NEGATIVE OVER POSITIVE

THE COHOMOLOGY CLASS

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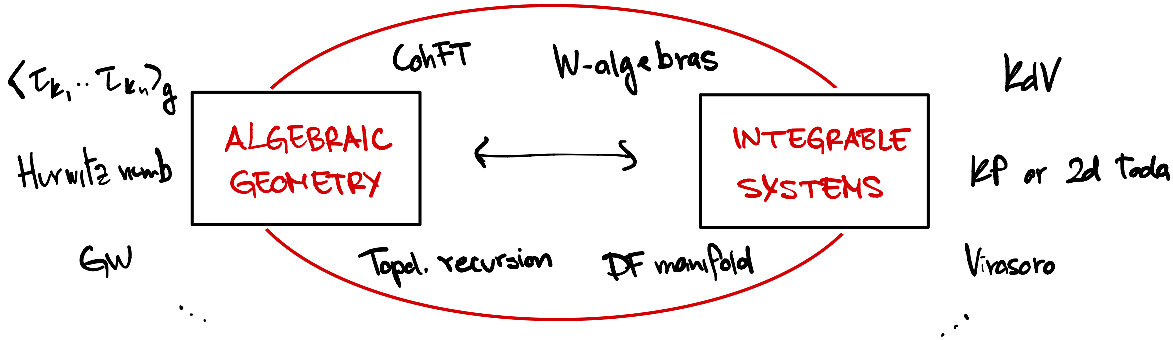
§1) MOTIVATION

Starting with Witten's approach to 2d quantum gravity in the early '90s



Thm (Witten, Kontsevich, Polishchuk-Vaintrob, Chiodo, Faber-Shadrin-Zvonkine, Givental, Adler-van Moerbeke)

① [CLASS]. Let $r \geq 2$, $a_1, \dots, a_n \in \{0, \dots, r-2\}$. Then $\exists$

$$W^r_{g,n}(a_1, \dots, a_n) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

a cohomology class (Witten r-spin class) satisfying certain axioms.

② [W-CNSTRNTS]. The descendant potential

$$Z^{W^r} = \exp\left[\sum_{g,n} \frac{\hbar^{g-1}}{n!} \sum_{\substack{k_1, \dots, k_n \geq 0 \\ 0 \leq a_1, \dots, a_n \leq r-1}} \int_{\overline{\mathcal{M}}_{g,n}} W^r_{g,n}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{k_i} \, t_{k_i, a_i} \right]$$

is the unique solution to certain W-constraints

$$W^i_k \, Z^{W^r} = 0 \qquad i = 1, \dots, r, \quad k \geq -1$$

③ [r-KdV] Z^{W^r} is the unique r-KdV τ-fnct satisfying a certain string eqn.

④ [MATRIX MODEL]. Kontsevich's r-Airy matrix integral

$$Z^{W^r} = \frac{1}{C_N} \int_{\mathcal{H}_N} e^{-\frac{1}{\hbar} \mathrm{Tr}\left(\frac{M^{r+1}}{r+1} - \Lambda M \right)} dM$$

GOAL: give a negative spin analogue of the above result.

For $r=2$, the class was constucted by P. Norbury, who conj Virasoro, KdV and connection to the Brézin-Gross-Witten model.

Thm (Chidambaram - Garcia-Failde - A.G.)

① [CLASS]. Let $r \geq 2$, $a_1, \dots, a_n \in \{1, \dots, r-1\}$. Then $\exists$

$$\Theta^r_{g,n}(a_1, \dots, a_n) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

② [W-CNSTRNTS]. The descendant potential Z^{Θ^r} is the unique solution to certain W-constraints

$$H^i_k Z^{\Theta^r} = 0 \qquad i = 1, \dots, r, \quad k \geq 2-i$$

Conj (CGG, proved for $r=2,3$)

③ [r-KdV] Z^{Θ^r} is the unique r-KdV τ-fnct satisfying a certain string eqn.

④ [MATRIX MODEL] BGW r-Bessel matrix integral

$$Z^{\Theta^r} = \frac{1}{C_N}\int_{\mathcal{H}_N} e^{-\frac{1}{\hbar}\mathrm{Tr}\left(\frac{M^{1-r}}{1-r} - \Lambda M + \hbar \log(M)\right)} dM$$

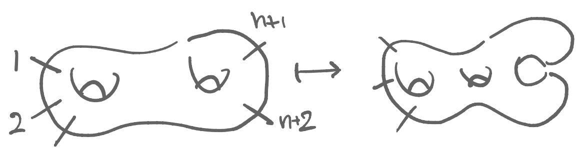
§2) MODULI SPACE OF CURVES & COHFTs

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C,p_1,\dots,p_n) \;\middle|\; \begin{array}{l} C \text{ cmlx, connect, cmct curve} \\ \text{genus } g\text{, with at worst nodal sing.} \\ p_1,\dots,p_n \text{ smooth dist pnts} \\ |\mathrm{Aut}| < \infty \end{array} \right\} \Big/ \sim$$

$\dim_{\mathbb{C}} = 3g-3+n$

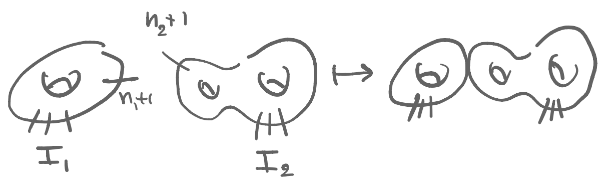
- ATTACHING MAPS:

$q: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$



$r: \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$

$g_1+g_2=g, \quad n_1+n_2=n$



- FORGETFUL MAPS:

$p: \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}, \qquad p_m: \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$

- NATURAL CLASSES: $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}, \qquad \mathbb{L}_i|_{(C,p_1,\dots,p_n)} = T^*_{p_i}C$

$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}), \qquad \kappa_a = p_*(\psi_{n+1})^{a+1} \in H^{2a}(\overline{\mathcal{M}}_{g,n})$

Rmk. The Poincaré dual of the fundamental class $1_{g,n} = [\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n})$ is COMPATIBLE with attaching maps:

$$q^* 1_{g,n} = 1_{g-1,n+2} \qquad\qquad r^* 1_{g,n} = 1_{g_1,n_1+1} \otimes 1_{g_2,n_2+1}$$

Defn (Kontsevich-Manin). A cohomological field theory is a triple

$$\boxed{(V, \eta, \Omega)} \quad \begin{cases} \bullet \ V \text{ is a finite dim } \mathbb{Q}\text{-v.s.} \\ \bullet \ \eta : V \times V \to \mathbb{Q} \text{ is a non-deg, symmetric, bilinear form} \\ \bullet \ \Omega_{g,n} : V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n}) \quad \text{linear} \end{cases}$$

satisfying axioms:

$\eta_{a,b} = \eta(v_a, v_b)$

1) S_n-EQUIVARIANCE

2) GLUING AXIOM:

Δ = dual to η
$= \sum \eta^{a,b} v_a \otimes v_b$

$$\begin{cases} q^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \Omega_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes \Delta) \\ r^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \left(\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}\right)(v_{I_1} \otimes \Delta \otimes v_{I_2}) \end{cases}$$

Example: • $V = \mathbb{Q}\langle v \rangle$, $\quad \eta(v,v) = 1$

• [TRIVIAL] $\qquad \Omega_{g,n}(v^{\otimes n}) = 1_{g,n}$

Hodge bundle

• [HODGE] $\qquad \Omega_{g,n}(v^{\otimes n}) = c(\mathbb{E}_{g,n}), \qquad \mathbb{E}_{g,n} \to \overline{\mathcal{M}}_{g,n} \leftarrow \mathrm{rk}_{\mathbb{C}} = g$

total Chern class

with $\mathbb{E}_{g,n}|_{C,p_1,\dots,p_n} =$ "holomorphic differentials on C"
$= H^0(C, \omega_C)$

• $V = \mathbb{Q}\langle v_0, \dots, v_{r-2} \rangle$, $\quad \eta(v_a, v_b) = \delta_{a+b,r-2}$, $\quad \Omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = W^r_{g,n}(a_1, \dots, a_n)$.

$\boxed{\text{QUESTION: How to compute } \Omega \text{ ?}}$

$(V,\eta,\Omega) \rightsquigarrow (V,\eta,\bullet)$ a Frobenius algebra:

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \in \mathbb{Q} \cong H^0(\overline{\mathcal{M}}_{0,3})$$

Thm (Teleman). If $(V,\eta,\bullet)$ is a SEMISIMPLE algebra, then

$$\Omega = \left\{ \begin{array}{l} \text{explicit combination of} \\ \psi\text{-}, \kappa\text{-}, \text{ and boundary classes} \end{array} \right.$$

$\leftarrow$ determined by
1. the product $\bullet$
2. rotation $R \in \mathrm{End}(V)[\![z]\!]$
3. translation $T \in V[\![z]\!]$

Example. $c(\mathbb{E}_{g,n})$ is semisimple and

$$c(\mathbb{E}_{g,n}) = \exp\left(\sum_{m \geq 1} \frac{B_{m+1}}{m(m+1)} \left(\kappa_m - \sum_{i=1}^{n} \psi_i^m + \delta_m \right) \right) \in H^{\leq 2g}(\overline{\mathcal{M}}_{g,n})$$

(Mumford)

$\Rightarrow$ TAUTOLOGICAL RELATIONS! e.g. $\kappa_1 - \sum_{i=1}^{n} \psi_i + \delta_1 = 0$ in $H^2(\overline{\mathcal{M}}_{0,n})$

§3) Θ^r-classes

For $r \geq 2$, $a_1, \dots, a_n \in \{0, \dots, r-1\}$

$$\overline{\mathcal{M}}^r_{g;a} = \left\{ (C, p_1, \dots, p_n, L) \;\middle|\; \begin{array}{c} (C,p_1,\dots,p_n) \text{ stable, genus } g, \text{ n-pointed curve} \\ L^{\otimes r} \cong \omega_{C,\log}^{-1}\left(-\textstyle\sum_i a_i p_i\right) \end{array} \right\} \Big/ \sim$$

$L \to C$ line bundle

$\omega_{C,\log} = \omega_C\left(\sum_i p_i\right)$

- **DEGREE CONDITION:**

$$r\cdot \deg(L) = -\deg(\omega_{C,\log}) - \sum_i a_i = -\left(2g-2+n+|a|\right) \in r\mathbb{Z}_{<0}$$

① $\deg L < 0$

② $r \mid 2g-2+n+|a|$

- **FORGETFUL MAP:**

$$f : \overline{\mathcal{M}}^r_{g;a} \to \overline{\mathcal{M}}_{g,n} \quad \text{is an } r^{2g}:1 \text{ branched cover}$$

<u>Defn.</u> Define $\mathbb{V}^r_{g;a} \to \overline{\mathcal{M}}^r_{g;a}$,

$$\mathbb{V}^r_{g;a}\big|_{(C,p_1,\dots,p_n,L)} = H^1(C,L)$$

<u>NB</u>: deg. condition $\Rightarrow$ ① $\mathbb{V}^r_{g;a}$ is well-defined

② $\mathrm{rk}_{\mathbb{C}} = h^1(C,L) \overset{RR}{=} -\deg(L)+g-1$

$$\overset{!}{=} \frac{(r+2)(g-1)+n+|a|}{r} = D^r_{g;a}$$

<u>Defn.</u> Define the r-spin Θ-class

$$\Theta^r_{g,n}(a_1,\dots,a_n) = (-1)^n \, r^{\frac{2g-2+n+|a|}{r}} f_* c_{top}(\mathbb{V}^r_{g;a}) \in H^{2D^r_{g;a}}(\overline{\mathcal{M}}_{g,n})$$

NB: $a \neq 0$

<u>Thm.</u> $V = \mathbb{Q}\langle v_1,\dots,v_{r-1}\rangle$, $\eta(v_a,v_b) = \delta_{a+b,r}$,

$$\Omega_{g,n} : V^{\otimes n} \to H^\bullet(\overline{\mathcal{M}}_{g,n}), \qquad \Omega_{g,n}(v_{a_1}\otimes\cdots\otimes v_{a_n}) = \Theta^r_{g,n}(a_1,\dots,a_n)$$

is a CohFT.

NB: $r=2$ gives Norbury's class: $\Theta^{r=2}_{g,n}(1,\dots,1) = \Theta_{g,n} \in H^{2(2g-2+n)}(\overline{\mathcal{M}}_{g,n})$

QUESTION: Can we compute Θ^r via Teleman's thm? I.e., is Θ^r semisimple?

Recall: The product is defined as $\eta(v_a \cdot v_b, v_c) = \Omega_{0,3}(v_a \otimes v_b \otimes v_c)$

However, $\Theta^r_{0,3}(a,b,c) = 0$ for some a,b,c

E.g. with $r=2$, $\Theta^2_{0,3}(1,1,1) = 0$ since $\in H^2(\overline{\mathcal{M}}_{0,3}) = 0$

Rmrk. A CohFT Ω on $V = \mathbb{Q}\langle v_1, \dots, v_r\rangle$ defines a Dubrovin-Frobenius structure with potential

$$F(t_1,\dots,t_r) = \sum_{k_1+\dots+k_r=n} \int_{\overline{\mathcal{M}}_{0,n}} \Omega_{0,n}(v_1^{\otimes k_1} \otimes \cdots \otimes v_r^{\otimes k_r}) \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!}$$

The DF mnfld associated to Θ^r is NOWHERE semisimple.

e.g. $F^{\Theta^2}(t_1) = 0$.

PROBLEM: Cannot apply Teleman's thm to Θ^r

$\downarrow$

GOAL: Deform Θ^r to something semisimple

Defn. $\forall\, \varepsilon \in \mathbb{C}$ a formal parameter, define for $a_i \in \{1,\dots,r-1\}$

$$\Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n) = \sum_{m\geq 0} \frac{\varepsilon^m}{m!}\, p_{m,*}\, \Theta^r_{g,n+m}(a_1,\dots,a_n,\underbrace{0,\dots,0}_{m\text{-times}})$$

$$p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$$

Thm

① [CohFT]. $\Theta^{r,\varepsilon}$ is well-defined (i.e. the sum is finite), and

$$V_{a_1}\otimes\cdots\otimes V_{a_n} \longmapsto \Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n)$$

is a CohFT.

② [DEFORMATION].

$$\Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n) = \Theta^r_{g,n}(a_1,\dots,a_n) + O(\varepsilon) \in H^{<2D^r_{g;a}}(\overline{\mathcal{M}}_{g,n})$$

③ [SEMISIMPLICITY]. $\forall\, \varepsilon \neq 0$, $\Theta^{r,\varepsilon}$ is semisimple

Rmrk. The DF mnfld structure gets deformed to a semisimple one.

E.g.

$$F^{\Theta^{2,\varepsilon}}(t_1) = \frac{\varepsilon^2}{2}\log(1-t_1) \neq 0 \quad \text{for } \varepsilon\neq 0.$$

SOLUTION: Apply Teleman's thm to $\Theta^{r,\varepsilon}$ for $\varepsilon\neq 0$ and take $\varepsilon\to 0$

Thm. From Teleman's thm:

$$\Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n) = \left\{ \begin{array}{l} \text{explicit combination of} \\ \psi\text{-}, \kappa\text{-}, \text{ and boundary classes} \end{array} \right. = \text{RHS}$$

involves coeffnts given by asymt exprs of higher Airy fnctns & Scorer fncts

Moreover:

① $[\deg_{\mathbb{C}} = D^r_{g;a}]\, \text{RHS} = \Theta^r_{g,n}(a_1,\dots,a_n)$

② $[\deg_{\mathbb{C}} = d]\, \text{RHS} = 0 \qquad \forall\, d > D^r_{g;a}$

TAUTOLOGICAL REL.

Examples. For $r=2$: $V = \mathbb{Q}\langle v_1\rangle$, $D^{r=2}_{g;1,\dots,1} = 2g-2+n$ and

① $[\deg_{\mathbb{C}} = 2g-2+n]\, \exp\left(\sum_{m>0} s_m \kappa_m\right) = \Theta_{g,n}$

② $[\deg_{\mathbb{C}} = d]\, \exp\left(\sum_{m>0} s_m \kappa_m\right) = 0 \qquad \forall d > 2g-2+n$

conjectures Kazarian & Norbury

where $\exp\left(-\sum_{m>0} s_m x^m\right) = \sum_{k\geq 0} (-1)^k (2k+1)!!\, x^k$

§4) SUMMARY

We defined

Θ^r = top Chern of some vector bundle

① Forms a CohFT (compatible with attaching maps)

② Not semisimple (cannot apply Teleman's thm)

We defined a deformation

$\Theta^{r,\varepsilon} = \Theta^r +$ corrections in ε of lower cohomological degree

① Forms a CohFT

② Semisimple $\forall \varepsilon \neq 0$ (can apply Teleman)

$\Rightarrow$ Expression for Θ^r & tautological relations

§5) QUESTIONS

① Is $\Theta^r_{g,n}$ trivial in $g=0$ for $r\geq 3$?

For some values of the primary fields $a_i \in \{1,\dots,r-1\}$, yes.

E.g. for $(g,n)=(0,3)$, we have

$$\Theta^r_{0,3}(a,b,c) = \begin{cases} -1 & \text{if } a+b+c = r-1 \\ 0 & \text{else} \end{cases}$$

The vanishing is resolved in the deformed class:

$$\Theta^{r,\varepsilon}_{0,3}(a,b,c) = \begin{cases} -1 & a+b+c = r-1 \\ -\varepsilon & a+b+c = 2r-2 \\ -\varepsilon^2 & a=b=c=r-1 \end{cases} \qquad (*)$$

② What's the product for $r\geq 3$?

From $(*)$ and the fact that $\eta(v_a,v_b) = \delta_{a+b,r}$ we deduce that the product associated to $\Theta^{r,\varepsilon}$ is

$$v_a \bullet v_b = \begin{cases} -v_{a+b+1} & a+b+c = r-1 \\ -\varepsilon\, v_{a+b+2-r} & a+b+c = 2r-2 \\ -\varepsilon^2 v_1 & a=b=c=r-1 \end{cases}$$

Notice that, for $\varepsilon=0$, we have that v_{r-1} is nilpotent. Thus, the algebra at $\varepsilon=0$ is NOT semisimple.

③ What we deduce from Teleman at $\varepsilon=0$?

The general shape of the formula given by Teleman's thm is:

$$\Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n) = \frac{1}{\varepsilon}\cdot\left(\text{class of } \deg_{\mathbb{C}} > D^r_{g;a}\right)$$

$$+ \quad \Theta^{r,\varepsilon}_{g,n}$$

$$+ \quad \varepsilon\cdot\left(\text{class of } \deg_{\mathbb{C}} < D^r_{g;a}\right)$$

Apparently, the orange term would blow up for $\varepsilon \to 0$. However, by construction $\Theta^{r,\varepsilon}_{g,n}(a_1,\dots,a_n)$ has degree $< D^r_{g;a}$. Thus, the class multiplying $\frac{1}{\varepsilon}$ is ZERO due to some tautological relations in $H^\bullet(\overline{\mathcal{M}}_{g,n})$.