Negative over positive II: Integrability

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I Motivation and Recap

$\boxed{\text{ALGEBRAIC GEOMETRY}} \longleftrightarrow \boxed{\text{INTEGRABLE SYSTEMS}}$

positive: Witten r-spin theory $\longleftrightarrow$ r-KdV heirarchy

negative: Θ^r-intersection theory $\longleftrightarrow$ r-KdV heirarchy

Recap: (Alessandro's talk)

① defined a CohFT Θ^r of rank $r-1$,

$$\Theta^r_{g,n}(a_1, \dots, a_n) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

$$1 \leq a_i \leq r-1$$

- not semi-simple

② defined a deformation

$$\Theta^{r,\epsilon} = \Theta^{r} + \varepsilon(\text{ lower degree terms})$$

- is semi-simple for $\varepsilon \neq 0$.

Descendant potential Z^{Θ^r}

$$Z^{\Theta^r} := \exp\left(\sum_{g,n} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \sum_{\substack{k_i \geq 0 \\ 1 \leq a_i \leq r-1}} \Theta^{r}_{g,n}(a_1, \dots, a_n) \prod_i \psi_i^{k_i} t_{rk_i + a_i} \right)$$

Conjecture (C - Garcia-Failde - Giacchetto) / Theorem for $r = 2, 3$

Z^{Θ^r} is the unique r-KdV τ-function satisfying the string equation

Θ^r

$$H^{\hat{s}}_{-s+2} Z^{\ominus} = 0$$

diff op of degree s in the times.

An approach to the conjecture

Step 1: Relate the descendant potential $Z^{\Theta^{g,\epsilon}}$ to topological recursion on a global spectral curve

Step 2: Take $\epsilon \to 0$ to relate $Z^{\Theta^{g}}$ to TR on the limit curve

Step 3: Realize TR as an equivalent set of W-algebra constraints.

Step 4: Show that these W-constraints characterize a r-KdV τ-function.

II Topological recursion (Steps 1 & 2)

TR (Chekov, Eynard, Orantin, Bouchard, ...)

input		output
spectral curve (C, x, y)	$\longrightarrow \boxed{\color{red}{TR}} \longrightarrow$	correlators $\omega_{g,n}$

C: Riemann surface; $\underbrace{x, y}_{\text{meromorphic functions}}$

$x : C \to \mathbb{P}^1$

$\omega_{g,n}$: symmetric meromorphic differentials on $C^{\times n}$

Often, $\omega_{g,n}$ encode interesting enumerative invariants: eg Hurwitz numbers, GW invariants, descendant integrals of CohFTs.

Theorem (CGG)

TR on $\left(\mathbb{P}^1, x = \underline{z^r} - \epsilon z, y = -\frac{1}{z}\right)$

computes descendant integrals of $\Theta^{r,\epsilon}$:

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{a_i=1}^{r-1} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^{r,\epsilon}_{g,n}(a_1,\dots,a_n) \prod_{i=1}^{n} \sum_{k_i \geq 0} \psi_i^{k_i} \, d\xi_i^{k_i,a_i}(z)$$

basis of differentials on $\mathbb{P}^1$

<u>Remarks</u>: ① proof uses thm of Dunin-Barkowski–Orantin–Shadrin–Spitz:

TR $\longleftrightarrow$ CohFTs.

② output of Teleman reconstruction related to higher Airy functions, uses work of Charbonnier

Now, $\epsilon \to 0$

Corollary: TR on $\underbrace{\left(\mathbb{P}^1,\ x = \frac{z^r}{r},\ y = -\frac{1}{z}\right)}_{r\text{-Bessel curve}}$

gives descendant integrals of Θ,

Remark: TR $\Rightarrow$ recursive formula for $\int \Theta^{r}_{g,n} \psi_1^{k_1} \cdots \psi_n^{k_n}$

III W-constraints (step 3)

relation TR $\Leftrightarrow$ W-constraints

(Milanov, Borot-Bouchard-C-Creutzig-Noshchenko)

W-algebra of interest: $\mathcal{W}(\mathfrak{gl}_r)$ with $c = r$.

$$\mathcal{W}(\mathfrak{gl}_r) = \langle W^1(z), \dots, W^r(z) \rangle$$

Eg: for $r = 2$ $\quad \mathcal{W}(\mathfrak{gl}_2) = \mathrm{Vir}^{c=2} \otimes \mathcal{H}$

Heisenberg algebra.

Idea: generating functions of TR correlators are highest weight vectors of $\mathcal{W}(\mathfrak{gl}_r)$-representations.

Consider $(\mathbb{P}^1, x = z^r, y = -z^{s-r})$, $1 \le s \le r+1$

Consider $\left(\mathbb{P}, x = \frac{z}{r}, y = -z \right)$

associate a $\mathcal{W}(\mathfrak{gl}_r)$-rep on

$$\mathbb{C}(\!(\hbar)\!)[t_1, t_2, t_3, \dots]$$

i.e. $W^i(z) = \sum W^i_k \, z^{-i-k}$

differential operators in t_i

$\underline{\text{Eg}}$: $J_k = \begin{cases} \hbar \dfrac{\partial}{\partial t_k} & k > 0 \\ -k t_{-k} & k < 0 \end{cases}$

$$W^1_k = J_{rk}$$

$$W^2_k = \frac{1}{2} \sum_{\substack{p_1, p_2 \in \mathbb{Z} \\ p_1 + p_2 = rk}} \underset{\in\, \mathbb{C}}{\psi(p_1, p_2)} : J_{p_1} J_{p_2} : \; + \frac{\hbar}{24}(r^2 - 1)\, \delta_{k,0}$$

$$H^i_k = W^i_k \Big|_{t_s \to t_s - \frac{1}{s}}$$ dilaton shift.

Theorem (BBCCN) TR is well-defined iff $r = \pm 1 \bmod s$. Then,

$$Z_{(r,s)} := \exp\left(\sum_{g,n} \frac{\hbar^{g-1}}{n!} \omega_{g,n} \right)$$ is the unique solution to $\mathcal{W}$-constraints:

$$H^i_k Z_{(r,s)} = 0 \quad \forall\ i = 1, \ldots, r \quad k \geq \left\lfloor \frac{s(i-1)}{r} \right\rfloor$$

with $t_i := \dfrac{dz}{z^{i+1}}$

<u>Remarks</u> ① if $s = r+1$, we get Witten r-spin Adler–van Moerbeke.

② $s = 1$ should correspond to Yang–Zhou

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③ $Z_{(r,s)}$ should all be r-KdV tau-functions

$$0 = H^1_k Z_{(r,s)} = \hbar \frac{\partial}{\partial t_{rk}} Z_{(r,s)} = 0 \quad \forall k \geq 1$$

r-th reduction condition.

Back to Θ^r, i.e., $s = r-1$

Theorem (CGG)

Z^{Θ^r} is the unique solution to W-constrs.

$$H^i_k Z^{\Theta^r} = 0 \quad \forall k \geq -i+2 \quad i=1,\dots,r.$$

IV Matrix models & r-KdV (step 4)

Goal: prove W-constraints characterize a r-KdV tau-function.

Proposition: Let Z be any function of the times s.t.

① [r-th reduction] $H^1_k Z = 0 \quad \forall k \geq 1$

② [string equation] $H^r_{-r+2} Z = 0$

Then $Z = Z^{\Theta^r}$

Remark: $H^r_{-r+2} = \hbar \frac{\partial}{\partial t_1} + O(2)$

controls t_1 dependance of τ-function.

Now, we find a candidate τ-function

$r=2$, Norbury conjectured Brezin-Gross-Witten τ-function for KdV.

$r>2$ "negative r" version, r-BGW τ-f"

Mironov- Morozov- Semenoff.

$$Z^{r\text{-BGW}}(\Lambda) =$$

$$\frac{1}{C_N}\int_{\mathcal{H}_N} e^{-\frac{1}{\hbar}\mathrm{Tr}\left(\frac{M^{1-r}}{1-r} - \Lambda M + \hbar\log(M)\right)} [dM]$$

$$t_k = \frac{1}{k}\mathrm{Tr}(\Lambda^{-k}) \quad \text{as} \quad N \to \infty.$$

Conjecture [negative Witten r-spin conj.]

$$Z^{\Theta^r} = Z^{r\text{-BGW}}$$

Remark: Conjecture $\Longleftrightarrow$ proving string eq

$$W^r_{-r+2} Z^{r\text{-BGW}} = 0$$

Theorem (CGG)

The negative Witten r-spin conjecture holds for $r = 2, 3$.

Remark: Proof uses symmetries of τ-function (i.e., Kac-Schwarz op) of Alexandrov

V Open questions

Summary

1. Z^{Θ^r} is the unique sol$^{\underline{n}}$ to W-constraints

2. Conjecturally Z^{Θ^r} is the r-BGW tau function (proved for $r = 2, 3$)

Questions

1. prove conjecture for $r \geq 4$

② Prove that $Z_{(r,s)}$ is a r-KdV tau function.

③ Is $Z_{(r,s)}$ the descendant potential of some CohFT $\Theta^{(r,s)}$?