ReNewQuantum workshop

# The negative Witten $r$-spin conjecture

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arXiv: AG/2205.15621

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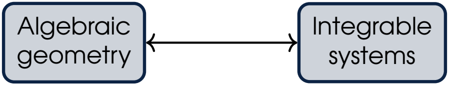
August 25th, 2022

# Motivation




- Started with Witten conjecture/Kontsevich theorem
- Other enumerative problem, e.g. Hurwitz numbers
- Generalised to Virasoro constraints in GW theory
- Involves other theories, e.g. topological recursion, Frobenius mnflds, etc

# The $r$-spin theorem

**Theorem (Witten, Kontsevich, Polischuk–Vaintrob, Givental, Adler–van Moerbeke, Faber–Shadrin–Zvonkine)**

- WITTEN $r$-SPIN CLASS. For $r \geqslant 2$ and $0 \leqslant a_i < r-1$,

$$W^r_{g,n}(a_1,\ldots,a_n) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

of pure degree, constructed from the moduli space of $r$-th roots and satisfying certain axioms.

- TR AND $W$-CNSTRNTS. The descendant potential

$$Z^{(r)} = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2}}{n!} \sum_{k_i,a_i} \int_{\overline{\mathcal{M}}_{g,n}} W^r_{g,n}(a_1,\ldots,a_n) \prod_{i=1}^{n} \psi_i^{k_i} t_{k_i,a_i}\right)$$

is computed from topological recursion on

$$\mathbb{P}^1, \qquad x(z) = \frac{z^r}{r}, \qquad y(z) = -z, \qquad B(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}.$$

Equivalently, $Z^{(r)}$ is the unique solution to certain W-cnstrnts:

$$W_{i,k}Z^{(r)} = 0, \qquad i = 1,\ldots,r,\ k \geqslant -1.$$

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**Theorem**

- $r$-KdV. $Z^{(r)}$ is the unique tau-fnctn of the $r$-KdV hierarchy satisfying the string equation $W_{2,-1}Z^{(r)} = 0$.
- Matrix models. $Z^{(r)}$ coincide with the higher Airy matrix integral:

$$Z^{(r)} = \frac{1}{C_N} \int_{\mathcal{H}_N} e^{-\frac{1}{\hbar} \mathrm{Tr}\left(\frac{M^{r+1}}{r+1} + \Lambda M\right)} dM$$

- Verlinde algebra. The correlators $W^r_{0,n}$ define an algebra isomorphic to the Verlinde algebra of $\mathrm{sl}_2(\mathbb{C})$ at level $r$.
- Hyperbolic geometry & JT gravity. For $r = 2$,

$$W^2_{g,n}(0, \ldots, 0) = 1.$$

When coupled with $\exp(2\pi^2\kappa_1)$, the recursion is a consequence of Mirzakhani's identity. It has connection with JT gravity.

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# The negative side of the story

**Theorem (CGG)**

- THETA $r$-SPIN CLASS. For $r \geqslant 2$ and $0 < a_i \leqslant r-1$,

$$\Theta^r_{g,n}(a_1,\ldots,a_n) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$$

of pure degree, constructed from the moduli space of negative $r$-th roots and satisfying certain axioms.

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**Conjecture (Proved for $r = 2, 3$)**

- $r$-KdV. $Z^{(-r)}$ is the unique tau-fnctn of the $r$-KdV hierarchy satisfying the string equation $\widehat{W}_{r,2-r} Z^{(-r)} = 0$.
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- Super hyperbolic geometry & super JT gravity. For $r = 2$,

$$\Theta^2_{g,n}(1, \dots, 1) = \Theta_{g,n}$$

was introduced by Norbury (who conjectured TR, Virasoro cnstrnts, KdV). when coupled with $\exp(2\pi^2 \kappa_1)$, the recursion is a consequence of super Mirzakhani's identity. It has connection with super JT gravity.

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# The moduli space of curves

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$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{c} C \text{ cmplx cmpct stbl curve} \\ \text{genus } g \text{ with at worst nodal sing.} \\ p_1, \ldots, p_n \text{ marked pnts} \end{array} \right\} / \sim$$

is a cmpct orbifold of $\dim_{\mathbb{C}} = 3g - 3 + n$.

- Attaching maps:

$$q\colon \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$
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- Forgetful maps:

$$p_m\colon \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$$

- Natural classes: consider the vector bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ with fibres $\mathbb{L}_i|_{(C,p_1,\ldots,p_n)} = T^*_{p_i}C$. Define

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}), \qquad \kappa_m = p_*(\psi_{n+1}^{m}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}).$$

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# More natural classes

Collection of cohomology classes that behaves well when restricted to the boundary are called cohomological field theories (CohFT).

Let $I$ be a finite set (colours/primary fields), $\eta = (\eta_{i,j})$ a non-degenerate bilinear form on $V = \mathrm{span}_{\mathbb{Q}} I$. A CohFT is a collection of cohomology classes

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A special class of CohFT, called semisimple, are fully understood.

**Theorem (Teleman's reconstruction theorem)**

If $\Omega$ is a semisimple CohFT, then

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# The Theta class

Fix $r \geqslant 2, 0 < a_i \leqslant r-1$. Consider the moduli space of negative $r$-th roots:

$$\overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n} = \left\{ (C, p_1, \dots, p_n, L) \mid L^{\otimes -r} \cong \omega_{\log,C}(\textstyle\sum_i a_i p_i) \right\} / \sim$$

It admits a forgetful map $f \colon \overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}$.

Define the vector bundle $\mathbb{V} \to \overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n}$

$$\mathbb{V}|_{(C,p_1,\dots,p_n,L)} = H^1(C, L).$$

and define the $r$-spin Theta class

$$\Theta^r_{g,n}(a_1, \dots, a_n) = \star\, f_* c_{top}(\mathbb{V}).$$

**Theorem**

The Theta $r$-spin class is a CohFT. However, it is not semisimple.

NB: excluding the primary field $a = 0$ is essential for the CohFT properties.

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$$\overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n} = \left\{ (C, p_1, \dots, p_n, L) \;\middle|\; L^{\otimes -r} \cong \omega_{\log,C}\left(\textstyle\sum_i a_i p_i\right) \right\} / \sim$$

It admits a forgetful map $f\colon \overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n} \to \overline{\mathcal{M}}_{g,n}$.

Define the vector bundle $\mathbb{V} \to \overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n}$

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and define the *r*-spin Theta class

$$\Theta^r_{g,n}(a_1, \dots, a_n) = \star\, f_* c_{top}(\mathbb{V}).$$

**Theorem**

The Theta $r$-spin class is a CohFT. However, it is not semisimple.

NB: excluding the primary field $a = 0$ is essential for the CohFT properties.

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The situation is even worst: the Frobenius mnfld associated to the $\Theta^r$ is *nowhere* semisimple.

**Idea**

Instead of moving inside the Frobenius mnfld, we can deform the full Frobenius mnfld structure.

For $\epsilon \in \mathbb{C}$, define the deformed $r$-spin Theta class

$$\Theta^{r,\epsilon}_{g,n}(a_1,\ldots,a_n) = \sum_{m\geqslant 0} \frac{\epsilon^m}{m!} p_{m,*}\Theta^r_{g,n+m}(a_1,\ldots,a_n,\underbrace{0,\ldots,0}_{m\text{ times}})$$

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The deformed Theta $r$-spin class is well-defined, it forms a CohFT, and

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# Tautological relations

We can apply Teleman's result to have an explicit expression of $\Theta^{r,\epsilon}$, then take a limit $\epsilon \to 0$.

**Corollary**

$$\Theta^{r,\epsilon}_{g,n}(a_1, \ldots, a_n) = \text{explicit expression in terms of } \psi\text{- and } \kappa\text{-classes}$$

Moreover, $\Theta^r_{g,n}$ is the constant coefficient in $\epsilon$ from the above expression, and all the terms in higher degree vanish in cohomology:

$$[\deg_{\mathbb{C}} = d] \, \mathrm{RHS} = 0$$

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Notation: $\Theta^2_{g,n}(1,\dots,1) = \Theta_{g,n}$ and similarly for the deformed class.

**Corollary**

$$\Theta^{\epsilon}_{g,n} = (-\epsilon^2)^{2g-2+n} \exp\left(-3\frac{\kappa_1}{\epsilon^2} - \frac{21}{2}\frac{\kappa_2}{\epsilon^4} - 69\frac{\kappa_3}{\epsilon^6} - \frac{2529}{4}\frac{\kappa_4}{\epsilon^8} + \dots\right)$$

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# Topological recursion and semisimple CohFTs

Topological recursion is procedure that takes a spectral curve $(\Sigma, x, y\colon \Sigma \to \mathbb{C}, B)$ and recursively construct meromorphic multidifferential forms on $\Sigma$:

$$(\Sigma, x, y\colon \Sigma \to \mathbb{C}, B) \longmapsto \{\omega_{g,n}(z_1, \dots, z_n)\}_{g,n}$$

**Theorem (Eynard, Dunin-Barkowski–Orantin–Shadrin–Spitz)**

There is an explicit correspondence:

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The descendant integrals $\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(a_1, \dots, a_n) \prod_i \psi_i^{k_i}$ are the expansion coefficients of $\omega_{g,n}(z_1, \dots, z_n)$ in a certain basis. The correspondence involves asymptotic expansion of exponential integrals.

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# TR for the deformed Theta

## Problem

Find the spectral curve computing the descendant integrals of $\Theta^{r,\epsilon}_{g,n}$ for $\epsilon \neq 0$.

## Theorem

The spectral curve associated to $\Theta^{r,\epsilon}_{g,n}$ is

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Moreover, taking the limit $\epsilon \to 0$, we obtain the *r*-Bessel curve which computes the descendant integrals of $\Theta^r_{g,n}$.

The exponential integral involved for $r = 2$ is

$$\frac{1}{u}\left(\frac{1}{\sqrt{2\pi u}}\int_\gamma \frac{1}{1-w} e^{-\frac{w^2}{2u}} dw - 1\right) \sim \sum_{k\geqslant 0}(2k+1)!!u^k$$

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# W-constraints

Packing the intersection numbers in generating series

$$Z^{(-r)} = \exp\left( \sum_{g,n} \frac{\hbar^{2g-2}}{n!} \sum_{k_i, a_i} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^r_{g,n}(a_1, \ldots, a_n) \prod_{i=1}^{n} \psi_i^{k_i} t_{k_i, a_i} \right)$$

we can restate topological recursion on the $r$-Bessel curve as a collection of differential operators in the times $t_{k_i,a_i}$ annihilating the partition function:

$$\widehat{W}_{i,k} Z^{(-r)} = 0 \qquad i = 1, \ldots, r, \; k \geqslant 2 - i.$$

E.g. $\widehat{W}_{1,k} = \frac{\partial}{\partial t_{k,0}}$, while $\widehat{W}_{2,k} = L_k$ form a representation of the Virasoro algebra as differential operators of order 2.

This was formalised by Borot–Bouchard–Chidambaram–Creutzig–Noshchenko through higher quantum Airy structures. The operators $\widehat{W}_{i,k}$ form a representation of the $\mathcal{W}^{1-r}(\mathfrak{gl}_r(\mathbb{C}))$.

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# More on W-constraints

1. The solution to the above W-cnstrnts is unique (up to multiplicative constant). Thus, any other solution must coincide with $Z^{(-r)}$.

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   - $r$th reduction condition:
$$\widehat{W}_{1,k} Z = 0$$
   - string equation:
$$\widehat{W}_{r,2-r} Z = 0$$

   then it must satisfy all W-cnstrnts
$$\widehat{W}_{i,k} Z^{(-r)} = 0 \qquad i = 1, \ldots, r, \; k \geqslant 2 - i.$$

**Idea**

Find an KP tau-fnctn that is a solution to $r$th reduction condition (i.e. an $r$-KdV tau-fnctn) and the string equation.

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# The Brézin–Gross–Witten model

The candidate tau-fnct is the higher BGW matrix model:

$$Z^{r\text{-BGW}} = \frac{1}{C_N} \int_{\mathcal{H}_N} e^{-\frac{1}{\hbar} \operatorname{Tr}\left( \frac{M^{-r+1}}{-r+1} - \Lambda M + \hbar N \log(M) \right)} \, dM.$$

It is easy to show that $Z^{r\text{-BGW}}$ is a KP tau-fnctn satisfying the $r$th reduction condition (i.e. an $r$-KdV tau-fnctn).

The string equation $\widehat{W}_{r,2-r} Z^{r\text{-BGW}} = 0$ is hard, since $\widehat{W}_{r,2-r}$ is of order $r$.

**Proposition**

The string equation holds for $Z^{2\text{-BGW}}$ and $Z^{3\text{-BGW}}$.

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# Evidence for the conjecture

1. Proved for $r = 2$ and 3.
2. We proved the W-cnstrnts
$$\widehat{W}_{i,k} Z^{r\text{-BGW}} = 0 \qquad i = 1, \ldots, r, \; k \geqslant 0.$$
However, the solution to this smaller set of cnstrnts is not unique.
3. The wave fnctn associated to $Z^{r\text{-BGW}}$ is a solution to the $r$-Bessel quantum curve:
$$\Big((-1)^r r\,\hat{y}^{r-1}\hat{x}\hat{y} - 1\Big)\psi(x) = 0.$$

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# Questions

1. How to prove that $Z^{r\text{-BGW}}$ satisfies the string equation?
2. Recently, Aggarwal proved the large genus asymptotic
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \sim \frac{(6g-5+2n)!!}{24^g g! \prod_i (2k_i+1)!!}(1+o(1)).$$
Is it possible to prove it using resurgence? Can this be generalised to higher spin (positive and negative)?
3. For $r=2$, the BGW matrix model can be analysed at the strong-field phase and at the weak-field phase. We saw that in the strong-field limit, the BGW integral is generates $\Theta$ descendant integrals:
$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{\epsilon=0} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$
On the other hand, the weak-field limit generates monotone Hurwitz numbers (Novak). These numbers have an ELSV formula (Alexandrov–Lewański–Shadrin), which can be written in terms of the deformed Theta class at $\epsilon = 1$:
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3. For $r=2$, the BGW matrix model can be analysed at the strong-field phase and at the weak-field phase. We saw that in the strong-field limit, the BGW integral is generates $\Theta$ descendant integrals:

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{\epsilon=0} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

   On the other hand, the weak-field limit generates monotone Hurwitz numbers (Novak). These numbers have an ELSV formula (Alexandrov–Lewański–Shadrin), which can be written in terms of the deformed Theta class at $\epsilon=1$:

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# Questions

1. How to prove that $Z^{r\text{-BGW}}$ satisfies the string equation?
2. Recently, Aggarwal proved the large genus asymptotic
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \sim \frac{(6g-5+2n)!!}{24^g g! \prod_i (2k_i+1)!!} \big(1+o(1)\big).$$
Is it possible to prove it using resurgence? Can this be generalised to higher spin (positive and negative)?
3. For $r=2$, the BGW matrix model can be analysed at the strong-field phase and at the weak-field phase. We saw that in the strong-field limit, the BGW integral is generates $\Theta$ descendant integrals:
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Thank you for the attention!