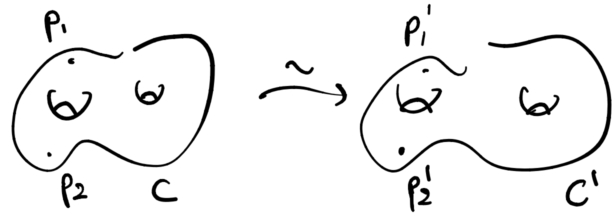
A 1-$\int$ Proof of the HZ Formula

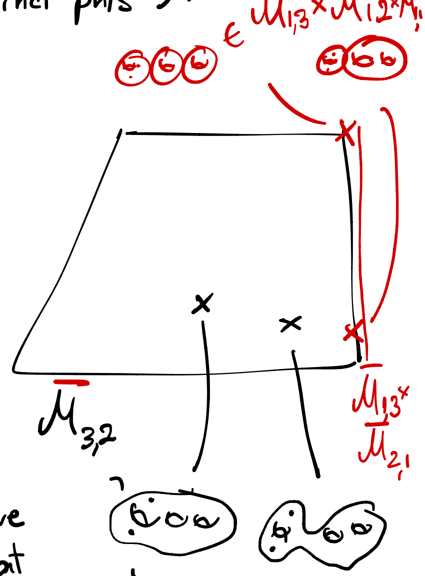
§1. Moduli space of curves

w/ D. Lewanski + P. Norbury

Fix $g, n \geq 0$ integers $(2g-2+n>0)$

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, \dots, p_n) \;\middle|\; \begin{array}{l} C \text{ smooth, cmct, cmlx} \\ \text{alg curve of genus } g \\ p_1, \dots, p_n \in C \text{ distinct pnts} \end{array} \right\} \Big/ \text{iso}$$





Fact: $\mathcal{M}_{g,n}$ is a smooth, connected orbifold of $\dim_{\mathbb{C}} = 3g-3+n$, not cmct.

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \dots, p_n) \;\middle|\; \begin{array}{l} C \text{ cmct, cmlx alg curve} \\ \text{arithm. genus } g \text{ with at} \\ \text{worst nodal sing, } p_1, \dots, p_n \in C \\ \text{distinct smooth pnts,} \\ |\mathrm{Aut}(C, p_1, \dots, p_n)| < +\infty \end{array} \right\} \Big/ \text{iso}$$

Fact: $\overline{\mathcal{M}}_{g,n}$ is a smooth, connected, cmct space of $\dim_{\mathbb{C}} = 3g-3+n$ and the "boundary" $\partial \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a nice $\mathrm{codim}_{\mathbb{C}} = 1$ closed subset (NCD) parametrised by $\overline{\mathcal{M}}_{g_i,n_i}$ of smaller dim.

$\Rightarrow$ 1) $\overline{\mathcal{M}}_{g,n}$ has a fundamental class:

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q} \qquad \forall \alpha \in H^{2(3g-3+n)}(\overline{\mathcal{M}}_{g,n})$$

2) PD holds

Natural classes.

- $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$, $i = 1, \dots, n$. $\quad \mathbb{L}_i \big|_{(C, p_1, \dots, p_n)} = T^*_{p_i} C$

  $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ $\quad$ ψ-classes

- $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, $\quad (C, p_1, \dots, p_n, p_{n+1}) \mapsto (C, p_1, \dots, p_n)$ forgetful map

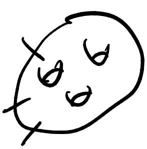
  $\kappa_a = p_*((\psi_{n+1})^{a+1}) \in H^{2a}(\overline{\mathcal{M}}_{g,n})$ $\quad$ κ-classes $\quad$ e.g. $\kappa_0 = 2g-2+n$

- $\underset{\substack{\uparrow \\ \mathrm{rk}_{\mathbb{C}} = g}}{\mathbb{H}} \to \overline{\mathcal{M}}_{g,n}$, $\quad \mathbb{H} \big|_{(C, p_1, \dots, p_n)} = \underset{\substack{\uparrow \\ \dim_{\mathbb{C}} = g}}{H^0(C, \omega_C)}$ = "$\mathbb{C}$-space of holom differentials"

  $\lambda_k = c_k(\mathbb{H}) \in H^{2k}(\overline{\mathcal{M}}_{g,n})$

  $\Lambda(t) = \sum_{k=0}^{g} t^k \lambda_k \in H^{2*}(\overline{\mathcal{M}}_{g,n})$ $\quad$ Hodge class

Heuristic: every curve-counting problem can be expressed as int. number on $\overline{\mathcal{M}}_{g,n}$:

$\#\Big\{$  $\Big| \; P$ holds $\Big\}$ $= \int_{\overline{\mathcal{M}}_{g,n}} \alpha^P$

$\rightsquigarrow$ motivation for studying $H^\bullet(\overline{\mathcal{M}}_{g,n})$

§2. Euler characteristic

$$\chi(\overline{\mathcal{M}}_{g,n}) = \sum_{\substack{\text{strata}\\ \Gamma}} \frac{\chi(\mathcal{M}_\Gamma)}{|\mathrm{Aut}(\Gamma)|}$$

<u>Question</u>. Compute $\chi_{g,n} = \chi_{orb}(\mathcal{M}_{g,n}) \in \mathbb{Q}$

<u>Thm</u> (Harer–Zagier '86)

$$\chi_{g,n} = \begin{cases} (-1)^n (n-3)! & g=0,\ n\geq 3 \\ (-1)^n \dfrac{(n-1)!}{12} & g=1,\ n\geq 1 \\ (-1)^n (2g-3+n)! \dfrac{B_{2g}}{2g(2g-2)!} & g\geq 2,\ n\geq 0 \end{cases}$$

B_m = mth Bernoulli number. $\quad \dfrac{t}{1-e^{-t}} = \sum_{m\geq 0} \dfrac{B_m}{m!} t^m$

<u>New proof</u>: Gauss-Bonnet formula + calculus on $H^*(\overline{\mathcal{M}}_{g,n})$.

<u>Prop</u> (GB for open orbifolds). $\overline{M}$ = smooth connected cmct orbifold, $\dim_{\mathbb{C}} = k$.
$D \subseteq \overline{M}$ nice (NCD), $M = \overline{M} - D$.

$$\chi_{orb}(M) = \int_{\overline{M}} c_{top}(\log T_{\overline{M},D})$$

log tangent bundle $\rightsquigarrow$ locally $D \cap U = \{z_1 = \dots = z_m = 0\}$

$\mathrm{rk}_{\mathbb{C}} = k$ vector bundle $\quad \log T^*_{\overline{M},D}(U) =$

$$= \left\langle \frac{dz_1}{z_1}, \dots, \frac{dz_m}{z_m}, dz_{m+1}, \dots dz_k \right\rangle$$

<u>Our case</u>: $\overline{M} = \overline{\mathcal{M}}_{g,n}$, $D = \partial\overline{\mathcal{M}}_{g,n}$ $\Rightarrow$ $M = \mathcal{M}_{g,n}$

Fact: $\log T^*_{g,n} = \log T^*_{\overline{\mathcal{M}}_{g,n},\partial\overline{\mathcal{M}}_{g,n}}$ = moduli space of quadratic diff with simple poles at marked pnts

$$\log T^*_{\overline{\mathcal{M}}_{g,n},\partial\overline{\mathcal{M}}_{g,n}} \Big|_{(C,p_1,\dots,p_n)} = H^0(C, \omega_C^{\otimes 2}(\textstyle\sum_i p_i))$$

quadratic Hodge bundle

Thm (Mumford, Bini, Chiodo)

$B_{m+1}(x)$ = Bernoulli poly

$$c(\log T_{g,n}) = \Lambda(-1)\cdot \exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$$

$$= \exp\left(\sum_{m\geq 1}(-1)^m\left(\frac{B_{m+1}(-1)}{m(m+1)}\kappa_m - \frac{B_{m+1}(0)}{m(m+1)}\sum_{i=1}^{n}\psi_i^m + \frac{B_{m+1}(0)}{m(m+1)}\,\frac{1}{2}\,\xi_*\frac{(\psi')^m-(-\psi'')^m}{\psi'-\psi''}\right)\right)$$

Mumford's formula

$+$

$B_{m+1}(0) = B_{m+1}$

$B_{m+1}(-1) = B_{m+1} - (-1)^m(m+1)$

Corollary.

$$\chi_{g,n} = \int_{\overline{M}_{g,n}} \Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$$

Prop. The HZ formula holds true.

$$\exp\left(-\sum\frac{u^m}{m}\right) = \exp(\log(1-u)) = 1-u$$

Proof. Claim 1): $\chi_{g,n+1} = -(2g-2+n)\chi_{g,n}$

Recall: $\overline{M}_{g,n+1} \xrightarrow{p} \overline{M}_{g,n}$ forgetful map

$\Lambda(-1) = p^*\Lambda(-1)$

$\kappa_m = p^*\kappa_m + \psi_{n+1}^m$

$$\chi_{g,n+1} = \int_{\overline{M}_{g,n+1}} \Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$$

$$= \int_{\overline{M}_{g,n+1}} \underbrace{p^*\left(\Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)\right)}_{\in H^{\leq 2(3g-3+n)}(\overline{M}_{g,n+1})}\underbrace{\exp\left(-\sum_{m\geq 1}\frac{\psi_{n+1}^m}{m}\right)}_{\cancel{1}-\psi_{n+1}}$$

$$= -\int_{\overline{M}_{g,n+1}} p^*\left(\Lambda(-1)\exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)\right)\cdot\psi_{n+1}$$

proj form. $\to \overset{!}{=} -\int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) \exp\left(-\sum_{m\geq 1} \frac{\kappa_m}{m}\right) \underbrace{p_* \psi_{n+1}}_{= (2g-2+n)}$

$$= -(2g-2+n)\cdot \chi_{g,n}$$

Claim 2). Base cases in g.

$\boxed{g=0}$ $(n=3)$. $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1 \quad \Rightarrow \quad \chi_{0,n} = (-1)^n (n-3)!$

dim = 0

$\boxed{g=1}$ $(n=1)$ $\int_{\overline{\mathcal{M}}_{1,1}} (1-\lambda_1)(1-\kappa_1) = -\int_{\overline{\mathcal{M}}_{1,1}} (\lambda_1 + \kappa_1) = -\frac{1}{12}$

dim = 1

$$\Rightarrow \quad \chi_{1,n} = (-1)^n \frac{(n-1)!}{12}$$

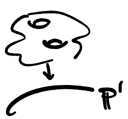
$\boxed{g\geq 2}$ $(n=0)$ $\int_{\overline{\mathcal{M}}_g} \Lambda(-1) \exp\left(-\sum_{m\geq 1} \frac{\kappa_m}{m}\right) =$

$$= \sum_{\ell\geq 0} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell \geq 1} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda(-1) \prod_{i=1}^{\ell} \psi_i^{\mu_i+1} = \frac{B_{2g}}{2g(2g-2)!}$$

Dubrovin-Yang-Zagier

Intermezzo:

simple Hurwitz numbers $\xleftrightarrow{\text{ELSV}}$ linear Hodge integrals

$\#\Big\{$  $\Big\}$

$$\int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-1) \prod_{i=1}^{n} \psi_i^{k_i}$$

Okounkov Pandharipande

integrable hierarchies

e.g. functional eqn for the gen. series of Hurwitz numbers / Hodge integrals (Toda eqn)

$$\mathcal{H}_g(z) = \sum_{d \geq 1} h_{g,d}\, z^d \qquad\qquad z = Te^{-T}$$

$$\mathcal{H}_g \underset{T \to \infty}{\sim} \begin{cases} \text{expr in linear Hodge from ELSV} \\ \text{expr with } B_{2g} \text{ from Toda} \end{cases}$$