THE NEGATIVE SIDE OF WITTEN'S CONJECTURE

j.w. in progress w/
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<u>Overview</u>. 1) Motivation: WK theorem
2) CohFTs and Θ-class
3) Integrability for Θ
4) Higher spin

§1) Motivation

$u(\hbar; t_0, t_1, ..)$ is a solution of KdV hierarchy if

$u(\hbar, t_0, 0, \ldots) = f(\hbar, t_0)$ initial cond.

$$\left.\begin{aligned} u_{t_1} &= u\, u_{t_0} + \frac{\hbar}{12} u_{t_0 t_0 t_0} \\ u_{t_2} &= \cdots \\ &\vdots \end{aligned}\right\} \text{ evolution eqns}$$

<u>Thm</u> (Witten-Kontsevich)

$$F^{WK}(\hbar, \underline{t}) = \sum_{g,n,k} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} 1_{g,n} \prod_{i=1}^{n} \psi_i^{k_i} t_{k_i}$$

$$\frac{1}{3!} \left(\int_{\overline{\mathcal{M}}_{0,3}} 1 \right) t_0^3 = \frac{t_0^3}{3!}$$

Then $u = \hbar F^{WK}_{t_0 t_0}$ is a solution of KdV with $u(\hbar, t_0, 0, ..) = t_0$

<u>E.g.</u> $\int_{\overline{\mathcal{M}}_{3,2}} \psi_1^8 = \frac{1}{93312}$ (1.6 secs)

$$F^{BGW}(\hbar,\underline{t}) = \frac{1}{8}\log(1-t_0) + O(\hbar)$$ (Brézin-Gross-Witten)

$u = F^{BGW}_{t_0t_0}$ is a sol of KdV with $u(\hbar,t_0,0,\ldots) = \dfrac{\hbar}{8(1-t_0)^2}$.

$\int_{\overline{\mathcal{M}}_{1,n}} \Theta_{1,n} = \frac{(n-1)!}{8}$

Thm (Conj by Norbury '17, CGG)

$$F^{BGW}(\hbar,\underline{t}) = \sum_{g,n,\underline{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{k_i} t_{k_i}$$

BGW appears:
- lattice YM theory
- as building block in MM
- volumes of moduli space of super RS
- super JT gravity
- phase transition to monotone Hurwitz #s

E.g. $\int_{\overline{\mathcal{M}}_{3,2}} \Theta_{g,n} \psi_1^2 = \frac{75}{1024}$ (0.01 secs)

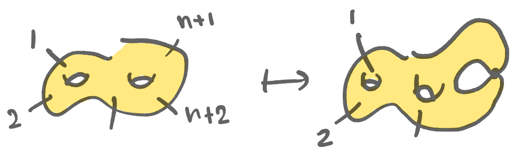
§2. CohFTs & $\Theta_{g,n}$

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C,p_1,\ldots,p_n) \;\middle|\; \begin{array}{l} C \text{ genus } g \text{ cmct stable curve} \\ p_1,\ldots,p_n \text{ smooth labled pnt} \end{array} \right\} \Big/ \sim$$

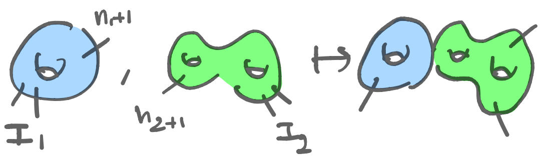
$\dim_{\mathbb{C}} = 3g-3+n$

- Attaching maps:

$q: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$



$r: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$



$g_1+g_2=g$, $I_1 \sqcup I_2 = \{1,\ldots,n\}$

$|I_1| = n_1$, $|I_2| = n_2$

- Forgetful maps:

$p_m: \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$

- $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$, $\quad \mathcal{L}_i|_{(C,p_1,\ldots,p_n)} = T^*_{p_i}C$

$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$, $\quad \kappa_m = p_* \psi_{n+1}^{m+1} \in H^{2m}(\overline{\mathcal{M}}_{g,n})$

Remrk: $1_{g,n} \in H^\circ(\overline{\mathcal{M}}_{g,n})$ is compatible with attaching / forgetful maps:

$$q^* 1_{g,n} = 1_{g-1,n+2}$$

$\vdots$

Defn (Kontsevich-Manin). A cohomological field theory is a triple

(V, η, Ω),

vector space / $\mathbb{Q}$

dim $< \infty$

$\eta : V \times V \to \mathbb{Q}$

non-deg. pairing

$\Delta \in V^{\otimes 2}$

dual bivector

$\Omega_{g,n} : V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$ linear, s.t.

- S_n-equiv.
- $q^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \Omega_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes \Delta)$
- $r^* \Omega_{g,n}(\; - \;) =$

$$= (\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+2})(v_{I_1} \otimes \Delta \otimes v_{I_2})$$

Ex. • $V = \mathbb{Q}v \quad \eta(v,v) = 1$

$$\Omega^{triv}_{g,n}(v^{\otimes n}) = 1_{g,n}$$

$$\Omega^{WP}_{g,n}(v^{\otimes n}) = \exp(2\pi^2 \kappa_1)$$

$$\Omega^{Hodge}_{g,n}(v^{\otimes n}) = c(\mathbb{E}_{g,n})$$

$\mathbb{E}_{g,n}|_{C,p_1,\ldots,p_n} = H^0(C,\omega)$, $\mathrm{rk} = g$

• GW classes

Q: how to compute Ω?

$(V,\eta,\Omega) \Rightarrow$ product on V $\quad \eta(v_1\cdot v_2, v_3) = \Omega_{0,3}(v_1\otimes v_2\otimes v_3)$

Thm (Teleman, Givental). If $(V,\cdot)$ is semisimple, then

$$\Omega = \begin{cases} \text{explicit expression} \\ \text{involving } \psi, \kappa, \text{ boundary divisors} \end{cases}$$

E.g. $\quad \Omega^{\text{Hodge}}_{g,n}(1^{\otimes n}) = \exp\left(\sum_{m\geq 1} \frac{B_{m+1}}{m(m+1)}\left(\kappa_m - \sum_{i=1}^{n}\psi_i^m + \delta_m\right)\right)$

$\Rightarrow$ terms in $\deg_{\mathbb{C}} > g$ vanish in cohom

(Notation: if $V = \mathbb{Q}v$, $\Omega_{g,n}(v^{\otimes n}) =: \Omega_{g,n}$

e.g. $\kappa_1 - \sum_{i=1}^{n}\psi_i + \delta_1 = 0$ in $H^2(\overline{\mathcal{M}}_{0,n})$

Q: other examples?

Fix g,n with $2g-2+n>0$, $a=(a_1,\dots,a_n)\in\{0,1\}^n$

$\overline{\mathcal{M}}^{\text{spin}}_{g;a} =$ moduli space of twisted spin curves

$$= \left\{ (C,p_1,\dots,p_n,L) \;\middle|\; \underbrace{L^{\otimes -2}} \cong \underbrace{\omega\left(\textstyle\sum_i (a_i+1)p_i\right)}_{\text{canonical}} \right\}/\sim$$

$2\mathbb{Z} \ni -2\deg L = 2g-2+n+|a| > 0$

- Forgetful map: $\overline{\mathcal{M}}^{\text{spin}}_{g;a} \xrightarrow{f} \overline{\mathcal{M}}_{g,n}$
- Universal root + universal curve: $\mathcal{L}_{g;a} \to \mathcal{C}_{g;a} \xrightarrow{\pi} \overline{\mathcal{M}}^{\text{spin}}_{g;a}$

Consider $\mathbb{V}_{g;a} = -R^\bullet\pi_*\mathcal{L}_{g;a}$ has fibers $\mathbb{V}_{g;a}|_{C,p_1,\dots,p_n,L} = H^1(C,L) - \cancel{H^0(C,L)}^{\,0}$

$\Rightarrow \mathbb{V}_{g;a}$ is a VB over $\overline{\mathcal{M}}_{g;a}$ of $\text{rk} = \dim H^1(C,L) \overset{RR}{=} -\deg(L)+g-1$

$= 2g-2+\frac{n+|a|}{2}$

Defn/Prop (Norbury). $V := \mathbb{Q}v$, $\eta(v,v) = 1$

$$\Theta_{g,n} = (-1)^n 2^{g-1+n} f_* c_{top}(\mathbb{V}_{g;\underbrace{(1,\dots,1)}_{n\text{-times}}}) \in H^{2(2g-2+n)}(\overline{\mathcal{M}}_{g,n})$$

$a = (\underbrace{1,\dots,1}_{n\text{-times}})$

Then (V, η, Θ) is a CohFT.

Rmk. $\Theta_{0,n} \in H^{2(n-2)}(\underbrace{\overline{\mathcal{M}}_{0,n}}_{\dim_{\mathbb{C}} = n-3 < n-2}) \Rightarrow \Theta_{0,n} = 0$

$\Rightarrow v_1 \bullet v_1 = 0$

$\Rightarrow (V, \bullet)$ is <u>not</u> semisimple

Defn/Thm (CGG). Deform the class: $\forall \varepsilon \in \mathbb{C}$

$$\Theta^{\varepsilon}_{g,n} := (-1)^n 2^{g-1+n} \sum_{m \geq 0} \frac{\varepsilon^m}{m!} p_{m,*} f_* c_{top}(\mathbb{V}_{g;1^n,0^m})$$

$p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$

Then Θ^{ε} is well-defined, $(V, \eta, \Theta^{\varepsilon})$ is a semisimple CohFT $\forall \varepsilon \neq 0$ and

$$\Theta^{\varepsilon}_{g,n} = \Theta_{g,n} + \underbrace{\varepsilon \cdot H^{<2(2g-2+n)}(\overline{\mathcal{M}}_{g,n})[\varepsilon]}_{\text{deformation}}$$

$\Longrightarrow$ Can apply Givental–Teleman

Corollary.

$$\Theta^{\varepsilon}_{g,n} = (-\varepsilon^2)^{2g-2+n} \exp\Big(\sum_{m>0} s_m (-\varepsilon^2)^m \kappa_m\Big)$$

$s_m \in \mathbb{Q}$, $\sum_{m>0} s_m x^m = -\log\Big(\sum_{k\geq 0} (-1)^k (2k+1)!!\, x^k\Big)$

Corollary.

1) $\Theta_{g,n}$ is tautological

2) $\Theta_{g,n} = [\deg_c = 2g-2+n] \exp\left(\sum_{m>0} s_m \kappa_m\right)$

3) $\underbrace{[\deg = d] \cdot \exp\left(\sum_{m>0} s_m \kappa_m\right) = 0}_{\text{tautological relations}} \quad \forall d > 2g-2+n$

Conj by Kazarian Norbury

§3) Integrability

Thm (Dunin-Barkowski-Orantin-Shadrin-Spitz). If (V, η, Ω) is a semisimple CohFT, then

$$\int_{\overline{M}_{g,n}} \Omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{i=1}^{n} \psi_i^{k_i}$$

are computed recursively by TR.

Thm (CGG). The intersection numbers $\int_{\overline{M}_{g,n}} \Theta_{g,n}^{\varepsilon} \prod_{i=1}^{n} \psi_i^{k_i}$ are computed by TR on $\left(\mathbb{P}^1, \ x(z) = \frac{z^2}{2} - \epsilon z, \ y(z) = \frac{1}{z}, \ \omega_{0,2} = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}\right)$

Consider the Virasoro reprs $m \geq -1$

$$L_m = \frac{\hbar}{8} \delta_{m,0} + \delta_{m,-1} \frac{t_0^2}{2} + \sum_{\substack{k,j \geq 0 \\ k-j=m}} \frac{(2k+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_k}$$

$$+ \frac{\hbar}{2} \sum_{\substack{i,j \geq 0 \\ i+j=m-1}} (2i+1)!!\,(2j+1)!! \frac{\partial^2}{\partial t_i \partial t_j}$$

Then $Z^{\Theta^\varepsilon}(\hbar,\underline{t}) = \exp\left(\sum \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^\varepsilon_{g,n} \prod_{i=1}^n \psi_i^{k_i} t_i^{k_i}\right)$

$$\left((2m+1)!! \frac{\partial}{\partial t_m} - \varepsilon L_{m-1} - L_m\right) Z^{\Theta^\varepsilon} = 0 \qquad \forall m \geq 0$$

<u>Corollary</u>. Z^Θ is the unique solution to

$$\left((2m+1)!! \frac{\partial}{\partial t_m} - L_m\right) Z^\Theta = 0 \qquad \forall m \geq 0. \qquad (*)$$

<u>Thm</u> (Mironov-Morozov-Semenoff, Alexandrov) $Z^{BGW} = \exp(F^{BGW})$ defined from the BGW matrix integral:

- is a KdV τ-fnct
- is the unique solution to $(*)$

<u>Corollary</u>. $Z^\Theta = Z^{BGW}$, i.e. Z^Θ defines a solution of the KdV hierarchy with initial condition $\frac{\hbar}{8(1-t_0)^2}$.

§4) Higher spin

<u>Thm</u> (CGG). Starting from the moduli space of higher twisted spin curves:

- define $\Theta^r_{g,n} : V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n})$ an $(r-1)$-dim CohFT of pure deg (NOT semisimple !!)

- define a deformation $\Theta^{r,\varepsilon}_{g,n}$, semisimple $\forall \varepsilon \neq 0$, and express it via Givental-Teleman obtaining taut. rels
- prove TR $\Leftrightarrow$ W-algebra cnstnts for Z^{Θ^r}

Missing: the rBGW model is a solution of the same (!) W-algebra constraints.

$$F^{\varepsilon}(\hbar;\underline{t}) = \sum \frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \Theta^{\varepsilon}_{g,n} \prod \psi_i^{k_i}\, t_{k_i}$$

$$f^{\varepsilon}(\hbar,t_0) = \hbar\, F^{\varepsilon}_{t_0t_0}(\hbar;t_0,t_1=0,t_2=0,\dots)$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta^{\varepsilon}_{g,n}$$

$-\varepsilon^2\frac{(n-1)!}{2}$

	$\dim = 3g-3+n$	$\dim \le 2g-2+n$
$g=0$	$n-3$	$n-2$
$g=1$	n	n
$\cancel{g=2}$	$\cancel{n+3}$	$\cancel{n+2}$

$$\int_{\overline{\mathcal{M}}_{1,n}} \Theta^{\varepsilon}_{1,n} = \frac{(n-1)!}{8}$$

$\boxed{g=0}$ $$-\varepsilon^2\sum_n \frac{1}{n!}\frac{(n-1)!}{2}t_0^n = +\frac{\varepsilon^2}{2}\log(1-t_0)$$

$\boxed{g=1}$ $$\hbar\sum_n \frac{1}{n!}\cdot\frac{(n-1)!}{8}t_0^n = \frac{\hbar}{8}\sum_{n\ge 1}\frac{t_0^n}{n} = -\frac{\hbar}{8}\log(1-t_0)$$

$\Rightarrow$ $u^{\Theta^{\varepsilon}}$ is a solution of KdV w/ i.c. $f(\hbar,t_0) = \left(\frac{\hbar}{8}-\frac{\varepsilon^2}{2}\right)\frac{1}{(1-t_0)^2}$