identified iff

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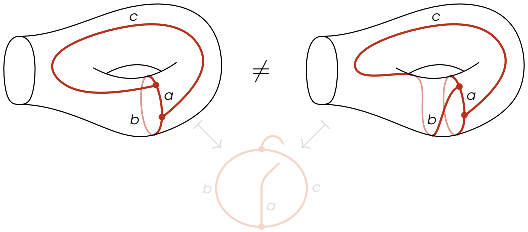

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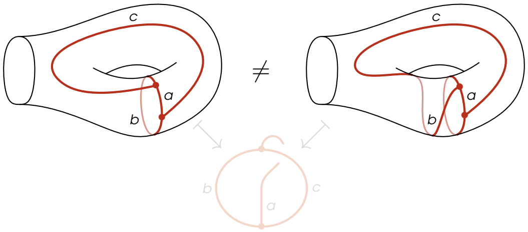

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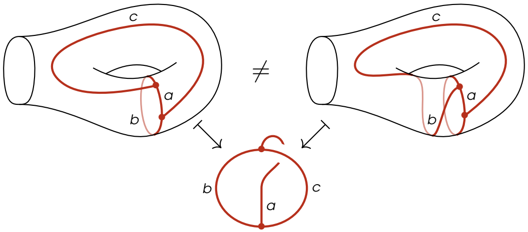

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Proposition

- $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$ is a real topological manifold of dimension $6g-6+2n$.
- The mapping class group $\mathrm{Mod}_{\Sigma} := \mathrm{Homeo}^{+}(\Sigma, \partial\Sigma)/\,\mathrm{Homeo}_{0}(\Sigma)$ is acting on $\mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L})$, and

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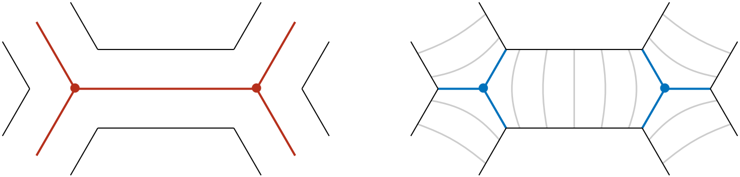
Embedded MRGs and measured foliations

Every embedded MRG $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$ defines an (isotopy class of) measured foliations on Σ. Locally:



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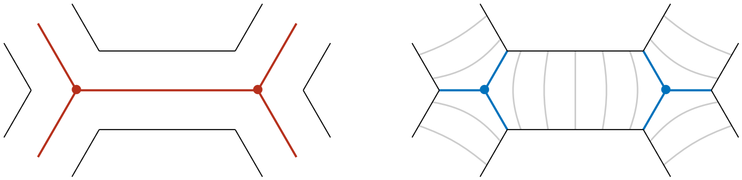

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Length of simple closed curve

Fix a simple closed curve γ in Σ, and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\mathrm{comb}}$. Define the length of γ with respect to $\mathbb{G}$:

- homotope γ to the embedded graph,
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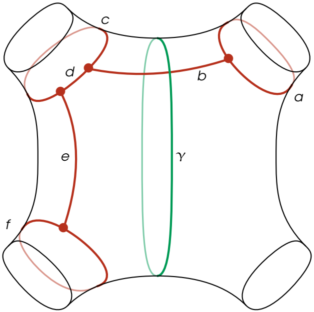

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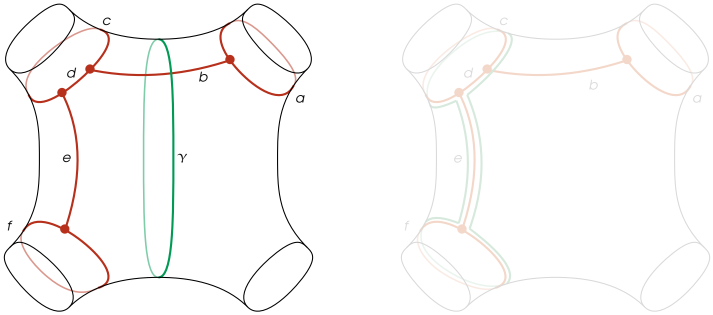

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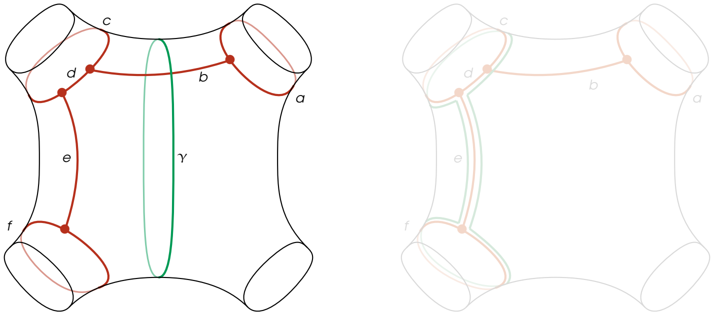

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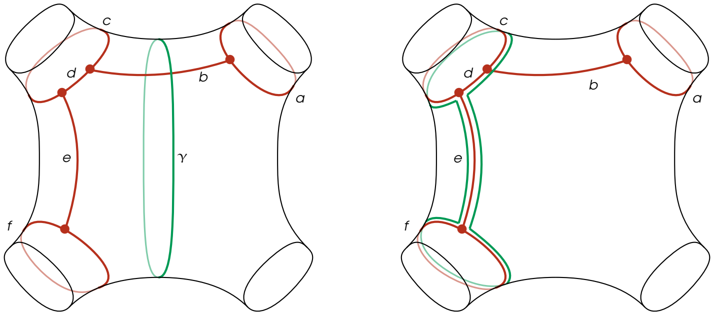

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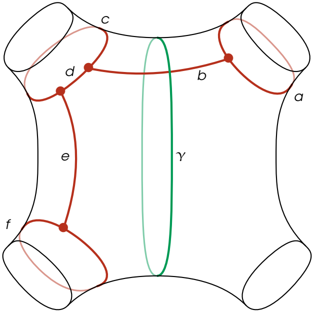

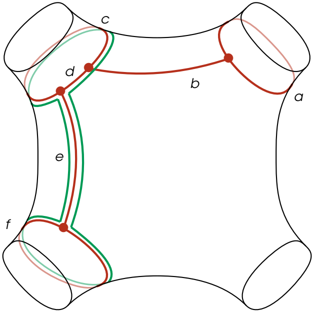

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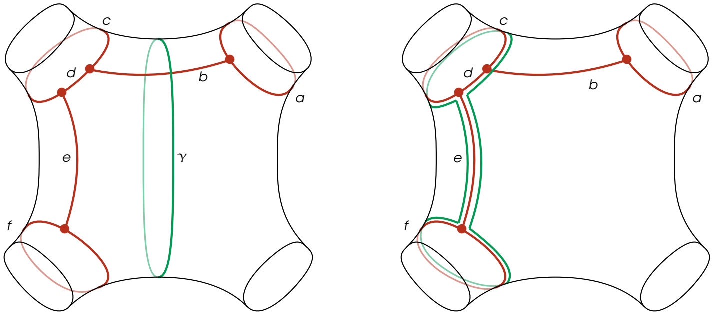

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Fix γ is a simple closed curve in Σ and $\mathbb{G} \in \mathcal{T}_{\Sigma}^{\text{comb}}$.

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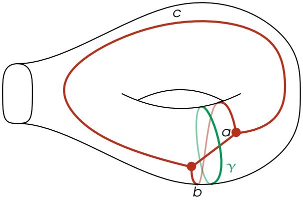

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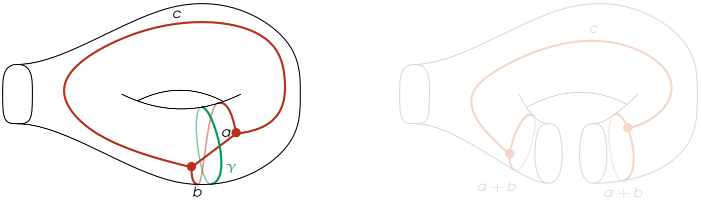

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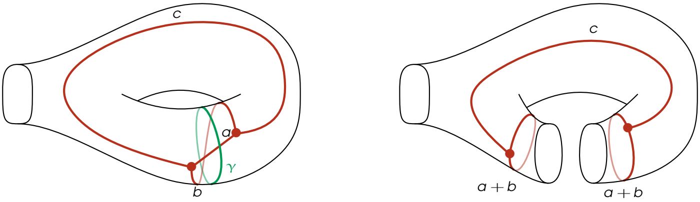

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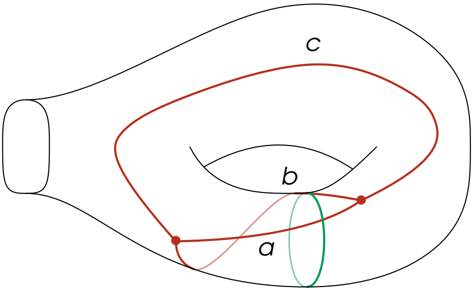

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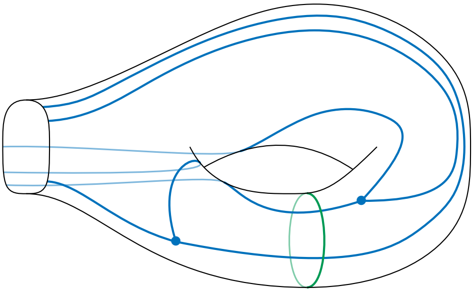

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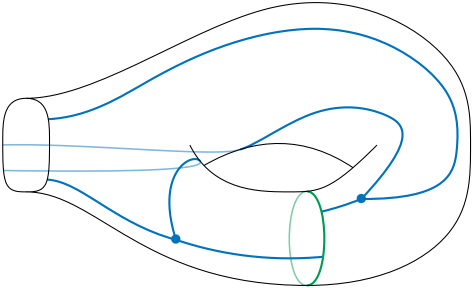

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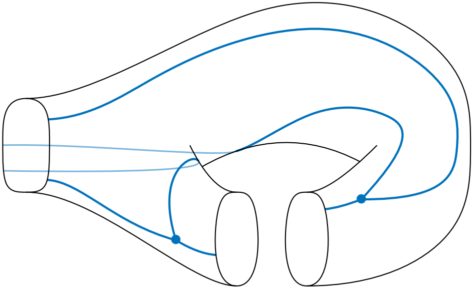

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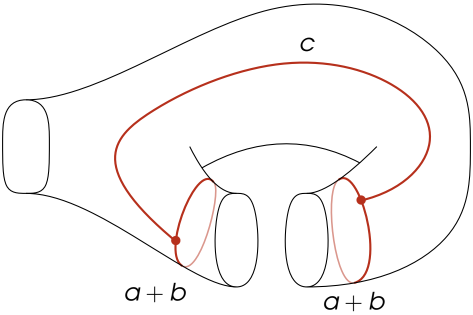

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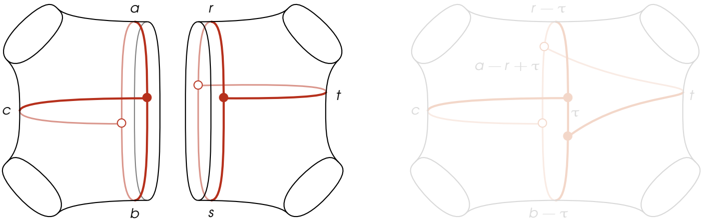

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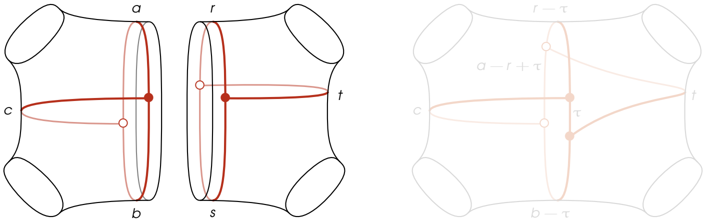

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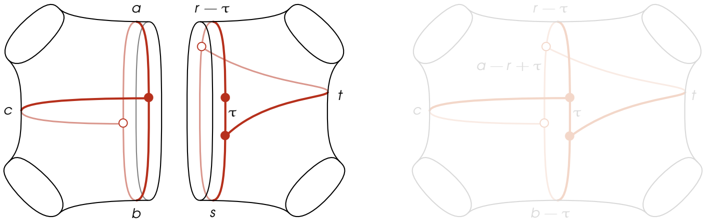

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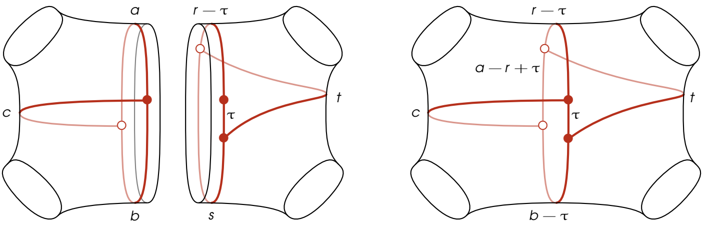

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Combinatorial Fenchel–Nielsen coordinates

Fix a pants decomposition $\mathscr{P} = (\gamma_1, \dots, \gamma_{3g-3+n})$ of Σ. We have a map

$$\begin{aligned} \mathrm{FN} \colon \mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L}) &\longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n} \\ \mathbb{G} &\longmapsto \big(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i)\big)_{i=1}^{3g-3+n} \end{aligned}$$

called the combinatorial Fenchel–Nielsen coordinates.

$\mathbb{G} =$

$$\mathrm{FN}(\mathbb{G}) = (\ell_{\mathbb{G}}(\gamma), \tau_{\mathbb{G}}(\gamma)) = (a+b, -a)$$

Question

Does $(\ell_{\mathbb{G}}(\gamma_i), \tau_{\mathbb{G}}(\gamma_i))$ determine $\mathbb{G}$?

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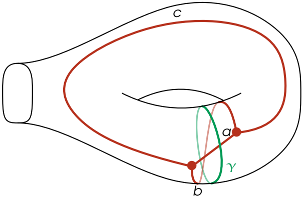
$\mathbb{G} =$ 

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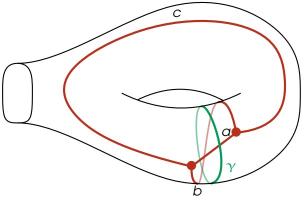
$\mathbb{G} =$ 

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Combinatorial Fenchel–Nielsen coordinates

Theorem (Andersen, Borot, Charbonnier, AG, Lewański, Wheeler)

For every choice of $\mathscr{P}$, the map

$$\mathrm{FN}\colon \mathcal{T}_{\Sigma}^{\mathrm{comb}}(\vec{L}) \longrightarrow (\mathbb{R}_{+} \times \mathbb{R})^{3g-3+n}$$

is a homeomorphism onto its image, with an open dense image.

The Kontsevich form

Define the Kontsevich 2-form ω_K on each cell of $\mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$ by

$$\omega_K := \sum_{i=1}^{n} \frac{L_i^2}{2}\Psi_i, \qquad \Psi_i := \sum_{e_i^{[a]} \prec e_i^{[b]}} \frac{d\ell_{e_i^{[a]}}}{L_i} \wedge \frac{d\ell_{e_i^{[b]}}}{L_i},$$

where $e_i^{[1]}, e_i^{[2]}, \ldots$ are the edges around the ith face of the ribbon graph underlying the cell, and $\prec$ is the order on the edges induced by the orientation of the surface.

$$\Psi_1 = \frac{2}{L^2}(da \wedge db + da \wedge dc + db \wedge dc)$$

$$\omega_K = da \wedge db + da \wedge dc + db \wedge dc$$

$e^{[1]} = a,\ e^{[2]} = b,\ e^{[3]} = c$

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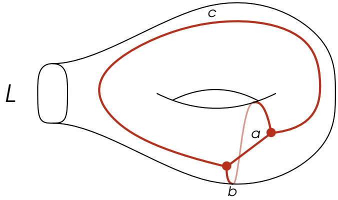

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The symplectic volumes

Theorem (Kontsevich '92, Zvonkine '02)

- The form ω_K on $\mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$ is symplectic and MCG invariant
- The symplectic volume $V_{g,n}(\vec{L})$ of $\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})$ is finite and given by

$$\int_{\mathcal{M}_{g,n}^{\mathrm{comb}}(\vec{L})} \exp(\omega_K) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\Big(\sum_{i=1}^{n} \frac{L_i^2}{2}\psi_i\Big).$$

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Theorem (ABCGLW '20)

For every choice of pants decomposition on Σ, we have a global coordinates $(\ell_i, \tau_i)_{i=1}^{3g-3+n}$ on $\mathcal{T}_{\Sigma}^{\text{comb}}(\vec{L})$. Then

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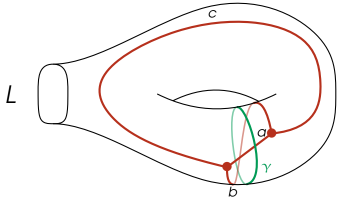

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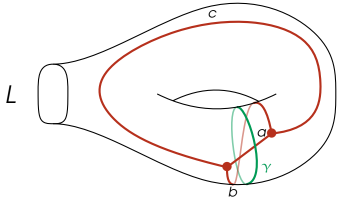

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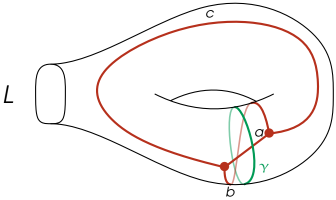

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Consider the following auxiliary functions $\mathcal{D}, \mathcal{R} \colon \mathbb{R}_+^3 \to \mathbb{R}_+$:

$$\mathcal{D}(L, \ell, \ell') \coloneqq [L - \ell - \ell']_+$$
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where $[x]_+ = \max\{x, 0\}$.

Theorem (ABCGLW '20)

For any $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$, we have

$$L_1 = \sum_{i=2}^{n} \sum_{\gamma} \mathcal{R}(L_1, L_i, \ell_{\mathbb{G}}(\gamma)) + \frac{1}{2} \sum_{\gamma,\gamma'} \mathcal{D}(L_1, \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma')).$$

Here, the first sum is over simple closed curves γ bounding a pair of pants with $\partial_1\Sigma$ and $\partial_i\Sigma$, and the second sum is over all pairs of simple closed curves γ, γ' bounding a pair of pants with $\partial_1\Sigma$.

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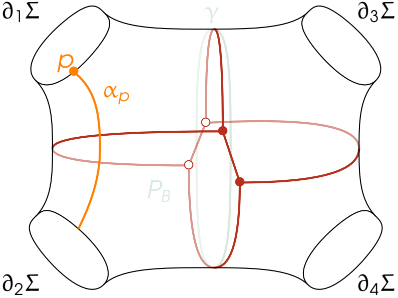
For any $\mathbb{G} \in \mathcal{T}_\Sigma^{\mathrm{comb}}(\vec{L})$, we have

$$1 = \sum_{i=2}^{n} \sum_{\gamma} \frac{\mathcal{R}\big(L_1, L_i, \ell_{\mathbb{G}}(\gamma)\big)}{L_1} + \frac{1}{2} \sum_{\gamma, \gamma'} \frac{\mathcal{D}\big(L_1, \ell_{\mathbb{G}}(\gamma), \ell_{\mathbb{G}}(\gamma')\big)}{L_1}.$$

Here, the first sum is over simple closed curves γ bounding a pair of pants with $\partial_1\Sigma$ and $\partial_i\Sigma$, and the second sum is over all pairs of simple closed curves γ, γ' bounding a pair of pants with $\partial_1\Sigma$.

Idea of the proof

- Fix a random point $p \in \partial_1\Sigma$ and shoot a geodesic arc α_p, i.e. a leaf in the dual foliation
- For a.e. point p, α_p will hit another boundary component
- The arc α_p determines a pair of pants P_B with $\partial P_B = (\partial_1\Sigma, \partial_i\Sigma, \gamma)$, or a pair of pants P_C with $\partial P_C = (\partial_1\Sigma, \gamma, \gamma')$



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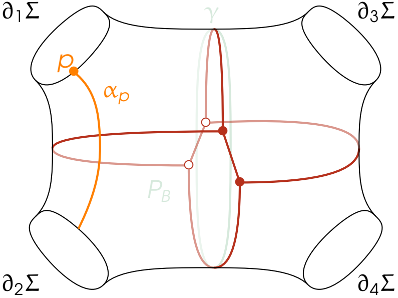

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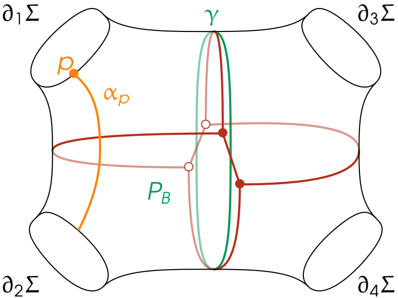

$$1 = \sum_{i=2}^{n} \sum_{P_B} \mathbb{P}\binom{\text{obtain}}{P_B} + \sum_{P_C} \mathbb{P}\binom{\text{obtain}}{P_C}$$

Witten–Kontsevich recursion

combinatorial Wolpert formula

combinatorial Mirzakhani identity

recursion for $V_{g,n}(\vec{L})$

The Kontsevich volumes are computed recursively by

$$
\begin{aligned}
V_{g,n}(L_1,\ldots,L_n) &= \sum_{i=2}^{n} \int_{\mathbb{R}_+} d\ell\, \ell\, \frac{\mathcal{R}(L_1,L_i,\ell)}{L_1}\, V_{g,n-1}(\ell, L_2,\ldots,\widehat{L_i},\ldots,L_n) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}_+^2} d\ell\, d\ell'\, \ell\ell'\, \frac{\mathcal{D}(L_1,\ell,\ell')}{L_1} \bigg(V_{g-1,n+1}(\ell,\ell',L_2,\ldots,L_n) \\
&\qquad + \sum_{\substack{h+h'=g \\ J\sqcup J'=\{L_2,\ldots,L_n\}}} V_{h,1+|J|}(\ell,J)\, V_{h',1+|J'|}(\ell',J') \bigg).
\end{aligned}
$$

with initial conditions $V_{0,3}(L_1,L_2,L_3) = 1$ and $V_{1,1}(L) = \frac{L^2}{48}$.

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combinatorial Wolpert formula + combinatorial Mirzakhani identity ⇒ recursion for $V_{g,n}(\vec{L})$

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Integral structure

Definition

A metric ribbon graph G is called integral if the length of every edge is a positive integer.

$$\mathbb{Z}\mathcal{M}_{g,n}^{\text{comb}}(\vec{L}) := \left\{ \begin{array}{c} \text{integral MRGs} \\ \text{type } (g,n) \text{ and boundary } \vec{L} \end{array} \right\} \subset \mathcal{M}_{g,n}^{\text{comb}}(\vec{L}).$$

We can count integral points as

$$N_{g,n}(\vec{L}) := \sum_{G \in \mathbb{Z}\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})} \frac{1}{\text{Aut}(G)}.$$

Idea. $N_{g,n}(\vec{L})$ is the volume of the combinatorial moduli space w.r.t the "counting measure", that is Dirac deltas at the integral points.

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Norbury recursion from Mirzakhani

integral structure $\mathbb{Z}\mathcal{M}_{g,n}^{\mathrm{comb}}$

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recursion for $N_{g,n}(\vec{L})$

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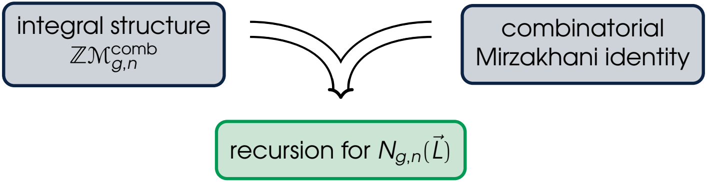

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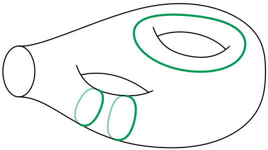
Multicurve count

Define $\mathcal{N}_\Sigma \colon \mathcal{T}_\Sigma^{\text{comb}} \times \mathbb{R}_+ \to \mathbb{N}$ the counting function,

$$\mathcal{N}_\Sigma(\mathbb{G}; t) \coloneqq \# \left\{ \gamma \;\middle|\; \substack{\text{multicurve in } \Sigma \\ \text{with } \ell_{\mathbb{G}}(\gamma) \leqslant t} \right\}.$$



Theorem (ABCGLW '20)

- The counting function $\mathcal{N}_\Sigma(\mathbb{G}; t)$ is computed by a Mirzakhani-type recursion (geometric recursion).
- It is MCG invariant, and its mean value

$$\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t) \coloneqq \int_{\mathcal{M}_{g,n}^{\text{comb}}(\vec{L})} \mathcal{N}_{g,n}(G; t) \, \frac{\omega_K^{3g-3+n}}{(3g-3+n)!}$$

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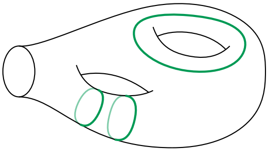

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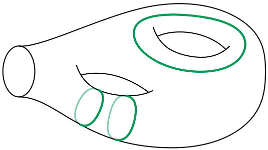

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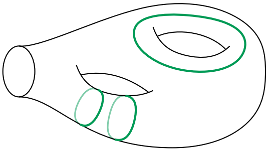

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To conclude we obtained:

- global length/twist coord's on $\mathcal{T}_{\Sigma}^{\text{comb}}(\vec{L})$
- a combinatorial Wolpert formula for ω_K
- a Mirzakhani identity, from which we gave a geometric proof of:
  - Witten–Kontsevich recursion for symplectic volumes/ψ-intersections
  - Norbury's recursion for lattice pnts
- a recursion for the multicurve counting and Masur–Veech volumes

∗ a PL manifold structure on $\mathcal{T}_{\Sigma}^{\text{comb}}(\vec{L})$

∗ a rescaling flow $\sigma^{\beta} : \mathcal{T}_{\Sigma}^{\text{hyp}}(\vec{L}) \to \mathcal{T}_{\Sigma}^{\text{hyp}}(\vec{L})$ that limits to $\mathcal{T}_{\Sigma}^{\text{comb}}(\vec{L})$

Thank you!

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2. J.E. Andersen, G. Borot, N. Orantin. "Geometric recursion". (2017) arXiv:1711.04729 [math.GT].

3. M. Kontsevich "Intersection theory on the moduli space of curves and the matrix Airy function". *Commun. Math. Phys.* 147 (1992).

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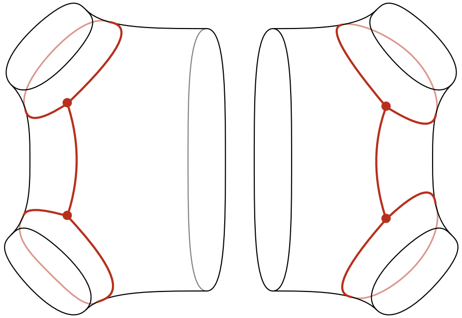
6. E. Witten "Two-dimensional gravity and intersection theory on moduli space". *Surv. Diff. Geom.* 1.1 (1990).

Non-admissible gluing



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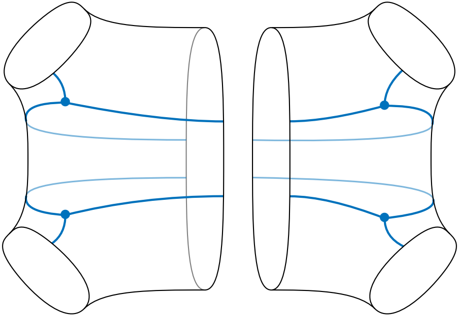

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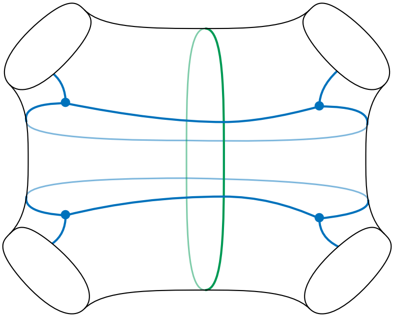

Geometric kernels

Lemma

For a fixed pair of pants P, identify $\mathbb{R}^3_+ \cong \mathcal{T}_P^{\mathrm{comb}}$.

- The function
$$\mathcal{D}(L, \ell, \ell') := [L - \ell - \ell']_+$$
associates to a point $(L, \ell, \ell') \in \mathcal{T}_P^{\mathrm{comb}}$ the fraction of $\partial_1 P$ that is not common with $\partial_2 P \cup \partial_3 P$ (once retracted to the graph).
- The function
$$\mathcal{R}(L, L', \ell) := \frac{1}{2}\Big([L - L' - \ell]_+ - [-L + L - \ell]_+ + [L + L' - \ell]_+\Big)$$
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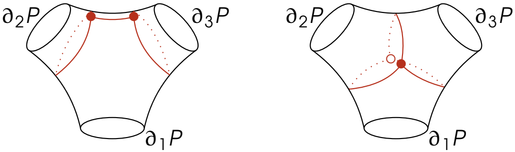

Spectral curves

- Symplectic volumes $V_{g,n}(\vec{L})$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = \frac{z^2}{2} \qquad y(z) = z \qquad B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- Lattice point count $N_{g,n}(\vec{L})$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = z + \frac{1}{z} \qquad y(z) = z \qquad B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- Average number of multicurves $\langle \mathcal{N}_{g,n} \rangle (\vec{L}; t)$ of length $\leqslant t$:

$$\mathcal{C} = \mathbb{C} \qquad x(z) = \frac{z^2}{2} \qquad y(z) = z$$

$$B(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \frac{(s\pi)^2}{\sin^2\big(s\pi(z_1 - z_2)\big)} \right) \frac{dz_1\, dz_2}{2}$$